Thermodynamics of Gauge Theory from Gradient Flow
For FlowQCD Collaboration
Thermodynamics of Gauge Theory from Gradient Flow

For FlowQCD Collaboration
$T_{\mu\nu}$
Poincare symmetry
Poincare symmetry

\[ T_{\mu\nu} \]

\[
\begin{bmatrix}
T_{00} & T_{01} & T_{02} & T_{03} \\
T_{10} & T_{11} & T_{12} & T_{13} \\
T_{20} & T_{21} & T_{22} & T_{23} \\
T_{30} & T_{31} & T_{32} & T_{33}
\end{bmatrix}
\]

energy  momentum  stress  pressure
Poincare symmetry

\[ T_{\mu\nu} \]

Einstein Equation

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \]

Hydrodynamic Eq.

\[ \partial_{\mu} T_{\mu\nu} = 0 \]
Definition of the operator is nontrivial because of the explicit breaking of Lorentz symmetry.

\[ T_{\mu\nu} = F_{\mu\rho} F_{\nu\rho} - \frac{1}{4} \delta_{\mu\nu} FF \]

Its measurement is extremely noisy due to high dimensionality and etc.
If we have $T_{\mu\nu}$
Thermodynamics

direct measurement of expectation values

$\langle T_{00} \rangle, \langle T_{ii} \rangle$

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Thermodynamics

direct measurement of expectation values

\( \langle T_{00} \rangle, \langle T_{ii} \rangle \)

Fluctuations and Correlations

viscosity, specific heat, ...

\[ c_V \sim \langle \delta T_{00}^2 \rangle \]
\[ \eta = \langle T_{12}; T_{12} \rangle \]

If we have

\( T_{\mu \nu} \)
Thermodynamics

direct measurement of expectation values
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Fluctuations and Correlations

viscosity, specific heat, ...
\[ c_V \sim \langle \delta T_{00}^2 \rangle \]
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If we have \[ T_{\mu\nu} \]

- confinement string
- EM distribution in hadrons

Hadron Structure

- vacuum configuration
- mixed state on 1\( ^{st} \) transition

Vacuum Structure
Measurement of Thermodynamics using gradient flow

Masakiyo Kitazawa (Osaka U.)

for FlowQCD Collaboration
Asakawa, Hatsuda, Iritani, Itou, MK, Suzuki

FlowQCD, arXiv:1312.7492[hep-lat]; to appear in PRD

LATTICE2014, 28 June 2014, New York
Flow QCD Collaboration

from

Gradient flow

\[ \partial_t B_\mu = D_\nu G_{\nu\mu} \]
Gradient flow
\[ \partial_t B_\mu = D_\nu G_{\nu \mu} \]

Hydrodynamic flow
\[ \partial_\mu T_{\mu \nu} = 0 \]

Thermodynamics
- direct measurement of expectation values
  \[ \langle T_{00} \rangle, \langle T_{ii} \rangle \]

Fluctuations and Correlations
- viscosity, specific heat, ...
  \[ c_v \sim \langle \delta T_{00}^2 \rangle \]
  \[ \eta = \langle T_{12}; T_{12} \rangle \]
Gradient Flow
YM Gradient Flow

\[ \partial_t A_\mu(t, x) = -\frac{\partial S_{YM}}{\partial A_\mu} \]

\[ A_\mu(0, x) = A_\mu(x) \]

\( t: \text{"flow time"} \)
\( \text{dim: [length}^2\text{]} \)
YM Gradient Flow

$$\partial_t A_\mu(t, x) = -\frac{\partial S_{YM}}{\partial A_\mu}$$

$$A_\mu(0, x) = A_\mu(x)$$

t: "flow time"

dim:[length$^2$]

- transform gauge field like diffusion equation
  $$\partial_t A_\mu = D_\nu G_{\mu\nu} = \partial_\nu \partial_\nu A_\mu + \cdots$$

- diffusion length \( d \sim \sqrt{8t} \)

- This is NOT the standard cooling/smearing

- All composite operators at \( t>0 \) are UV finite Luescher, Weisz, 2011
Applications of Gradient Flow

① scale setting
② running coupling
③ topology
④ operator relation
⑤ autocorrelation, etc.

A. Ramos, plenary, Friday
Small Flow Time Expansion of Operators and EMT
Operator Relation

Luescher, Weisz, 2011

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) O_i^R(x) \]

an operator at \( t > 0 \)

remormalized operators of original theory
Operator Relation

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) O_i^R(x) \]

- \( \tilde{O}(t, x) \): an operator at \( t > 0 \)
- \( c_i(t) \): coefficients
- \( O_i^R(x) \): remormalized operators of original theory

Luescher, Weisz, 2011
Operator Relation

\[ \tilde{\mathcal{O}}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) \mathcal{O}_i^R(x) \]

an operator at \( t>0 \)

remormalized operators of original theory

Luescher, Weisz, 2011
Constructing EMT

\[ \tilde{O}(t, x) \xrightarrow{t \to 0} \sum_i c_i(t) \mathcal{O}_i^R(x) \]

- **gauge-invariant dimension 4 operators**

\[
\begin{align*}
U_{\mu\nu}(t, x) &= G_{\mu\rho}(t, x)G_{\nu\rho}(t, x) - \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x) \\
E(t, x) &= \frac{1}{4}\delta_{\mu\nu}G_{\mu\nu}(t, x)G_{\mu\nu}(t, x)
\end{align*}
\]
Constructing EMT 2

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T^R_{\mu\nu}(x) - \frac{1}{4} \delta_{\mu\nu} T^R_{\rho\rho}(x) \right] + \mathcal{O}(t) \]

\[ E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) T^R_{\rho\rho}(x) + \mathcal{O}(t) \]

Suzuki coeffs.

\[
\begin{align*}
\alpha_U(t) &= g^2 \left[ 1 + 2b_0 s_1 g^2 + O(g^4) \right] \\
\alpha_E(t) &= \frac{1}{2b_0} \left[ 1 + 2b_0 s_2 g^2 + O(g^4) \right]
\end{align*}
\]

\[ g = g(1/\sqrt{8t}) \]

\[ s_1 = 0.03296 \ldots \]

\[ s_2 = 0.19783 \ldots \]

See also, Patella, Parallel7E, Thu.
Constructing EMT 2

\[ U_{\mu\nu}(t, x) = \alpha_U(t) \left[ T_{\mu\nu}^R(x) - \frac{1}{4} \delta_{\mu\nu} T_{\rho\rho}^R(x) \right] + \mathcal{O}(t) \]

\[ E(t, x) = \langle E(t, x) \rangle + \alpha_E(t) T_{\rho\rho}^R(x) + \mathcal{O}(t) \]

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Remormalized EMT

\[
T_{\mu\nu}^R(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t, x)_{\text{subt.}} \right]
\]
Numerical Analysis on the Lattice
Gradient Flow Method

lattice regularized gauge theory

T_{\mu\nu}

continuum theory (with dim. reg.)

analytic (perturbative)

U_{\mu\nu}, E

continuum theory (with dim. reg.)

gradient flow
Gradient Flow Method

- **Lattice regularized gauge theory**
  - Gradient flow

- **Continuum theory (with dim. reg.)**
  - Perturbative
  - Analytic
  - Gradient flow

- Measurement on the lattice
Caveats

- Gauge field has to be sufficiently smeared!
- Perturbative relation has to be applicable: $\sqrt{8t} \ll \Lambda^{-1}, T^{-1}$
- Measurement on the lattice
- Perturbative relation has to be applicable!
- $a \ll \sqrt{8t}$
Caveats

Gauge field has to be sufficiently smeared!

\[ a \ll \sqrt{8t} \]

Perturbative relation has to be applicable!

\[ \sqrt{8t} \ll \Lambda^{-1}, T^{-1} \]

Continuum theory (with dim. reg.)

\[ T_{\mu\nu}^R \]

Analytic (perturbative)

Gradient flow

\[ U_{\mu\nu}, E \]

Continuum theory (with dim. reg.)

\[ a \ll \sqrt{8t} \ll \Lambda^{-1} \]
\[ \tilde{T}_{\mu\nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu\nu}(t) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t)_{\text{subt.}}. \]

\[ T_{\mu\nu}^R = \lim_{t \to 0} \tilde{T}_{\mu\nu}(t) \]

\[ \langle \tilde{T}_{\mu\nu}(t) \rangle \] in continuum

non-perturbative region

\[ \langle T_{\mu\nu}^R \rangle \]

O(t) effect

classical
\[ \tilde{T}_{\mu\nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu\nu}(t) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t)_{\text{subt.}} \]

\[ T_{\mu\nu}^R = \lim_{t \to 0} \tilde{T}_{\mu\nu}(t) \]

\[ \langle \tilde{T}_{\mu\nu}(t) \rangle \text{ in continuum} \]

\[ \langle T_{\mu\nu}^R \rangle \text{ non-perturbative region} \]

\[ \langle T_{\mu\nu} \rangle \text{ O(t) effect} \]

\[ t \to 0 \text{ limit with keeping } t \gg a^2 \]

on the lattice

classical

\[ t \to 0 \text{ limit with keeping } t \gg a^2 \]
Numerical Simulation

- SU(3) YM theory
- Wilson gauge action

Simulation 1
(arXiv:1312.7492)
- lattice size: $32^3 \times N_t$
- $N_t = 6, 8, 10$
- $\beta = 5.89 - 6.56$
- $\sim 300$ configurations

using SX8 @ RCNP
SR16000 @ KEK

Simulation 2
(new, preliminary)
- lattice size: $64^3 \times N_t$
- $N_t = 10, 12, 14, 16$
- $\beta = 6.4 - 7.4$
- $\sim 2000$ configurations

using BlueGeneQ @ KEK
efficiency $\sim 40\%$

twice finer lattice!
Numerical Simulation

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\( \varepsilon-3p \) at \( T=1.65T_c \)

\[
\tilde{T}_{\mu \nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu \nu}(t) + \frac{\delta_{\mu \nu}}{4\alpha_E(t)} E(t)_{\text{subt.}}
\]

\[
T_{\mu \nu}^R = \lim_{t \to 0} \tilde{T}_{\mu \nu}(t)
\]

\begin{align*}
N_t &= 6, 8, 10 \\
\sim 300 \text{ confs.}
\end{align*}

Emergent plateau!

\[
2a \lesssim \sqrt{8t} \lesssim 0.4T^{-1}
\]

the range of \( t \) where the EMT formula is successfully used!
\( \varepsilon - 3p \) at \( T = 1.65T_c \)

\[
\tilde{T}_{\mu \nu}(t) = \frac{1}{\alpha_U(t)} U_{\mu \nu}(t) + \frac{\delta_{\mu \nu}}{4\alpha_E(t)} E(t)_{\text{subt.}}
\]

\( T_{\mu \nu}^R = \lim_{t \to 0} \tilde{T}_{\mu \nu}(t) \)

Emergent plateau!

\[ 2a \lesssim \sqrt{8t} \lesssim 0.4T^{-1} \]

the range of \( t \) where the EMT formula is successfully used!

\( Nt = 6, 8, 10 \)

\(~300\) confs.
Emergent plateau!

\[ 2a \lesssim \sqrt{8t} \lesssim 0.4T^{-1} \]

Direct measurement of e+p on a given T!

NO integral / NO vacuum subtraction
Continuum Limit

32^3 \times N_t

N_t = 6, 8, 10

T/T_c = 0.99, 1.24, 1.65
Continuum Limit

Comparison with previous studies

Boyd+1996

$32^3 \times N_t$

$N_t = 6, 8, 10$

$T/T_c = 0.99, 1.24, 1.65$
Numerical Simulation

- SU(3) YM theory
- Wilson gauge action

Simulation 1

(\text{arXiv:1312.7492})

- lattice size: $32^3 \times N_t$
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Simulation 2

(new, preliminary)

- lattice size: $64^3 \times N_t$
- $N_t = 10, 12, 14, 16$
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using BlueGeneQ @ KEK
efficiency $\sim 40\%$

twice finer lattice!
Entropy Density on Finer Lattices

\[ T = 2.31T_c \]
\[ 64^3 \times N_t \]
\[ N_t = 10, 12, 14, 16 \]
\[ 2000 \text{ confs.} \]

- The wider plateau on the finer lattices
- Plateau may have a nonzero slope

FlowQCD, 2013
\[ T = 1.65T_c \]
Continuum Extrapolation

- $T = 2.31 T_c$
- 2000 confs
- $N_t = 10 \sim 16$

Continuum extrapolation is stable

$a \to 0$ limit with fixed $t/a^2$

\[
\sqrt{8tT} = 1312.7492
\]

\[
t/a^2 = 1.3
\]

\[
t/a^2 = 1.0
\]

$N_t = 16, 14, 12, 10$
Summary

$T_{\mu\nu}^R(x)$
Summary

EMT formula from gradient flow

$$T^R_{\mu\nu}(x) = \lim_{t \to 0} \left[ \frac{1}{\alpha_U(t)} U_{\mu\nu}(t, x) + \frac{\delta_{\mu\nu}}{4\alpha_E(t)} E(t, x)_{\text{subt.}} \right]$$

This formula can successfully define and calculate the EMT on the lattice.

This operator provides us with novel approaches to measure various observables on the lattice!

This method is direct, intuitive and less noisy.
Many Future Studies!!

Thermodynamics

direct measurement of expectation values
\[ \langle T_{00}, T_{ii} \rangle \]

Fluctuations and Correlations

viscosity, specific heat:
\[
\frac{c_v}{\langle \delta T_{00}^2 \rangle} = \eta = \langle T_{12}; T_{12} \rangle
\]

Now we have \( T_{\mu\nu} \)

- confinement string
- EM distribution in hadrons

Hadron Structure

- vacuum configuration
- mixed state on 1st transition

Vacuum Structure

Other observables

full QCD Makino, Suzuki, 2014
non-pert. improvement Patella 7E(Thu)

O(a) improvement
Nogradi, 7E(Thu); Sint, 7E(Thu)
Monahan, 7E(Thu)
and etc.
One More Thing...
One More Thing...

Fluctuations and Correlations

viscosity, specific heat, ...

\[ c_V \sim \langle \delta T^2_{00} \rangle \]

\[ \eta = \langle T_{12}; T_{12} \rangle \]
Energy Correlation Function

\[ \langle T_{00}(\tau) T_{00}(0) \rangle \]

\( T = 2.31T_c \)

\( b = 7.2, \ N_t = 16 \)

2000 confs

p = 0 correlator

\( \tau < 2\sqrt{2t} \)

smeared
Energy Correlation Function

\[ \langle T_{00}(\tau) T_{00}(0) \rangle \]

T=2.31Tc
b=7.2, Nt=16
2000 confs
p=0 correlator

\[ \tau < 2\sqrt{2t} \]
Smeared

\[ \tau \text{ independent const.} \rightarrow \text{energy conservation} \]
Energy Correlation Function

\[ \langle T_{00}(\tau)T_{00}(0) \rangle \]

- \( T = 2.31T_c \)
- \( b = 7.2, \ N_t = 16 \)
- 2000 confs
- \( p = 0 \) correlator

- \( \tau \) independent const.
- \( \Rightarrow \) energy conservation

- specific heat

\[ c_V = \frac{\langle \delta E^2 \rangle}{VT^2} \]

- \( \Rightarrow \) Novel approach to measure specific heat!
Keep your attention to this new flow just like...
Keep your attention to this new flow
just like...
Thank you for your attention!