Phase Diagram of Non-Degenerate Twisted Mass Fermions

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Based on work done in collaboration with Steve Sharpe
[Paper in Progress]

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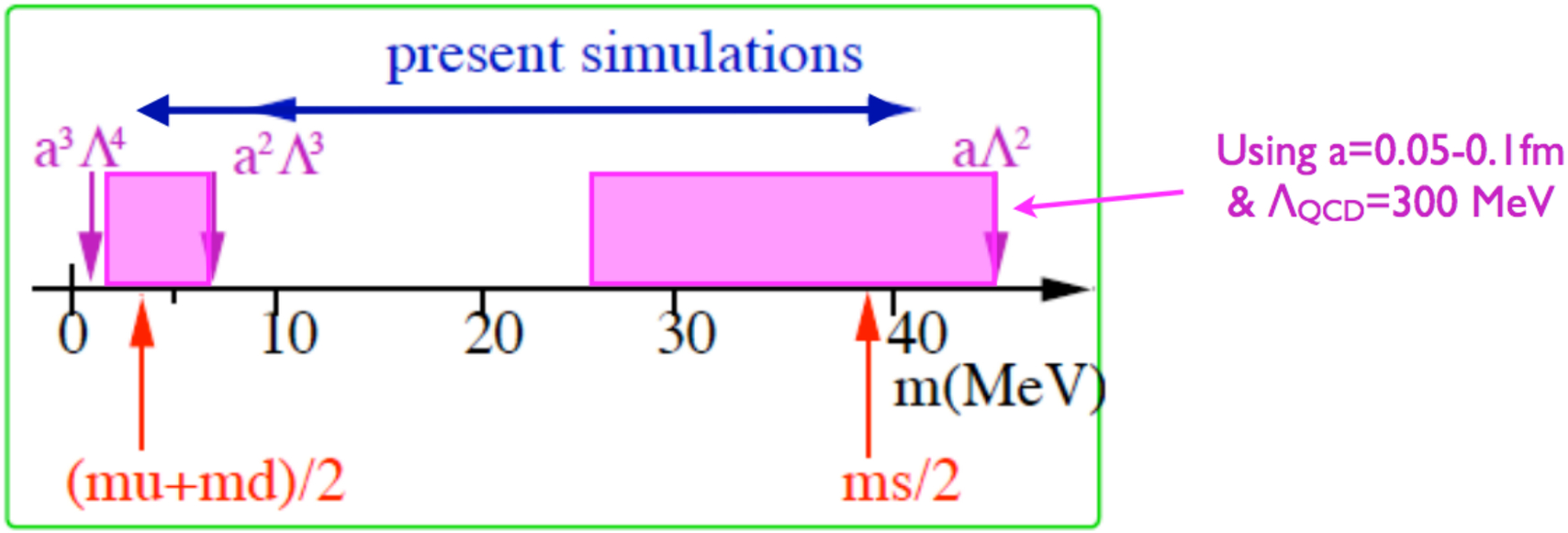
Outline

- Motivation
- Background
- Non-Degenerate Untwisted Quarks at Finite Lattice Spacing
- Non-Degenerate Maximally Twisted Quarks
- Critical Manifold for Arbitrary Twist
- Conclusions

Motivation

- Present simulations are frequently in the "Aoki regime" ($m \sim a^2$)



[Sharpe 2013]

- Increasingly, non-degenerate up and down quarks are used
- How is the competition between quark mass and discretization effects impacted by this non-degeneracy?
- In this talk, I study this question for Wilson and twisted mass fermions in lattice chiral perturbation theory (ChPT) focusing on the phase diagram and pion spectrum

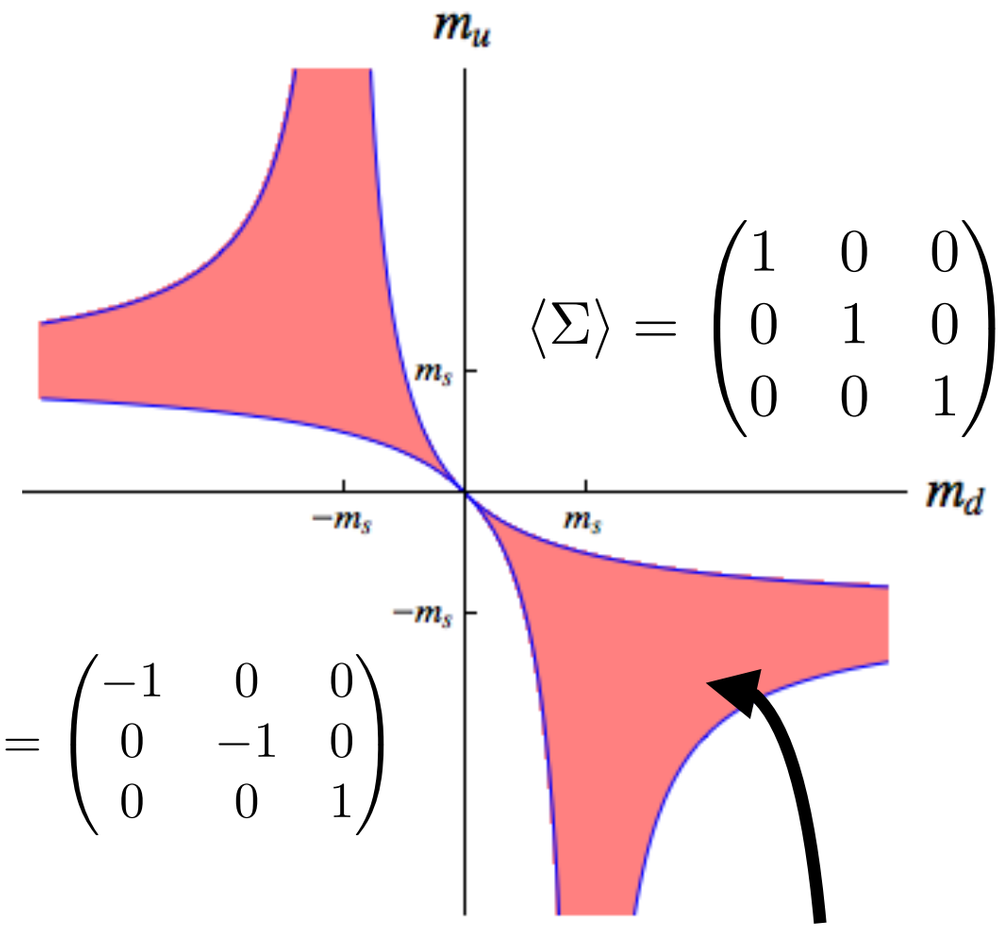
Background

Continuum SU(3) ChPT

- In leading order (LO) SU(3) ChPT, CP is spontaneously violated due to the condensate, <Σ>, becoming complex
  [Dashen 1971], [Creutz 2004]

- The transition occurs where the π_0 is massless

$$\mathcal{V}_{SU(3)LO} = -\frac{f^2}{4}\mathrm{tr}(\chi\Sigma^\dagger + \chi^\dagger\Sigma)$$



$$\langle\Sigma\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\langle\Sigma\rangle = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\langle\Sigma\rangle = \begin{pmatrix} e^{i\phi_1} & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{-i(\phi_1+\phi_2)} \end{pmatrix}$$

$$m_{\pi_0}^2 = \frac{2}{3}B_0\left(m_u + m_d + m_s - \sqrt{m_u^2 + m_d^2 + m_s^2 - m_u m_d - m_u m_s - m_s m_d}\right)$$

Vanishes at: $$m_u = \frac{-m_s m_d}{m_s + m_d}$$

LO SU(2) Lattice ChPT

- Introducing discretization errors into the SU(2) ChPT potential, a term proportional to a^2 with low energy coefficient W' is added to the potential [Symanzik 1983], [Sharpe & Singleton 1998]

- The phase structure depends on the sign of W'

$$\mathcal{V}_{A^2} = -\frac{f^2}{4}\mathrm{tr}(\chi\Sigma^\dagger + \chi^\dagger\Sigma) - W'\left(\mathrm{tr}(A\Sigma^\dagger + A\Sigma)\right)^2$$

$$A = 2W_0 b_1 a \qquad \widetilde{\chi} \equiv 2B_0 M + A \equiv 2B_0\widetilde{M}$$

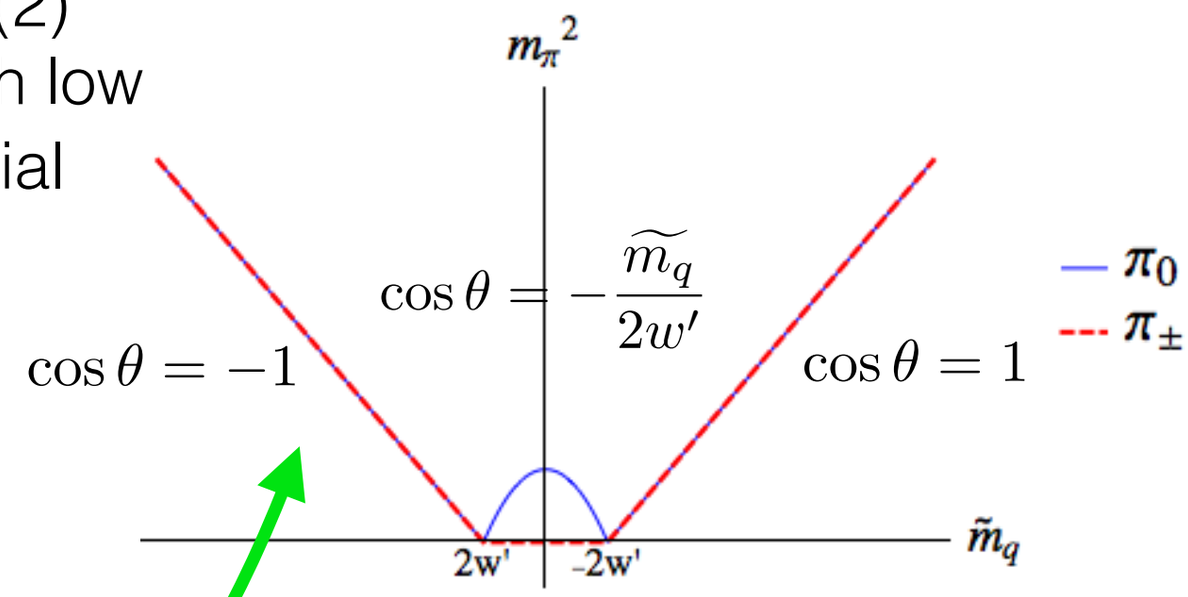
Parametrizing: $\langle\Sigma\rangle = e^{i\theta\hat{n}\cdot\vec{\tau}}$

Minimized at

$$\frac{\mathcal{V}_{A^2}}{f^2} = -\widetilde{m}_q\cos\theta - w'\cos^2\theta$$

$$w' \equiv \frac{16W'A^2}{f^2}$$

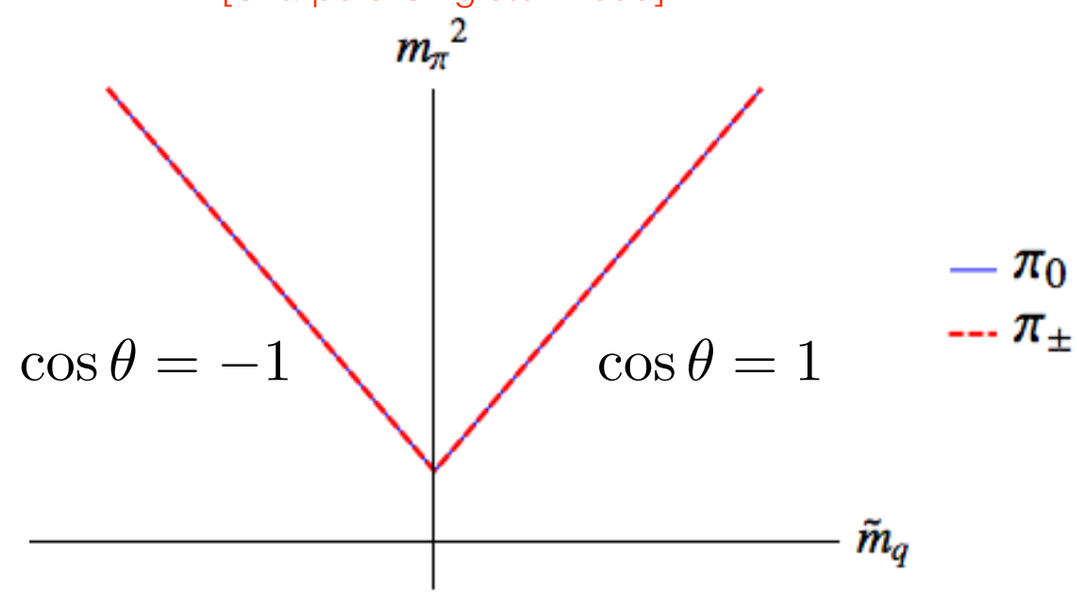
Aoki Scenario [Aoki 1984]



w'<0

First-Order Scenario

[Sharpe & Singleton 1998]



w'>0

Non-Degenerate Quarks In ChPT

- LO results in SU(2) ChPT are unaffected by non-degeneracy; the effect seen at LO in SU(3) enters at NLO in SU(2)
- SU(2) and SU(3) theories can be matched, expanding in inverse strange quark mass
- At leading order in $1/m_s$, only the l_7 Gasser-Leutwyler coefficient is necessary
  [Gasser & Leutwyler 1984]

$$\mathcal{V}_{SU(2)} \supset \frac{l_7}{16}[\mathrm{tr}(\chi^\dagger\Sigma - \Sigma^\dagger\chi)]^2$$

$$m_q \equiv B_0(m_u + m_d) \qquad \epsilon \equiv B_0(m_u - m_d)$$

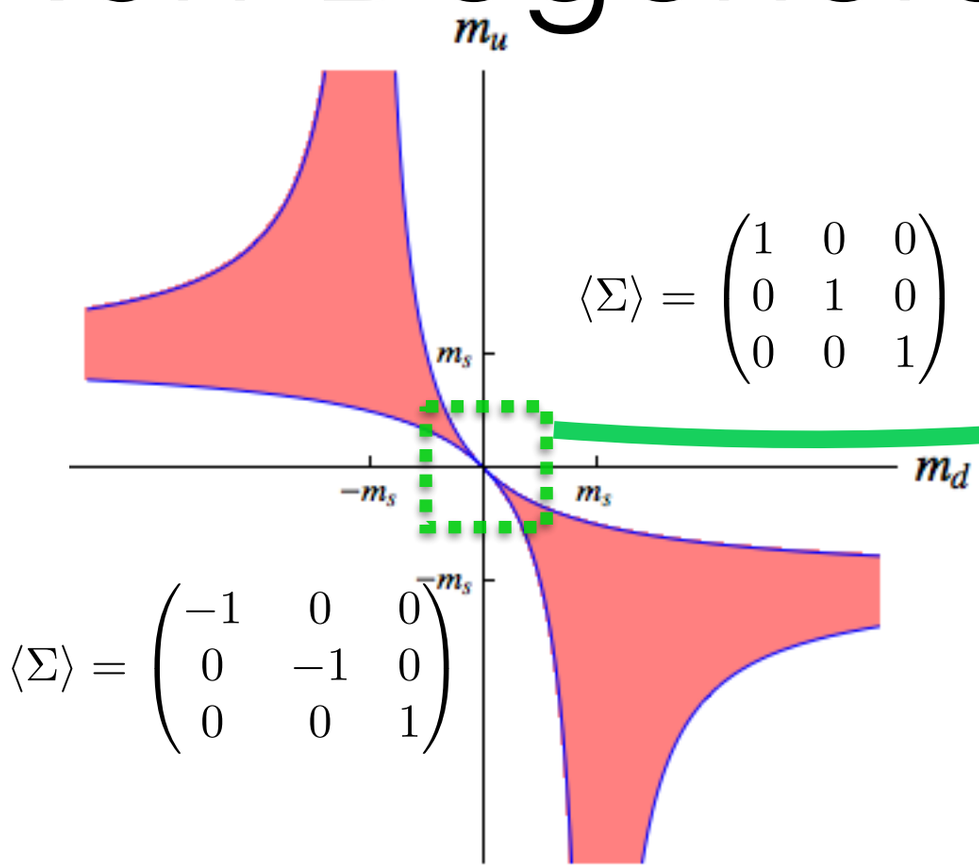
$$m^2_{\pi^0 SU(3)LO} = m_q - \frac{\epsilon^2}{4B_0 m_s} + \mathcal{O}\left(\frac{m^4_{u,d}}{m^3_s}\right)$$

$$l_7 = \frac{f^2}{8B_0 m_s} + \mathcal{O}\left(\frac{m_{u,d}}{m^2_s}\right)$$

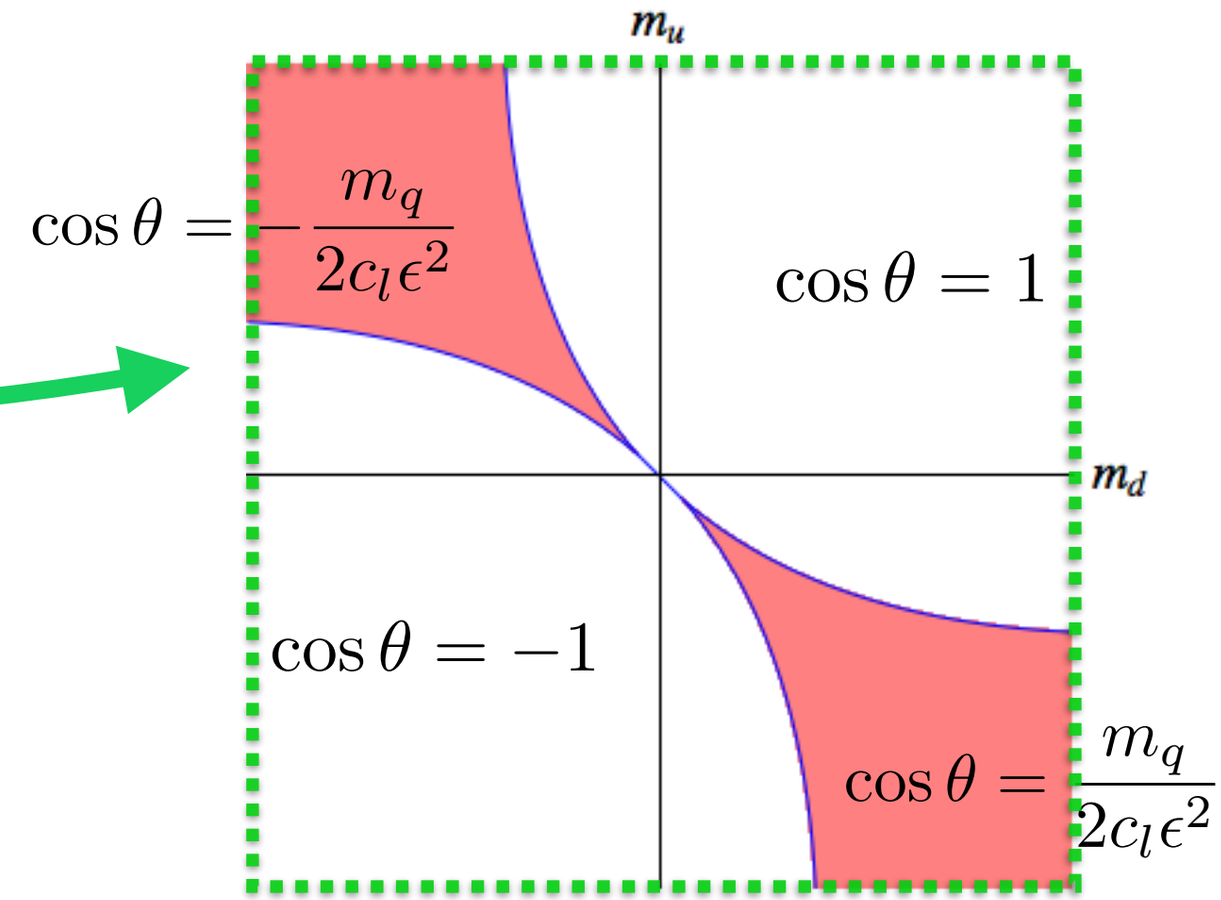
$$m^2_{\pi^0 SU(2)NLO} = m_q - \frac{l_3 m_q^2}{4f^2} - \frac{2l_7\epsilon^2}{f^2} + \mathcal{O}\left(m^3_{u,d}\right)$$

New Results

Non-Degenerate Quarks In ChPT



LO SU(3)



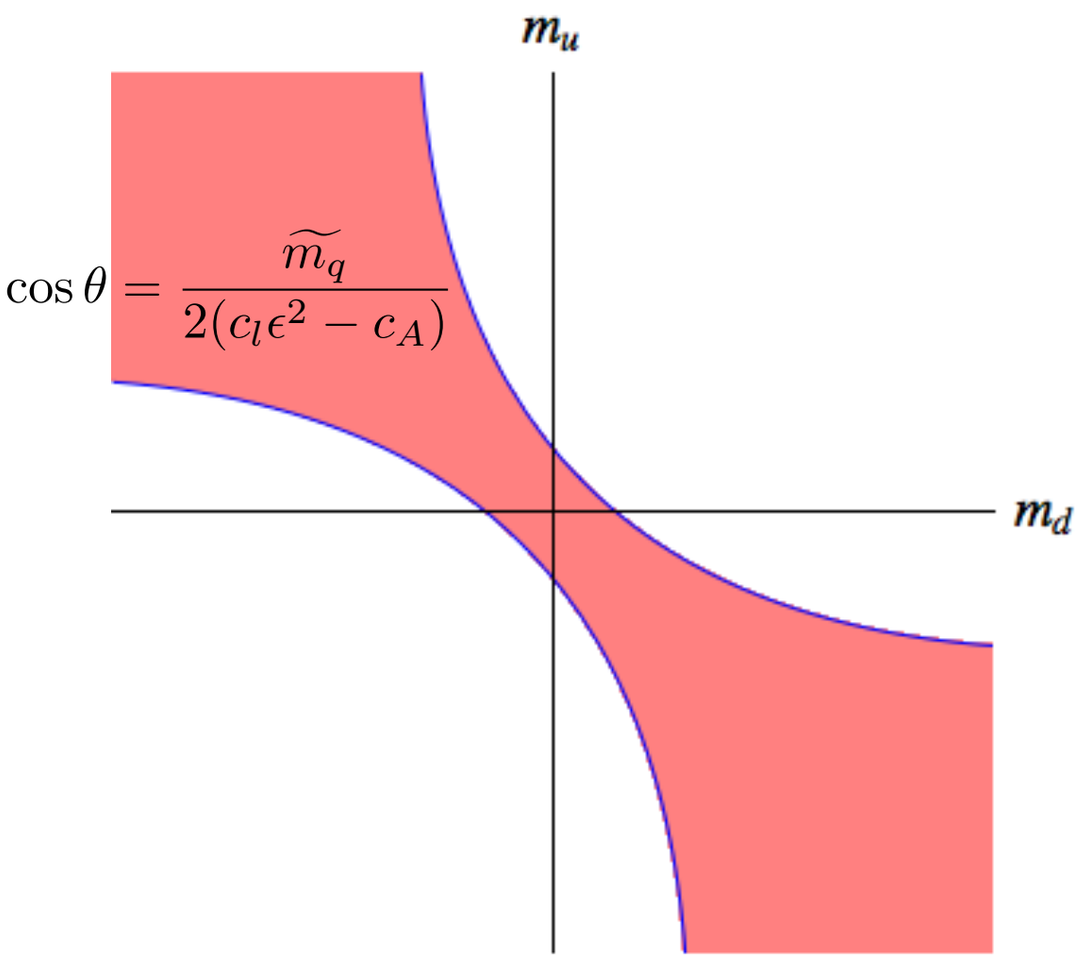
NLO SU(2)

$$\langle\Sigma\rangle = e^{i\theta\hat{n}\cdot\vec{\tau}}$$

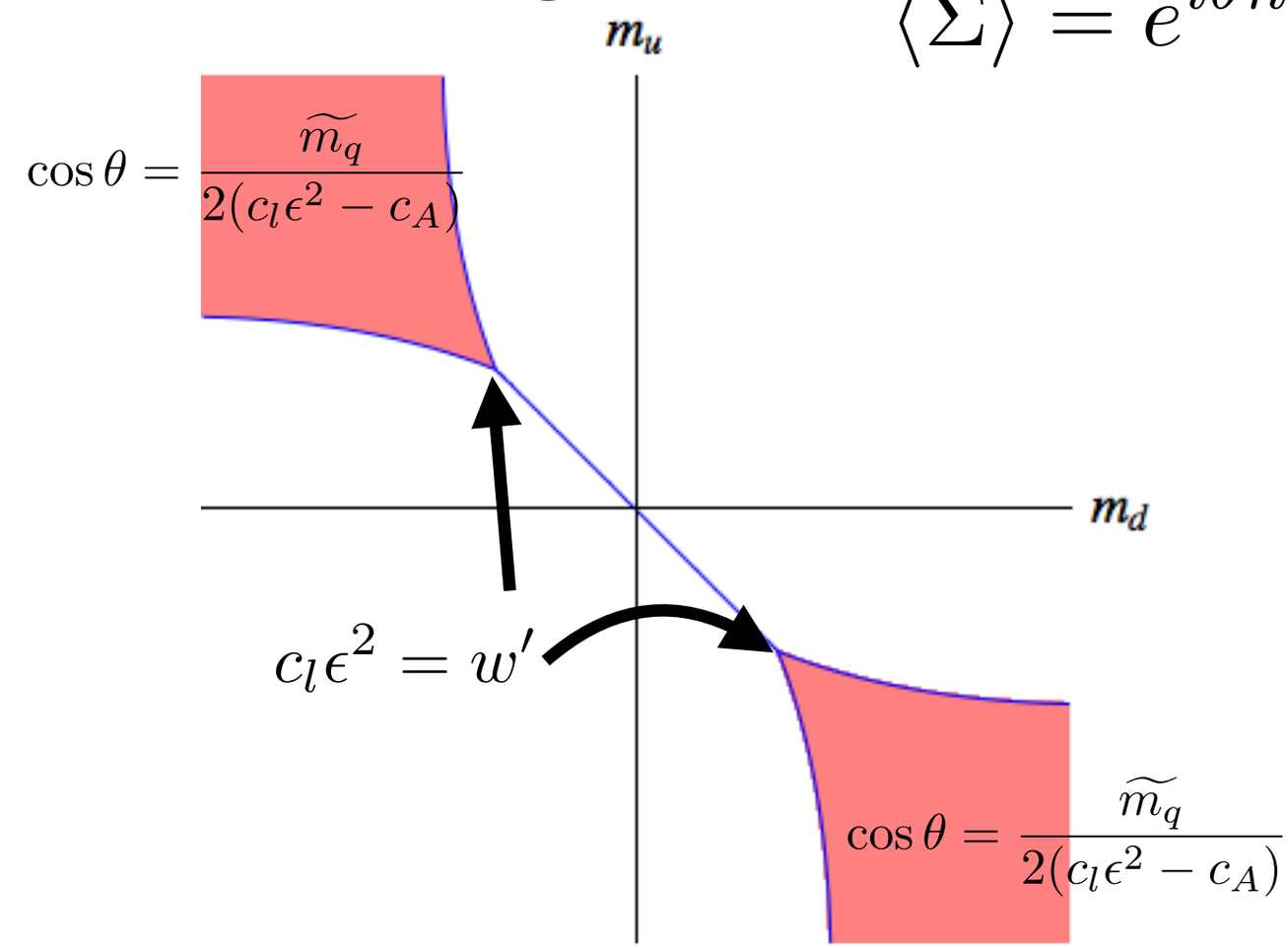
$$\frac{\mathcal{V}_{SU(2)}}{f^2} = -m_q\cos\theta - c_l\epsilon^2 n_3^2\sin^2\theta \qquad c_l \equiv \frac{l_7}{f^2}$$

Non-Degenerate Quarks with Finite Lattice Spacing

$$\langle \Sigma \rangle = e^{i\theta \hat{n}\cdot\vec{\tau}}$$



Aoki Scenario (w'<0)



First-Order Scenario (w'>0)

$$\frac{\mathcal{V}_{A^2}}{f^2} = -\widetilde{m_q}\cos\theta - w'\cos^2\theta - c_l\epsilon^2\sin^2\theta$$

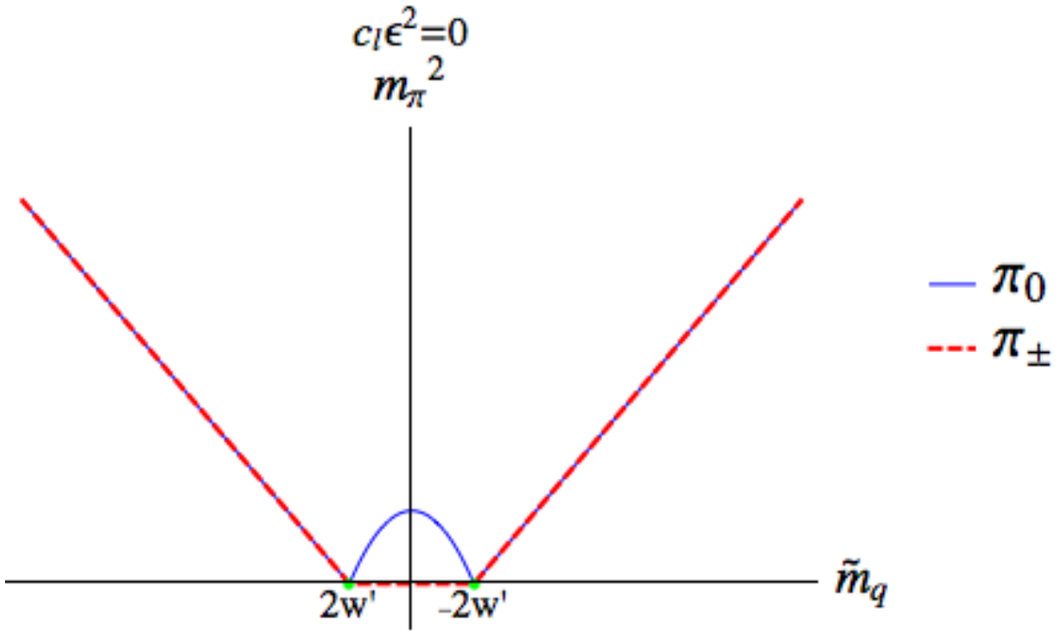
Critical points at:

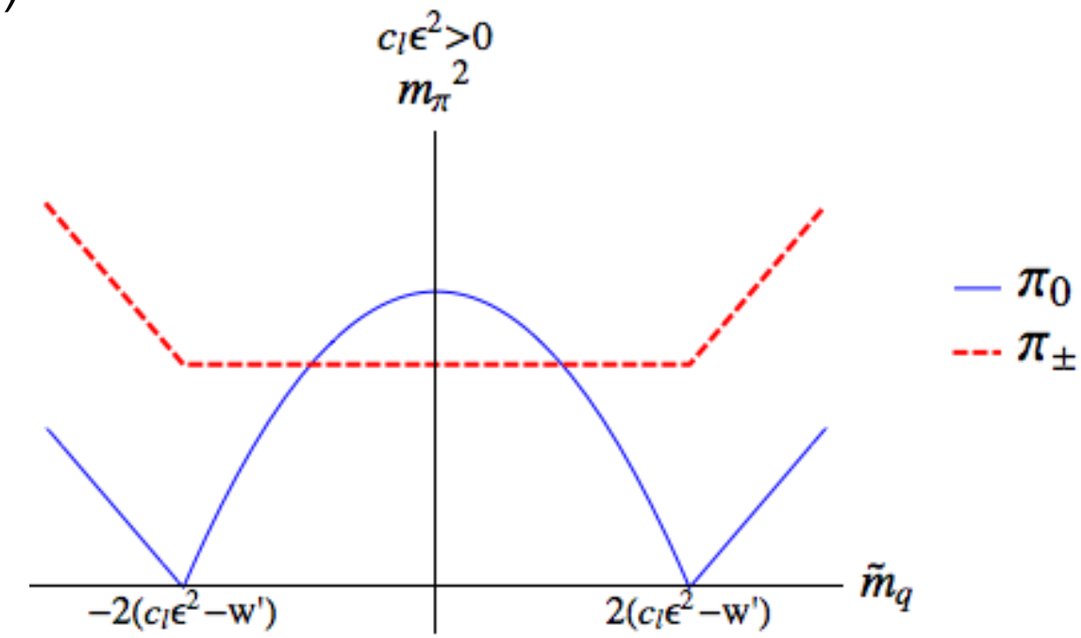
$$c_l\epsilon^2 = w' \rightarrow \frac{\epsilon^2}{m_s} \sim W'A^2$$

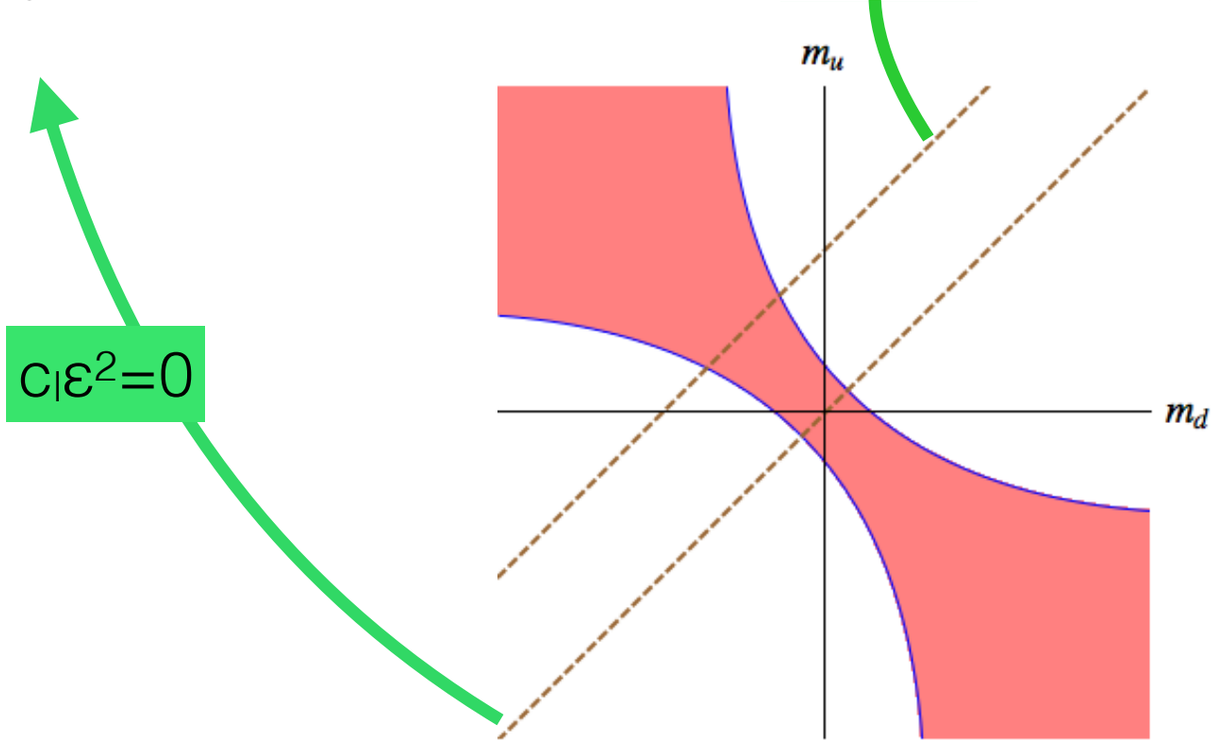
The Aoki phase is continuously connected to the continuum CP violating phase!

Pion Masses

Aoki Scenario (w'<0)





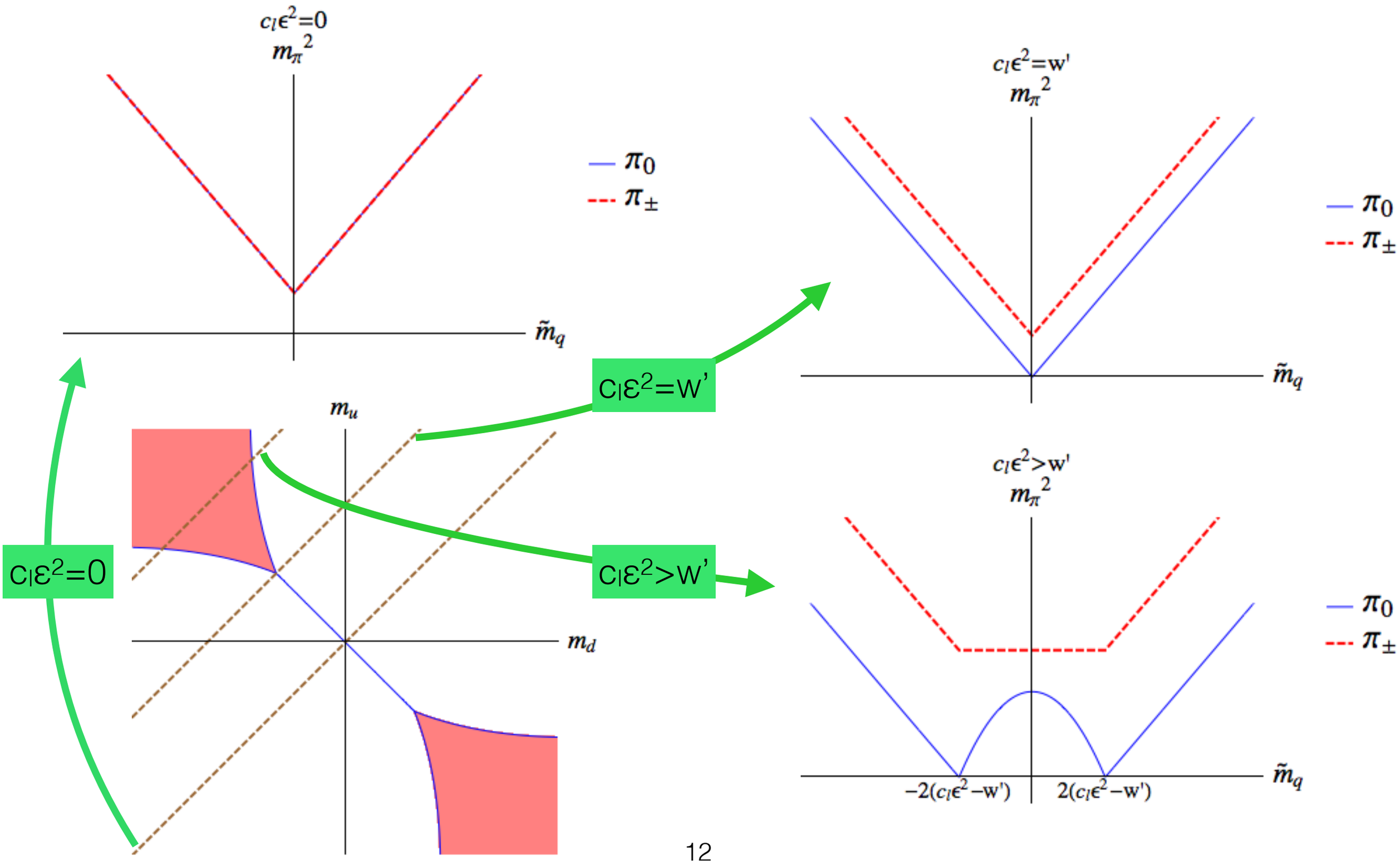


Parametrizing the pion fields: $\Sigma = e^{i\theta\vec{n}\cdot\vec{\tau}} e^{2\pi i/f}$

Where: $\pi = \begin{pmatrix} \frac{\pi_0}{2} & \frac{\pi_+}{\sqrt{2}} \\ \frac{\pi_-}{\sqrt{2}} & -\frac{\pi_0}{2} \end{pmatrix}$

Pion Masses

First-Order Scenario (w'>0)



Power Counting

- We have considered the l_7 term in the SU(2) $O(m^2)$ potential but the comparable l_3 term has been neglected

$$\mathcal{V}_{SU(2)} \supset -\frac{l_3}{16}[\mathrm{tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi)]^2 + \frac{l_7}{16}[\mathrm{tr}(\chi^\dagger\Sigma - \Sigma^\dagger\chi)]^2$$

- The l_3 term does not qualitatively alter the phase diagram near m_q=0 and is not necessary in qualitatively matching the SU(2) and SU(3) phase diagrams

- In the Aoki regime ($m \sim A^2$), if an m^2 term is included then mA, A^3 and A^4 terms should also be included

- As I have checked, while these terms do qualitatively change the phase diagram, they do not change it as significantly as the l_7 term

Twisted Quark Mass

$$B_0\widetilde{M} = \widetilde{m_q}\mathbb{1} + \epsilon\tau_3 \rightarrow \widetilde{m_q}e^{i\tau_1\omega} + \epsilon\tau_3 \equiv m\mathbb{1} + i\mu\tau_1 + \epsilon\tau_3$$

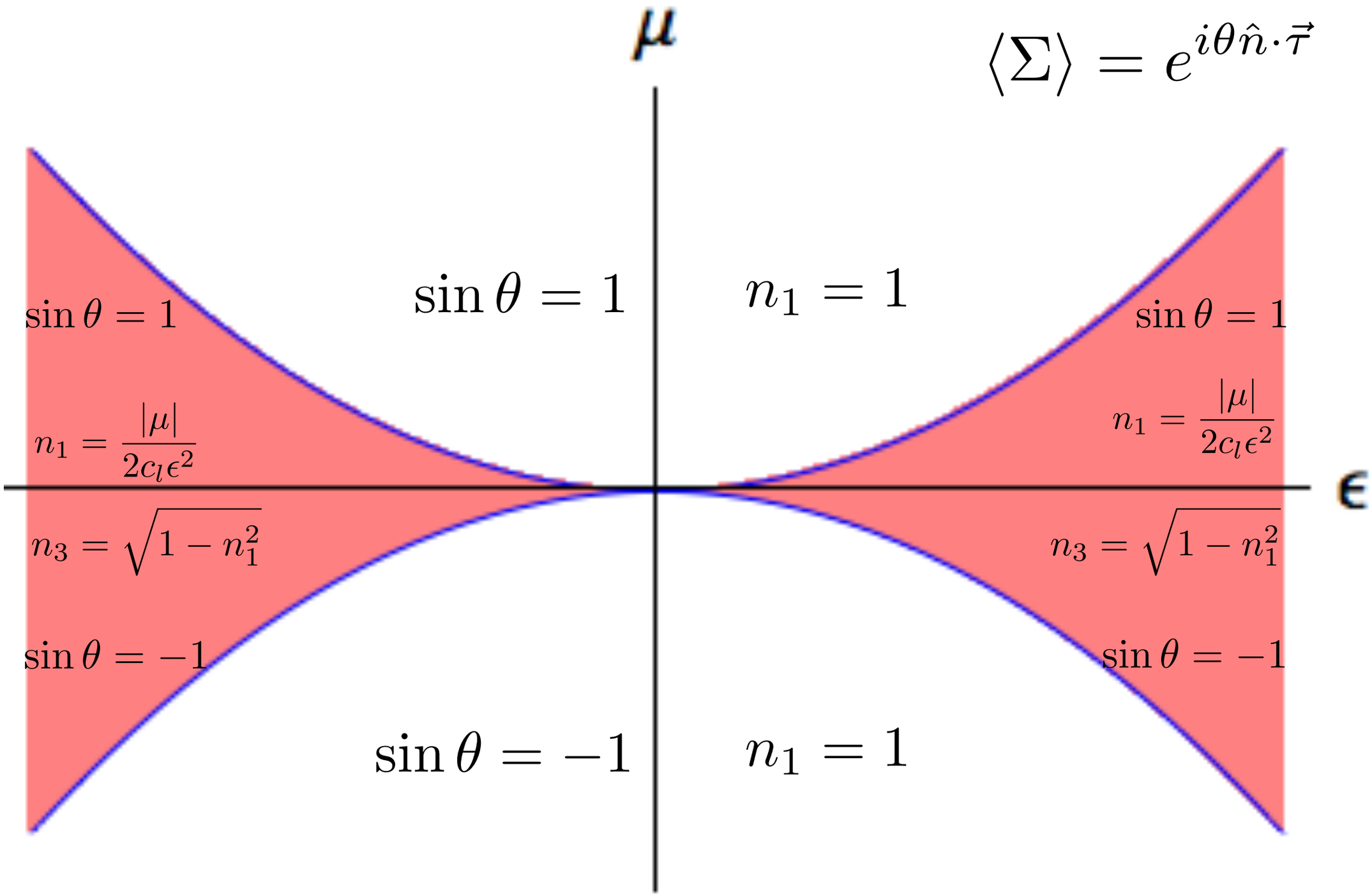
- With non-degenerate quarks, the twist cannot be in the τ_3 direction to maintain a real fermion determinant [Frezzotti and Rossi 2003]

$$\frac{\mathcal{V}_{A^2}}{f^2} = \boxed{-m\cos\theta} \boxed{-\mu n_1 \sin\theta} \boxed{-w'\cos^2\theta} \boxed{-c_l n_3^2 \epsilon^2 \sin^2\theta}$$

Untwisted Term **Twist Term** **Discretization Term** **Isospin Breaking Term**

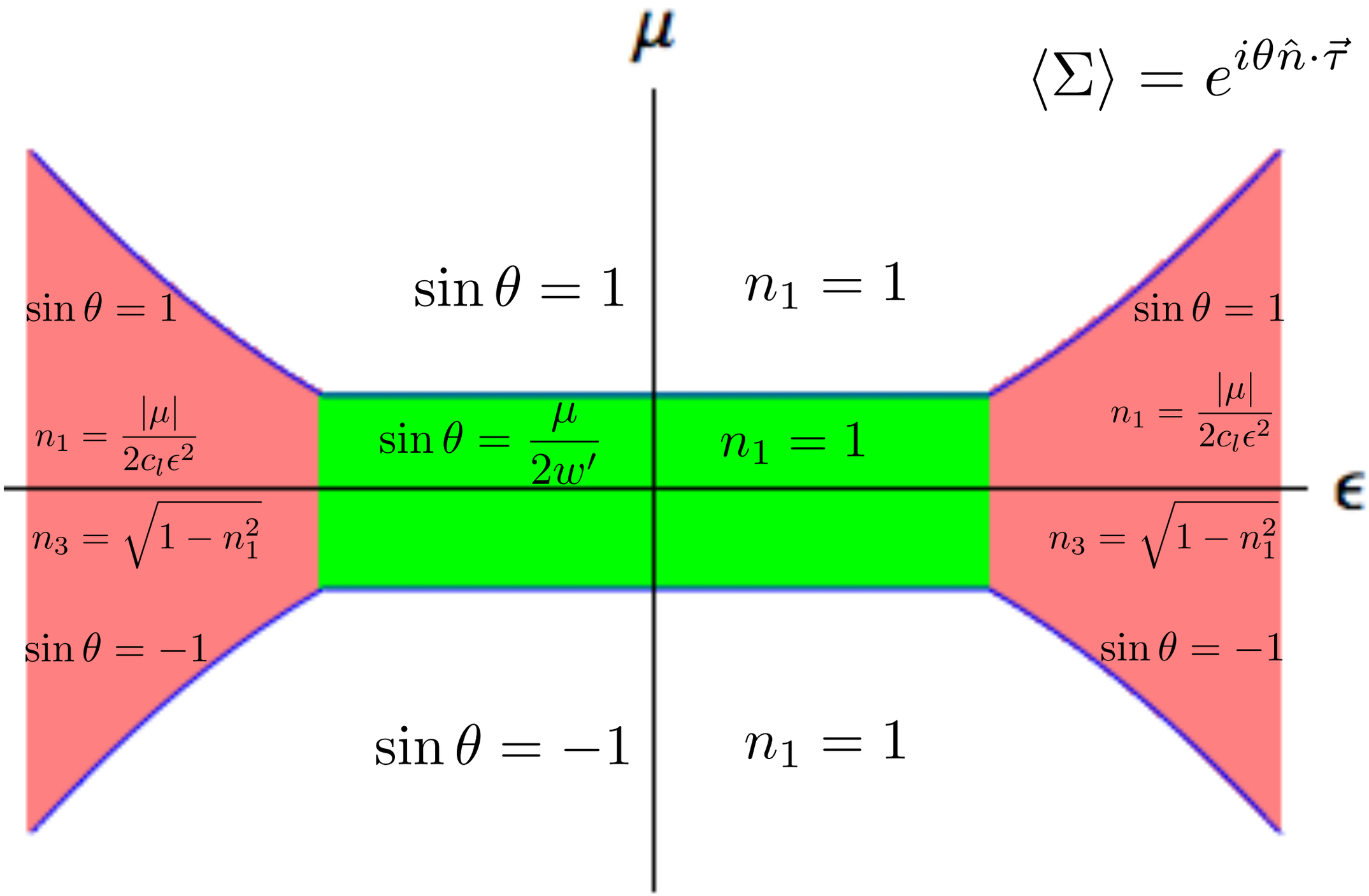
- In practice maximal twist (m=0) is most interesting and we consider this first

Maximal Twist (m=0) Phase Diagram



Aoki Scenario and the continuum (w'≤0)

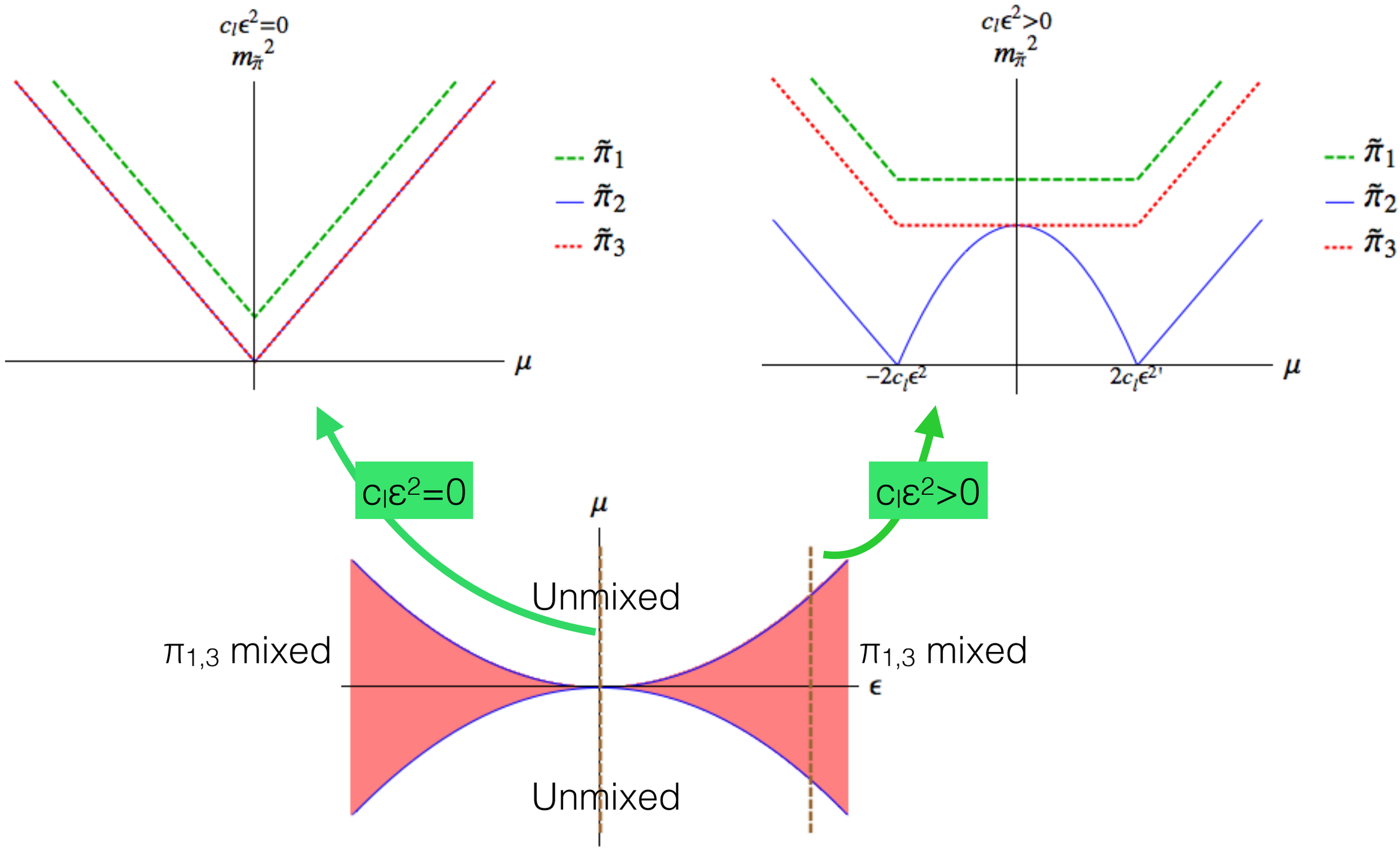
Maximal Twist Phase Diagram



First-order Scenario (w'>0)

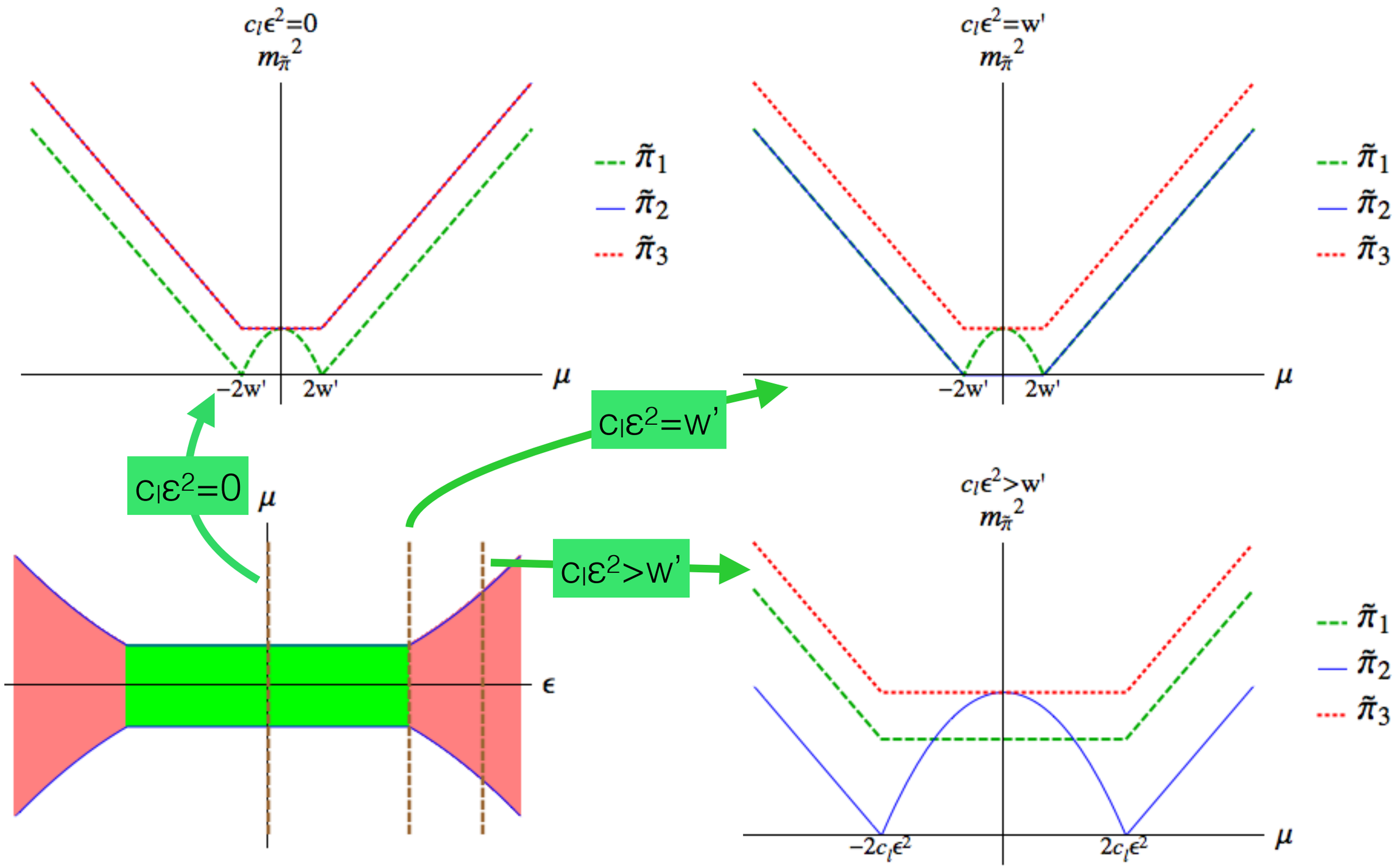
Maximal Twist Pion Masses

Aoki Scenario (w'<0)



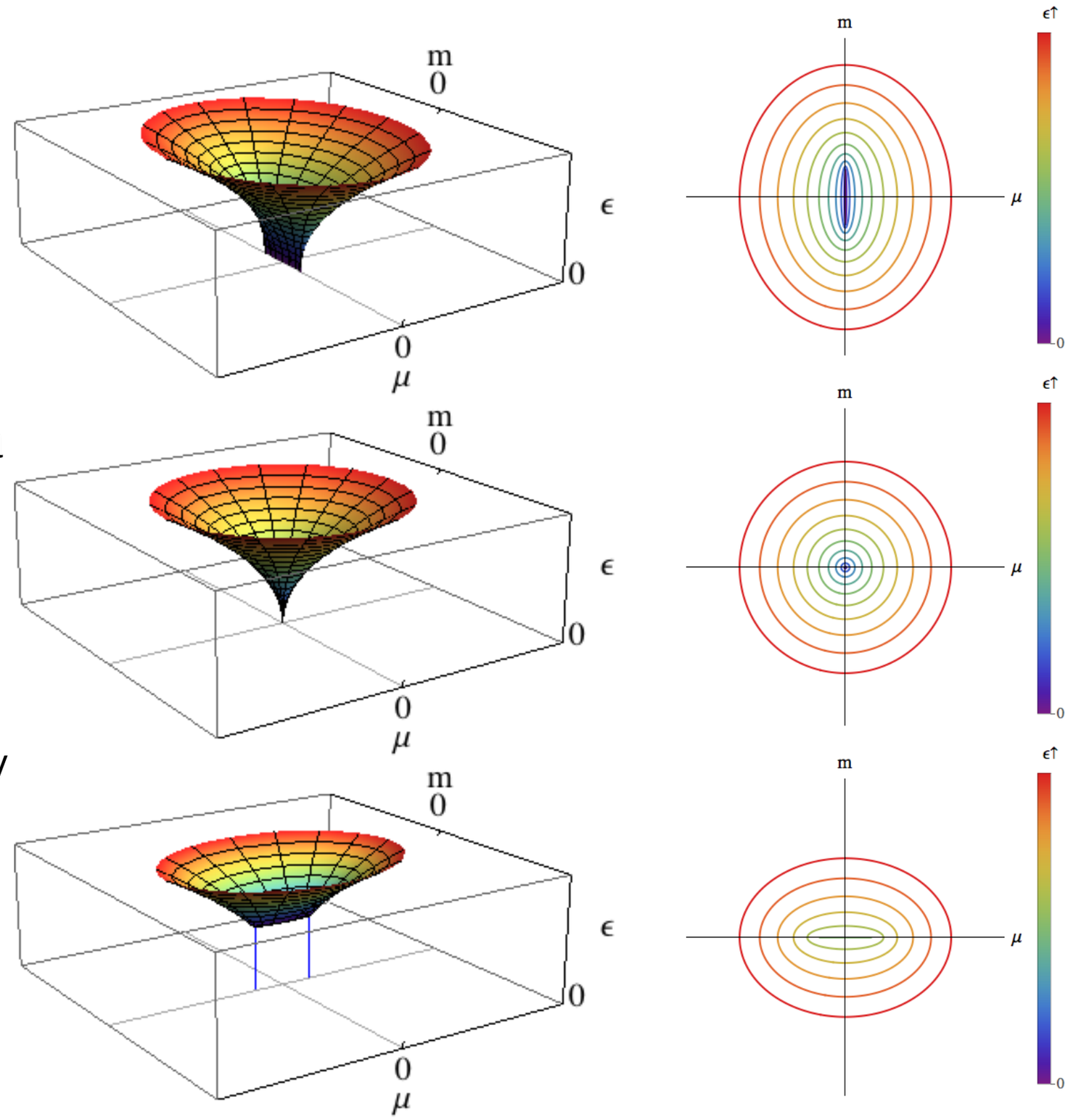
Maximal Twist Pion Masses

First-Order Scenario (w'>0)



Critical Manifold For Arbitrary Twist

- Minimizing the full, arbitrary twist potential results in a critical manifold of second-order transition.
- Along this boundary at least one pion is massless



Conclusions

- The continuum CP violating phase and the Aoki phase are continuously connected
- The critical surface of this phase can be closer to the physical point when discretization effects are considered
- When non-degenerate quarks are introduced, generic twist, including maximal twist, results in a more complicated phase diagram and non-degenerate pions
- Higher order terms modify the phase diagram but the effects are not significant

References

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- K. Symanzik. Continuum Limit and Improved Action in Lattice Theories. 2. O(N) Nonlinear Sigma Model in Perturbation Theory. Nucl.Phys., B226:205, 1983.

Extra Slides
(Higher order)

Order ma~a^3

- To see how higher order terms in discretization error effect this picture, we need to include two more terms in the potential,

$$\mathcal{V}_{A^3} = -W\,\mathrm{tr}(\chi^\dagger\Sigma + \Sigma^\dagger\chi)\,\mathrm{tr}(A^\dagger\Sigma + \Sigma^\dagger A) - \frac{W_{3,3}}{f^2}\,\mathrm{tr}((A^\dagger\Sigma)^3 + (\Sigma^\dagger A)^3)$$

- Parametrizing Σ and absorbing the part of $W_{3,3}$ proportional to cosθ into m yields:

$$\frac{\mathcal{V}_{A^3}}{f^2} = -(m\cos\theta + \mu n_1 \sin\theta)(1 + \delta_w \cos\theta) - w'\cos^2\theta - c_l n_3^2 \epsilon^2 \sin^2\theta - w_3 \cos^3\theta$$

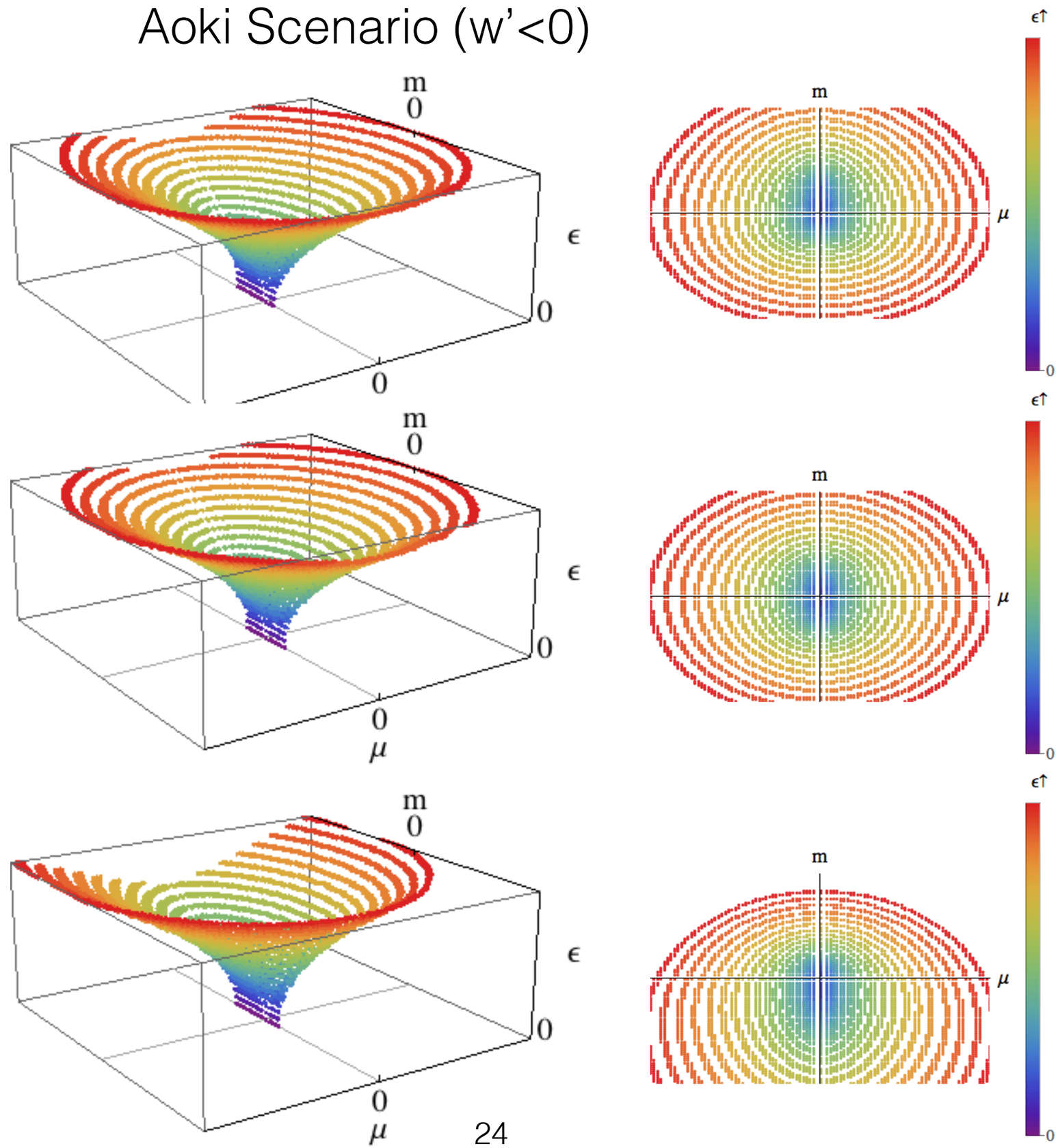
- The w_3 term which is cubic in cosθ which will introduce asymmetry between positive and negative m. Both the w_3 and δ_w terms will shift the w'>0 first-order transition line off the imaginary mass axis. These terms also make the potential more difficult to analyze exactly.

A detailed discussion of these effects for ε=0 can be found in [Sharpe 2008]

Critical Manifold O(ma~a^3)

Aoki Scenario (w'<0)

- There is very little qualitative difference introduced by the higher-order terms.
- For $w_3>0$, there is a shift toward -m.
- For $\delta_w>0$, there is a shift and an overall slant toward -m.



$\delta_w=0$
$w_3>0$

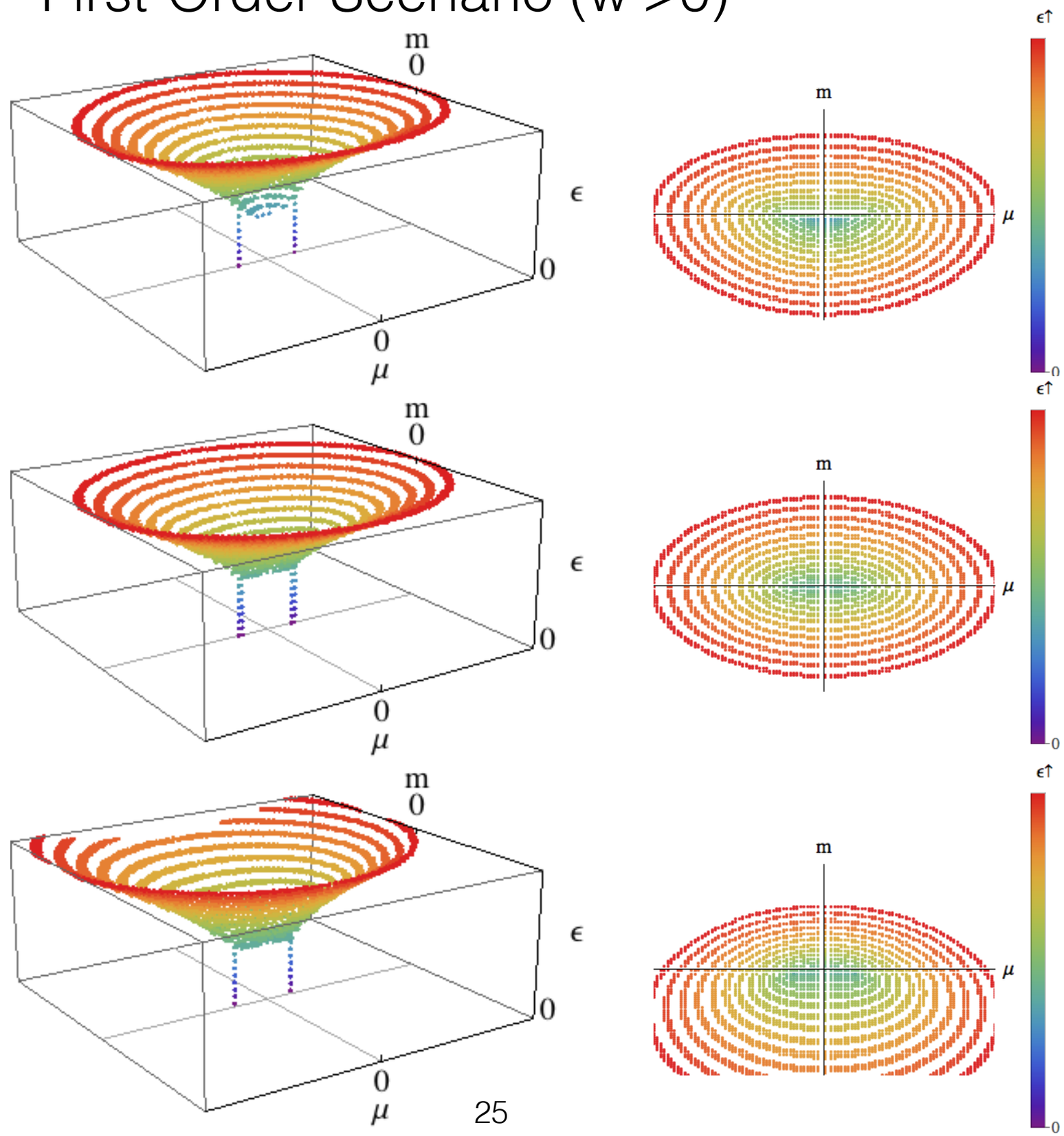
$w_3=\delta_w=0$

$\delta_w>0$
$w_3=0$

Critical Manifold O(ma~a^3)

First-Order Scenario (w'>0)

- For $w_3>0$, there is a discontinuity in the manifold, due to the asymmetry in positive and negative m.
- For $\delta_w>0$, is a shift in the second-order endpoints to greater m and an overall slant in towards larger m.



$\delta_w=0$
$w_3>0$

$w_3=\delta_w=0$

$\delta_w>0$
$w_3=0$