A construction of the Schrödinger Functional for Möbius Domain Wall Fermions

Yuko Murakami

K-I. Ishikawa

(Hiroshima University)

Contents

1.Background

The Schrödinger Functional Scheme on the lattice has the following merits and demerits.

- Merits
  - We can calculate a running coupling and running mass non-perturbatively using the step scaling function in the region from high energy to low energy.
  - We can also get renormalization constants for various operators non-perturbatively.
- Demerit
  - Lattice regularization introduces cut-off errors. We need to tune the lattice action including the SF boundary parameters.
  - There is no continuum chiral symmetry on the lattice.

The lattice chiral symmetry can be realized by the Ginsparg Wilson relation.

- Domain Wall Fermion(DWF) [Y.Shamir, NPB406 (1993)]
- overlap Fermion [H.Neuberger, PLB427 (1998)]

Can we apply the SF scheme to the lattice Chiral Fermions ?

1. Background

The SF boundary term must break the chiral symmery on the SF temporal boundary. [M.Luscher, JHEP 05 (2006)]

$$\gamma_5 S(x,y) + S(x,y)\gamma_5 = \int_{z_0=0} d^3z\, S(x,z)\gamma_5 P_- S(z,y) + \int_{z_0=T} d^3z\, S(x,z)\gamma_5 P_+ S(z,y)$$

The overlap fermion and the DWF with SF boundary term should reproduce this relation in the continuum limit.

There are following works for the construction.

- Y.Taniguchi : Schrödinger functional formalism with Ginsparg-Wilson fermion [JHEP0512(2005)]
- M.Lüscher :The Overlap fermion based on the Universality arguments.[JHEP 05 (2006)]
- Y.Taniguchi : The DWF based on the temporal orbifording [JHEP 0610 (2006)] .
- S.Takeda : The DWF based on the Universality arguments[PRD 87 (2010)].

In this talk, I would like to talk about the construction of the Schrodinger Functional Scheme for the Mobius Domain wall fermion.

2. A construction of the MDWF with the SF boundary condition

- The Definition of the MDWF [R.C.Brower,et al, NPPS140 (2005)]

$$D_{MDWF} = \begin{pmatrix} D_1^+ & D_1^- P_L & 0 & 0 & 0 & m_f P_R \\ D_2^- P_R & D_2^+ & D_2^- P_L & 0 & 0 & 0 \\ 0 & D_3^- P_R & D_3^+ & D_3^- P_L & 0 & 0 \\ 0 & 0 & D_4^- P_R & D_4^+ & D_4^- P_L & 0 \\ 0 & 0 & 0 & D_5^- P_R & D_5^+ & D_5^- P_L \\ m_f P_L & 0 & 0 & 0 & D_6^- P_R & D_6^+ \end{pmatrix}$$

For example, $N_5 = 6$

$$D_s^+(n;m) = (D_{WF} b_s + \mathbf{1})(n;m)$$
$$D_s^-(n;m) = (D_{WF} c_s - \mathbf{1})(n;m)$$
$$P_{R/L} = \frac{1 \pm \gamma_5}{2}$$

- $N_5 \to \infty$,Effective operator
- b_s, c_s are tunable parameters, improve the chiral symmetry

	Standard Shamir	Borici	Shamir Optimal	Chiu Optimal
(b_s, c_s)	$(a, 0)$	(a, a)	$(\omega_s + a, \omega_s - a)$	(ω_s, ω_s)
5th dimansional parameter	Constant	Constant	The Zolotarev approximation	The Zolotarev approximation

If we apply the SF boundary condition to this operator naively, we cannot reproduce the proper continuum chiral symmetry .

$$\gamma_5 S(x,y) + S(x,y)\gamma_5 = \int_{z_0=0} d^3z\, S(x,y)\gamma_5 P_- S(x,y) + \int_{z_0=T} d^3z\, S(x,y)\gamma_5 P_+ S(x,y)$$

In order to recover the continuum chiral symmetry, a temporal boundary operator has been introduced by Takeda for the Standard DWF.

We will apply the Takeda's approach to the MDWF by adding the temporal boundary term .

2. A construction of the MDWF with the SF boundary condition

- The SF construction for the Standard DWF [S.Takeda,PRD 87 (2010)]

$$D_{DWF}^{SF} = (D_{MDWF} + B_{SF}) =$$

$$\begin{pmatrix} D_{WF}+1 & P_R & 0 & 0 & 0 & c_{SF}B + m_f P_R \\ P_L & D_{WF}+1 & P_R & 0 & c_{SF}B & 0 \\ 0 & P_L & D_{WF}+1 & c_{SF}B + P_R & 0 & 0 \\ 0 & 0 & -c_{SF}B + P_L & D_{WF}+1 & P_R & 0 \\ 0 & -c_{SF}B & 0 & P_L & D_{WF}+1 & P_R \\ -c_{SF}B + m_f P_L & 0 & 0 & 0 & P_L & D_{WF}+1 \end{pmatrix}$$

$$b_s = 1.0$$
$$c_s = 0.0$$

$$B_{SF}(n, s_5; m, t_5) = f(s_5) B(n,m) \delta(s_5, N_5 - t_5 + 1)$$

$$B(n,m) = \delta(n,m)\delta(n_4, m_4)\gamma_5 (P_L \delta(n_4, a) + P_R \delta(n_4, T-a))$$

$$f(s_5) = \begin{cases} c_{SF} & (1 \le s_5 \le \frac{N_5}{2}) \\ -c_{SF} & (1 + \frac{N_5}{2} \le s_5 \le N_5) \end{cases}$$

- B_{SF} only has support at $t = a\,, T - a$, the time.
- The 5th dimensional structure of the B_{SF} is similar to the mass term and breaks the lattice chiral symmetry explicitly.
- D_{DWF}^{SF} satisfies the lattice discrete symmetry $(\mathrm{C}, \mathrm{P}, \mathrm{T}, \Gamma_5)$.

We apply B_{SF} to the MDWF Operator.

2. A construction of the MDWF with the SF boundary condition - The Application of the SF boundary term B_{SF} to the MDWF

The SF boundary term for the Mobius DWF ($N_5 = 6$)

$$D_{MDWF}^{SF} = (D_{DWF} - D^- B_{SF}) =$$

$$\begin{pmatrix} D_1^+ & D_1^- P_L & 0 & 0 & 0 & -D_1^- c_{SF} B - m_f D_1^- P_R \\ D_2^- P_R & D_2^+ & D_2^- P_L & 0 & -D_2^- c_{SF} B & 0 \\ 0 & D_3^- P_R & D_3^+ & -D_3^- c_{SF} B + D_3^- P_L & 0 & 0 \\ 0 & 0 & D_4^- c_{SF} B + D_4^- P_R & D_4^+ & D_4^- P_L & 0 \\ 0 & D_5^- c_{SF} B & 0 & D_5^- P_R & D_5^+ & D_5^- P_L \\ D_6^- c_{SF} B - m_f D_6^- P_L & 0 & 0 & 0 & D_6^- P_R & D_6^+ \end{pmatrix}$$

In order to satisfy the discrete symmetries($\mathrm{C}, \mathrm{P}, \mathrm{T}, \Gamma_5$), b_s and c_s must have the 5th direction parity symmetry (Palindrome).

$$\begin{pmatrix} D_1^+ & D_1^- P_L & 0 & 0 & 0 & -D_1^- c_{SF} B - m_f D_1^- P_R \\ D_2^- P_R & D_2^+ & D_2^- P_L & 0 & -D_2^- c_{SF} B & 0 \\ 0 & D_3^- P_R & D_3^+ & -D_3^- c_{SF} B + D_3^- P_L & 0 & 0 \\ 0 & 0 & D_3^- c_{SF} B + D_3^- P_R & D_3^+ & D_3^- P_L & 0 \\ 0 & D_2^- c_{SF} B & 0 & D_2^- P_R & D_2^+ & D_2^- P_L \\ D_1^- c_{SF} B - m_f D_1^- P_L & 0 & 0 & 0 & D_1^- P_R & D_1^+ \end{pmatrix}$$

What to do in the Optimal DWF type for cost reduction?

2. The Application of the SF boundary term B_{SF} to the MDWF
- The Optimal Coefficients for the MDWF with the SF boundary term

We give up the optimality to maintain the discrete symmetry.

The Sign functional approximation by Zolotarev (Optimal)

$$R(x, N_5) = \frac{\prod_{j=1}^{N_5}(1+\omega_j x) - \prod_{j=1}^{N_5}(1-\omega_j x)}{\prod_{j=1}^{N_5}(1+\omega_j x) + \prod_{j=1}^{N_5}(1-\omega_j x)} \xrightarrow[N_5\to\infty]{} \mathrm{Sign}(x)$$

$$b_s = \begin{pmatrix} b_1 & & & & & \\ & b_2 & & & & \\ & & b_3 & & & \\ & & & b_4 & & \\ & & & & b_5 & \\ & & & & & b_6 \end{pmatrix}$$

- All numbers of b_s are different ($b_i = c_j$ for $i \neq j$)
- No 5th dimensional parity Symmetry

Introduce the 5th dimensional parity symmetry

$$\omega_j = \omega_{N_5-1+j}$$

$$\tilde{R}(x, N_5) = \frac{\prod_{j=1}^{N_5/2}(1+\omega_j x)^2 - \prod_{j=1}^{N_5/2}(1-\omega_j x)^2}{\prod_{j=1}^{N_5/2}(1+\omega_j x)^2 + \prod_{j=1}^{N_5/2}(1-\omega_j x)^2}$$

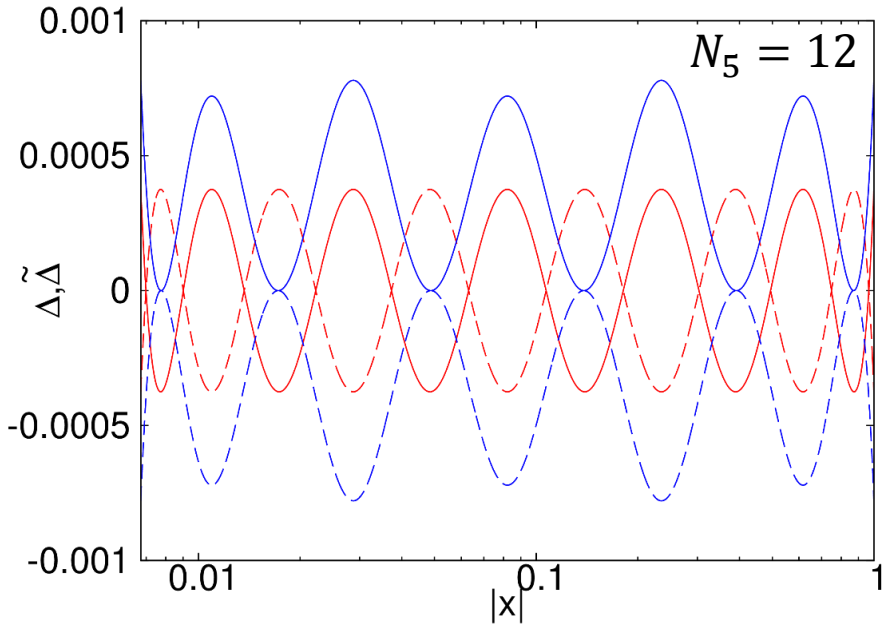
$$b_s = \begin{pmatrix} b_1 & & & & & \\ & b_2 & & & & \\ & & b_3 & & & \\ & & & b_3 & & \\ & & & & b_2 & \\ & & & & & b_1 \end{pmatrix}$$

- Loses Optimality.
- Poses the discrete symmetry

We compare the Optimal approximation and the doubled Zolotarev approximation.

2. The Application of the SF boundary term B_{SF} to the MDWF - The error analysis of the doubled Zolotarev approximation

The observation of the approximation error for the Zolotarev approximation and for the doubled Zolotarev approximation.



- Type A The Optimal Zolotarev approx.

$$\Delta(x) = \mathrm{sign}(x) - R_{N_5}(x)$$

$x > 0$: the red solid line

$x < 0$: the red dashed line

- Type B The doubled Zolotarev approx.

$$\tilde{\Delta}(x) = \mathrm{sign}(x) - \tilde{R}_{N_5}(x)$$

$x > 0$: the blue solid line

$x < 0$: the blue dashed line

Type	The region of the error bound
Type A	Oscillation between plus and minus
Type B	Oscillation in one side

$$\widetilde{R_{N_5}}(x) = \frac{2R_{N_{5/2}}}{R_{N_{5/2}}^2 + 1}$$

- We find the relation

$$\mathrm{Max}|\tilde{\Delta}(x)| \approx 2\mathrm{Max}|\Delta(x)|$$

We employ the type B coefficients for the MDWF with the SF boundary term.

3.Numerical results

We check the Universality of the MDWF with the SF boundary term in the following set up.

- Common parameter

m_f	M	c_{SF}	θ
0.0	1.0	1.0	0.0

  - Choice of a set of b_s and c_s $\quad b_s + c_s = \omega_s, b_s - c_s = 1$

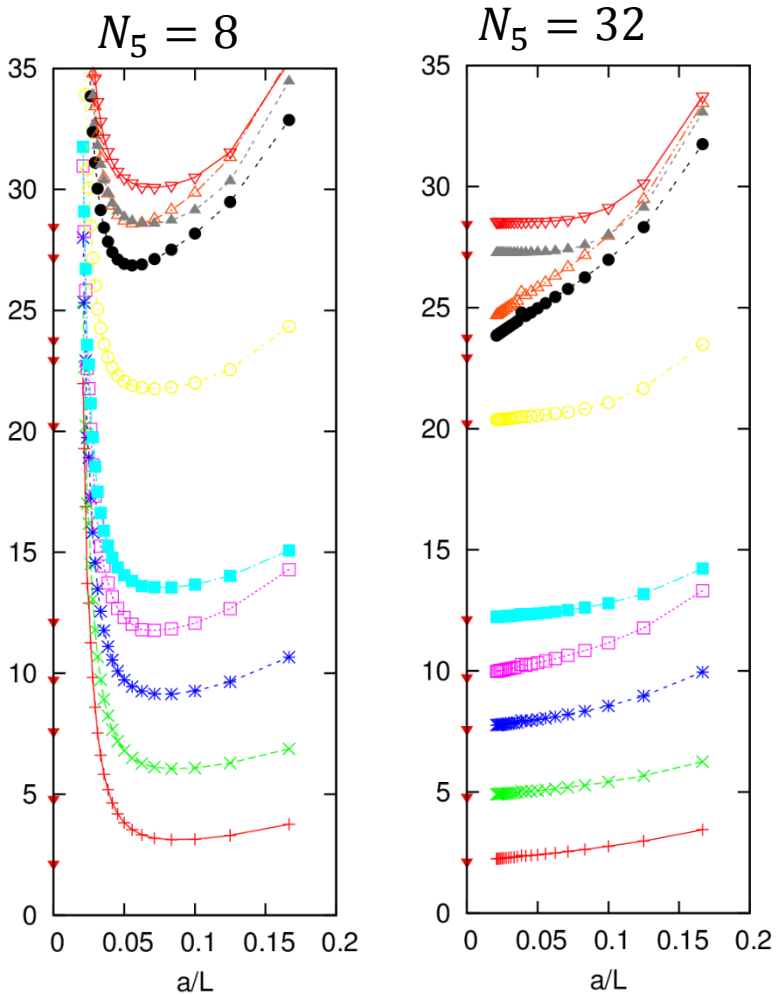
    ω_s:typeB

  - We call this Quasi-Optimal Shamir DWF
- Universality Check $\quad D_{eff} = \epsilon^\dagger P^\dagger D_{PV}^{-1} D_{MDWF} P \epsilon$ [A.Borici, NPPS 83 (2000)]
  - The lowest eigenvalues of the Hermitian operator $L^2 D_{eff}^\dagger D_{eff}$
  - The GW relation of the Propagator
- The comparison the GW relation violation between the Quasi-Optimal Shamir and the Standard DWF

3.Neumerical results

- The lowest eigenvalues of the Hermitian operator $L^2 D_{eff}^{\dagger} D_{eff}$

$(6 \leq L \leq 48)$



- $N_5 = 8$

  Eigenvalues increase away from eigenvalues of the continuum limit operator.

  This is because the accuracy of the sign function approximation becomes worse as the N_5 is fixed at constant.

- $N_5 = 32$

  Eigenvalues approach to the continuum limit values.

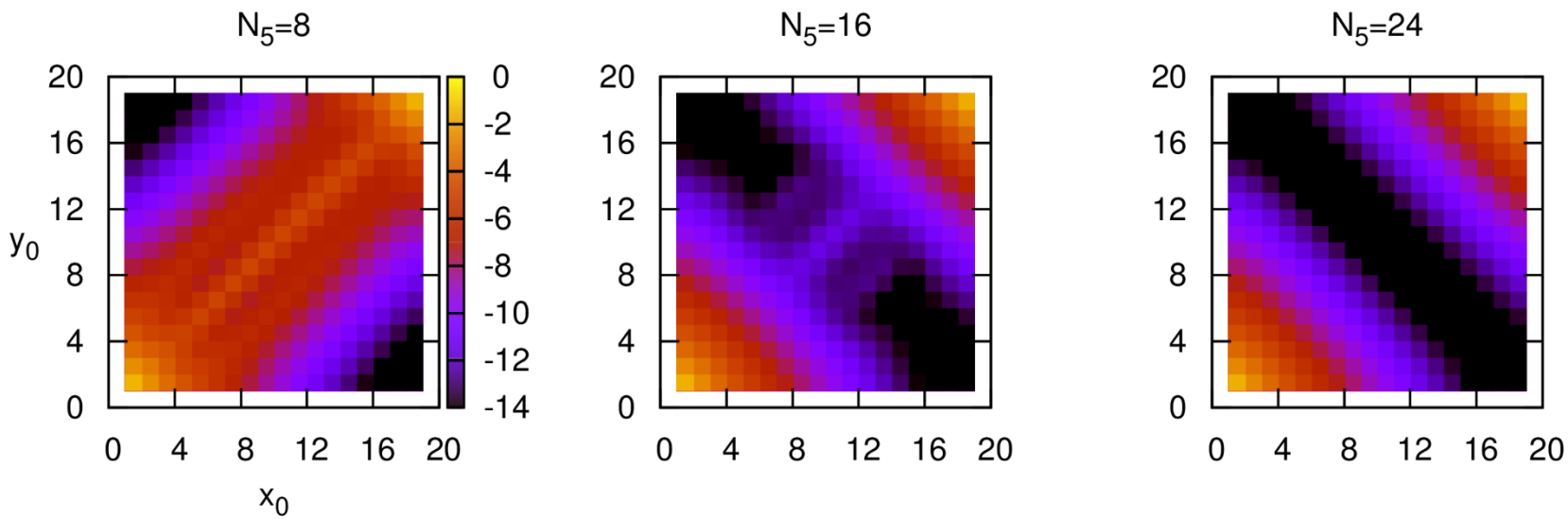
- The Universality is realized if the accuracy of the sign function approximation is good enough.

The Continuum limit from "S. Sint, R. Sommer, NPB 465 (1996)"

3. Numerical results - The Ginsparg Wilson Relation

We check the GW relation violation

$$|\delta(x_0,y_0)| = \left|\gamma_5 D_{eff}(\boldsymbol{p}=\boldsymbol{0},x_0,y_0) + D_{eff}(\boldsymbol{p}=\boldsymbol{0},x_0,y_0)\gamma_5 - 2\sum_{z_0=1}^{T-1} D_{eff}(\boldsymbol{p}=\boldsymbol{0},x_0,z_0)\,\gamma_5 D_{eff}(\boldsymbol{p}=\boldsymbol{0},z_0,y_0)\right|$$



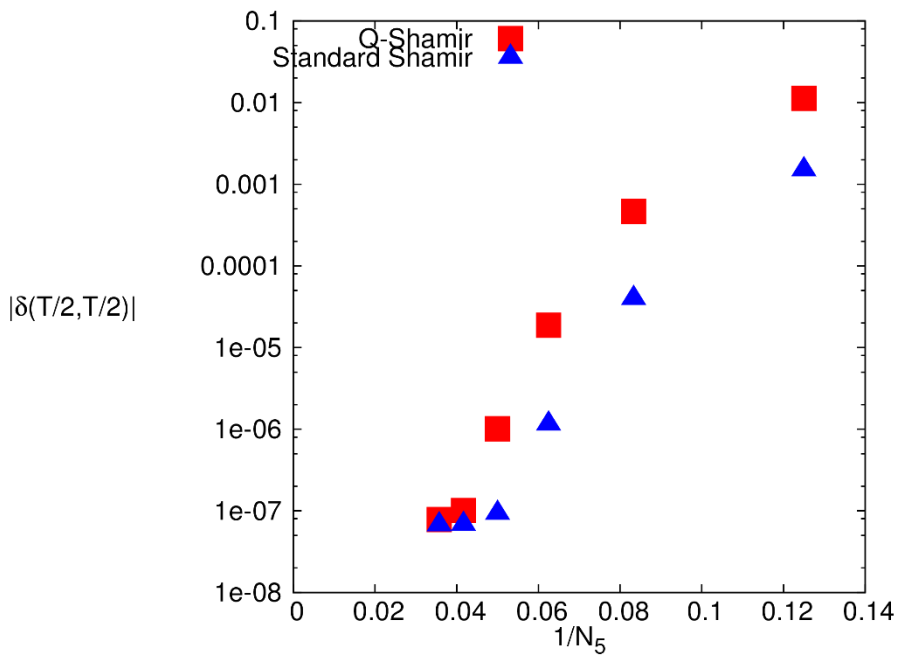
- $N_5 \to \infty$, the bulk chiral symmetry restores.
- The chiral symmetry violation remains at the temporal boundaries.
- The Universality is seems to be realized.

parameter
$L = 6, T = 20$

$$\gamma_5 S(x,y) + S(x,y)\gamma_5 = \int_{z_0=0} d^3z\, S(x,z)\gamma_5 P_- S(z,y) + \int_{z_0=T} d^3z\, S(x,z)\gamma_5 P_+ S(z,y)$$

3.Numerical results - Standard vs. Quasi-Optimal Shamir DWF

We compare the GW relation violation between the Quasi Optimal Shamir DWF and the standard DWF.



parameter

$L = 6, T = 20$

$N_5 = 8,12,16,20,24,32$

The Standard DWF has a smaller bulk GW relation violation than the Quasi-Optimal DWF.

The computational cost of the Standard DWF seems to be better than that of the Quasi-Optimal DWF.

4.Summary

In this talk, we show the Construction of the SF scheme for the MDWF.

- We introduce the 5th parity symmetry on the MDWF parameter b_s and c_s in order to hold the discrete symmetries (C,P,T,Γ5) with the SF boundary term.
- We propose the choice of the MDWF parameter to improve the lattice chiral symmetry with the SF boundary term.
- We Check the universality of the Optimal type DWF with SF scheme

We find

- The proposal for the MDWF coefficients is as good as the Zolotarev optimal one.
- If the accuracy of the sign function approximation is enough, our approach satisfies the universality.
- the Standard DWF seems to be better than the Quasi-Optimal Shamir DWF in view of the computational cost.

We conclude that the construction of SF scheme is available by the imposition the 5th dimensional Symmetry on the MDWF.

Future Work

- Tuning the boundary coefficient c_{SF}
- Checking the one loop beta function

Thank you !

Back ups

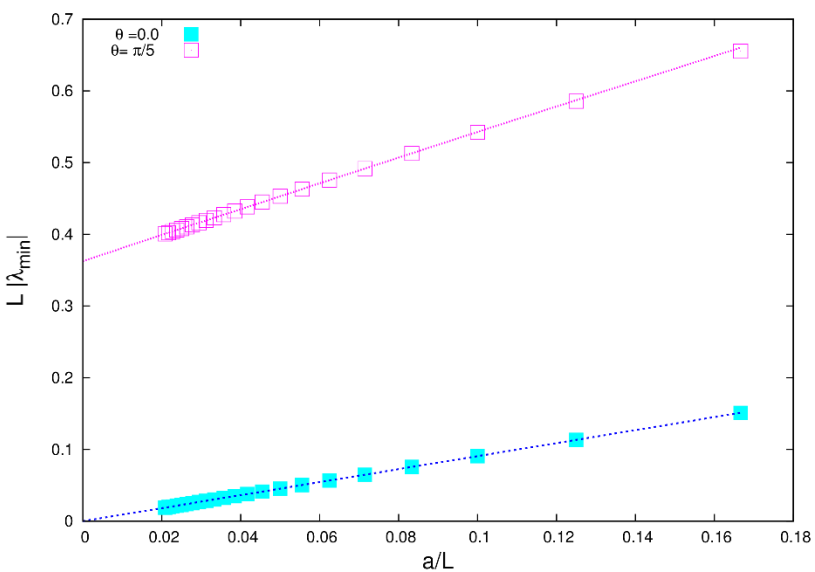
Contimuum Limit of λ_{min} , λ_{max}

We need $N_5, \lambda_{min}, \lambda_{max}$ to chose the 5th dimensional parameter b_s, c_s
$\lambda_{min}, \lambda_{max}$ are the highest and the lowest eigenvalue of the DWF kernel operator

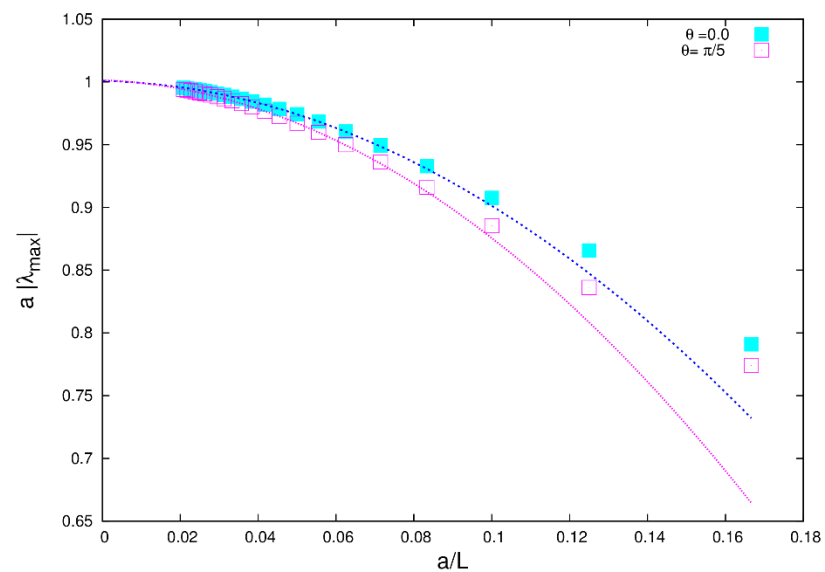
$$H_w = \frac{\gamma_5 D_{WF}}{\gamma_5 D_{WF}+2} \quad : \text{ Shamir type}$$
$$H_w = \gamma_5 D_{WF} \quad : \text{ Borici type}$$

When we use the Zolotarev approximation, the range of the approximate is given by these eigenvalues $\lambda_{min} \leq x \leq \lambda_{max}$

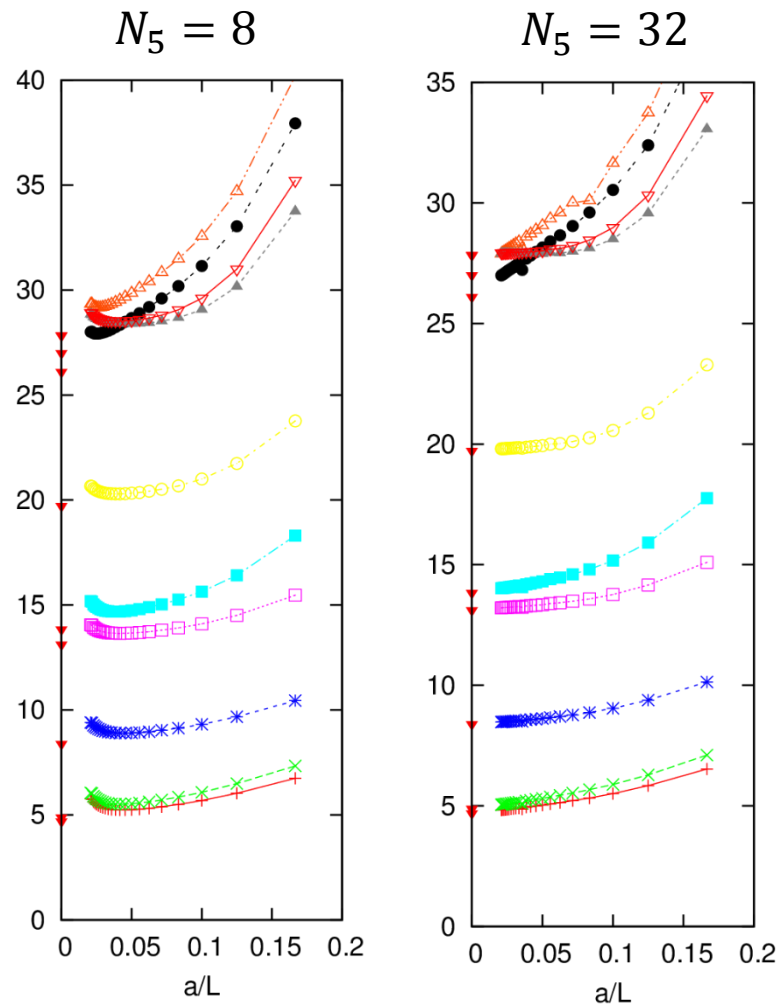
- $|\lambda_{min}|$



- $|\lambda_{max}|$

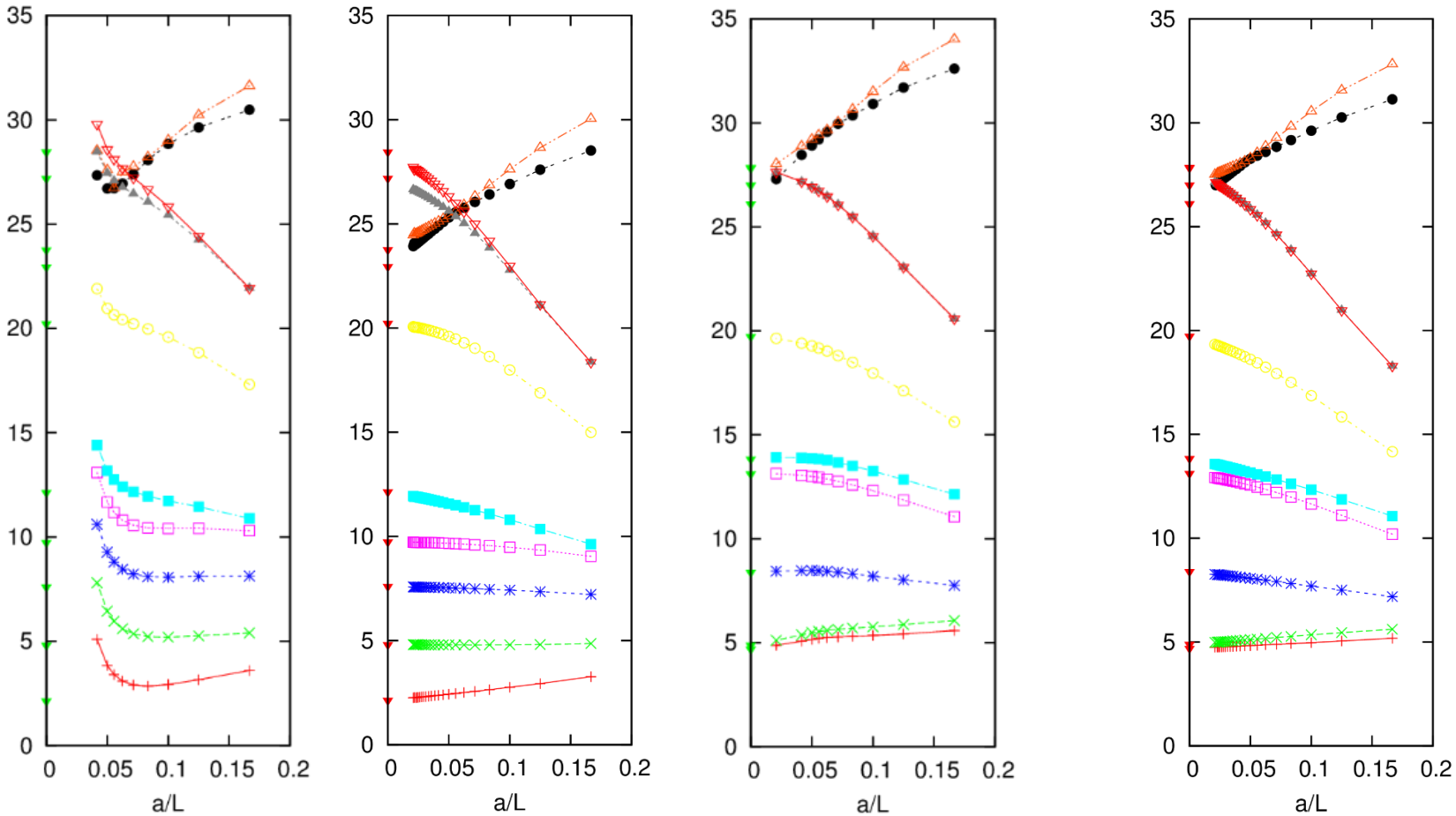


The Quasi-Optimal Shamir DWF $(\theta = \pi/5)$



The Quasi-Optimal Chiu DWF

$\theta = 0$ $\theta = \pi/5$

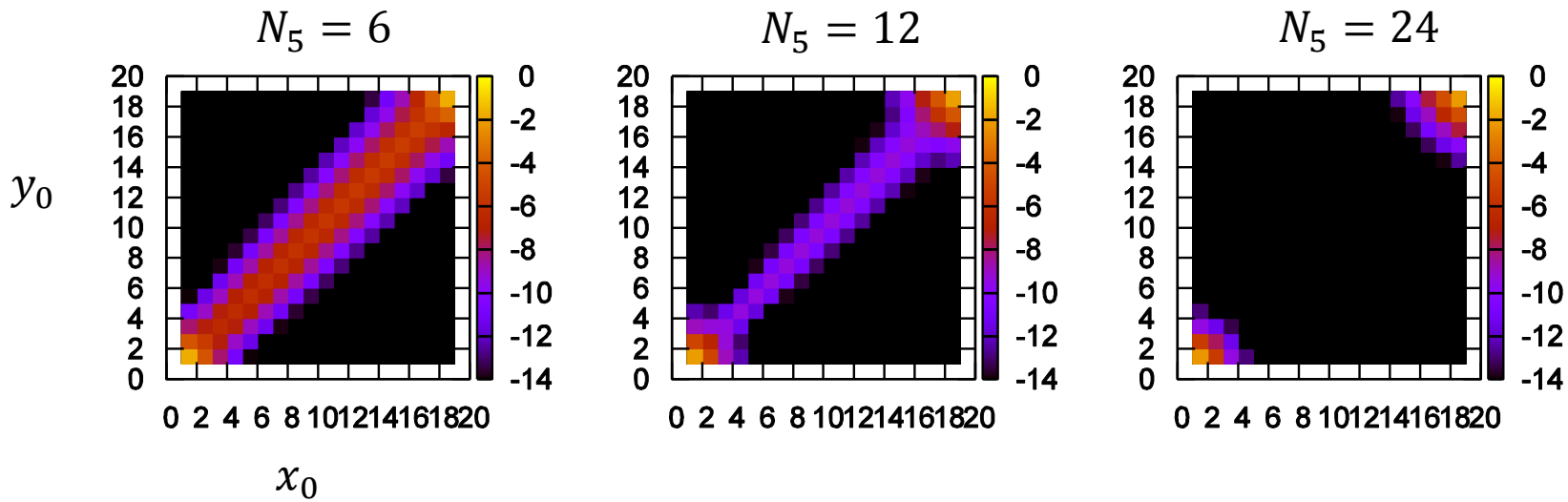


The Ginsparg Wilson Relation

For the Quasi-Optimal Chiu DWF

- parameter

$L = 6, T = 20$



The arrangement of b_s, c_s

$L = 6, N_5 = 8$

s	b_s	c_s
1	1.2563778	0.2563778
2	2.4594867	1.4594867
3	6.9085802	5.9085802
4	17.102189	16.102189
5	17.102189	16.102189
6	6.9085802	5.9085802
7	2.4594867	1.4594867
8	1.2563778	0.2563778