Multi-channel 1 to 2 matrix elements in finite volume

arXiv:1406.5965
Raúl Briceño, Máx Hansen & André Walker-Loud

The College of William & Mary

Jefferson Lab
Motivation

Many interesting quantities to compute with LQCD which involve multiple hadrons in initial/final state

\[ K \rightarrow \pi\pi \]
\[ B \rightarrow K^*\ell^+\ell^- \rightarrow K\pi\ell^+\ell^- \]
\[ pp \rightarrow de^+\nu_e \]
\[ \gamma\pi \rightarrow \pi\pi \]
\[ \gamma N \rightarrow \Delta \rightarrow N\pi \]

SM/BSM

“calibrate the sun”

chiral dynamics

...
Motivation

Unlike the single hadron ground state spectrum or matrix elements, NO simple relation between finite-volume (FV) matrix-elements and infinite-volume (∞V) transition amplitudes.

We were motivated to determine a “master formula” with as few approximations as possible: in this work - focus on transition form-factors between (pseudo)-scalar states.
Motivation

Master formula: finite-volume matrix element of a current that

- can inject arbitrary four-momentum and angular momentum
- includes all inelastic coupled channels, “a”
- incorporates partial-wave mixing (from box and/or physics)

\[ \langle E_{\Lambda_f,n_f} P_f; L | \tilde{\mathcal{J}}[J,P,|\lambda|] (0, P_f - P_i) | E_{\Lambda_i,0} P_i; L \rangle = \frac{1}{\sqrt{2E_{\Lambda_i,0}}} \sqrt{[A^\dagger_{\Lambda_f,n_f;\Lambda\mu} R_{\Lambda_f,n_f} A_{\Lambda_f,n_f;\Lambda\mu}]} \]

\[ \langle a, P_f, Jm_J; \infty | \tilde{\mathcal{J}}[J,P,|\lambda|] (0, Q; \infty) | P_i; \infty \rangle = [A_{\Lambda\mu;Jm_J}]_a (2\pi)^3 \delta^3(P_f - P_i - Q) \]

\( A \) column vector in angular-momentum/channel space
\( \Lambda_f \) denotes the projection onto the finite volume irrep. \( \Lambda \) row \( \mu \)
\( R_{\Lambda_f,n_f} \) matrix: related to the residues of FV two-particle propagators of state \( n_f \)
$C^{(1)}(x_0 - y_0, k) \equiv \langle 0 | \varphi(x_0, k) \varphi^\dagger(y_0, -k) | 0 \rangle$

$$= e^{-E^{(1)}_k(x_0 - y_0)} | \langle 0 | \varphi(0, k) | E^{(1)}_k; L \rangle |^2 + \mathcal{O} \left( L^3 \frac{e^{-E^{(1)}_{3, th}(x_0 - y_0)}}{E^{(1)}_{3, th}} \right),$$

$$= \quad \text{correction to } E_k \quad \text{E}_{3, \text{th}}$$
1 AND 2 HADRON CORRELATORS

\[ C_{\lambda \mu}^{(2)}(x_0 - y_0, P) = \langle 0 | O_{\lambda \mu}(x_0, P) O_{\lambda \mu}^\dagger(y_0, -P) | 0 \rangle \]

\[ O_{\lambda \mu}(x_0, P, |P - k|, |k|) = \sum_{R \in LG(P)} C(P \Lambda \mu; Rk; R(P - k)) \varphi(x_0, Rk) \bar{\varphi}(x_0, R(P - k)) \]

\[ R \text{ element of } LG(P), \text{ little group of rotations leaving } P \text{ invariant} \]

\[ C(P \Lambda \mu; Rk; R(P - k)) \equiv \langle \Lambda(P), \mu | \Lambda_1(\{P - k\}_P), R(P - k); \Lambda_2(\{k\}_P), Rk \rangle \]

projection onto \( \Lambda \)

\[ \text{eg } O_{A_1^+}(x_0, 0) = \frac{\sigma}{\sqrt{6}} \sum_{\hat{i} = \{\hat{x}, \hat{y}, \hat{z}\}} \left[ \varphi(x_0, q(1)\hat{i}) \bar{\varphi}(x_0, -q(1)\hat{i}) + \varphi(x_0, -q(1)\hat{i}) \bar{\varphi}(x_0, q(1)\hat{i}) \right] \]

\[ q(1) = \frac{2\pi}{L} \]

\[ O_{A_1}(x_0, q(1)\hat{z}) = \frac{1}{2} \sum_{\hat{i} = \{\hat{x}, \hat{y}\}} \left[ \varphi(x_0, q(1)\hat{i}) \bar{\varphi}(x_0, -q(1)(\hat{z} - \hat{i})) + \varphi(x_0, -q(1)\hat{i}) \bar{\varphi}(x_0, q(1)(\hat{z} + \hat{i})) \right] \]
\[ C^{(2)}_{\Lambda \mu}(x_0 - y_0, \mathbf{P}) = \langle 0 | \mathcal{O}_{\Lambda \mu}(x_0, \mathbf{P}) \mathcal{O}^\dagger_{\Lambda \mu}(y_0, -\mathbf{P}) | 0 \rangle \]

\[
\int \frac{dP_0}{2\pi} \frac{dk_0}{2\pi} e^{iP_0(x_0 - y_0)} \left\{ \begin{array}{c}
\begin{array}{c}
P - k \\
k
\end{array} \\
\end{array} + \cdots \right\}
\]

The integration over \( k_0 \) puts one hadron on-shell:
The integration over \( P_0 \) can not be performed until non-perturbatively summing over all diagrams.
\[ C^{(2)}_{\Lambda\mu}(x_0 - y_0, \mathbf{P}) = \langle 0 | \mathcal{O}_{\Lambda\mu}(x_0, \mathbf{P}) \mathcal{O}^\dagger_{\Lambda\mu}(y_0, -\mathbf{P}) | 0 \rangle \]

\[ \int \frac{dP_0}{2\pi} e^{iP_0(x_0 - y_0)} \left\{ \sum \omega_k, \mathbf{k} \right\} \]

\[ E_{\text{free}} = \omega_k + \omega \mathbf{k} - \mathbf{P} \]

Bethe-Salpeter Kernel: explicit dependence upon off-shell scattering

Related to the K-matrix
Real part of inverse scattering amplitude

Kim, Sachrajda and Sharpe
K-matrix differs from infinite volume K-matrix only by terms exponentially suppressed by $m\pi L$
1 and 2 hadrons

\[ C_{\Lambda\mu}^{(2)}(x_0 - y_0, P) = \langle 0 | \mathcal{O}_{\Lambda\mu}(x_0, P) \mathcal{O}_{\Lambda\mu}^\dagger(y_0, -P) | 0 \rangle \]

\[
\int \frac{dP_0}{2\pi} e^{iP_0(x_0-y_0)} \left\{ \begin{array}{c}
\cdot \\
+ \text{Intermediate state can go } on-shell \\
+ \text{power-law volume dependence (L"uscher)} \\
+ \cdots \end{array} \right\}
\]

Kim, Sachrajda and Sharpe

Intermediate state can go \textit{on-shell} and feel the boundary of the box
\[ C_{\Lambda\mu}^{(2)}(x_0 - y_0, P) = \langle 0 | \mathcal{O}_{\Lambda\mu}(x_0, P) \mathcal{O}_{\Lambda\mu}^\dagger(y_0, -P) | 0 \rangle \]

\[ \int \frac{dP_0}{2\pi} e^{iP_0(x_0 - y_0)} \left\{ + \quad + \quad V \right\} \]

Poles of this infinite series lead to quantization condition that determines spectrum of interacting system

\[ \det[M(E_n)] = \det \left[ \mathbb{K}(E_n) + (\mathbb{F}V(E_n))^{-1} \right] = 0 \]

If we only cared about the spectrum and scattering - we would be done - this is a generalization of the Lüscher formula relating finite-volume energy levels to infinite volume scattering phase shifts.
$C^{(2)}_{\Lambda \mu}(x_0 - y_0, \mathbf{P}) = \langle 0 | \mathcal{O}_{\Lambda \mu}(x_0, \mathbf{P}) \mathcal{O}^\dagger_{\Lambda \mu}(y_0, -\mathbf{P}) | 0 \rangle$

Kim, Sachrajda and Sharpe

For our work - we also need to know the residues of the poles

$$R_{\Lambda, n} = \text{adj}[\mathcal{M}(P_0, M)] \text{ tr } \left[ \text{adj}[\mathcal{M}(P_0, M)] \frac{\partial \mathcal{M}(P_0, M)}{\partial P_0, M} \right]^{-1} \bigg|_{P_0, M = E_{\Lambda, n}}$$

$\text{adjugate of a matrix: } \frac{1}{\mathcal{M}(P_0, M)} \equiv \frac{1}{\text{det}[\mathcal{M}(P_0, M)]} \text{adj}[\mathcal{M}(P_0, M)]$

diverges at eigen-energies

finite at eigen-energies
\[ C^{(2)}_{\Lambda \mu}(x_0 - y_0, P) = \langle 0 | O_{\Lambda \mu}(x_0, P) O^{\dag}_{\Lambda \mu}(y_0, -P) | 0 \rangle \]

\[ \int \frac{dP_0}{2\pi} e^{iP_0(x_0 - y_0)} \{ \ldots \} \]

\[ = L^3 \sum_n e^{-E_{\Lambda,n}(x_0 - y_0)} V^{\dag}_{O,\Lambda,n} R_{\Lambda,n} V_{O,\Lambda,n} \]

A vector in angular momentum and open channels; encodes off-shell artifacts

Residue of two-particle propagator

Sum over "n" runs over all energies below the N>2 inelastic threshold
\[ C_{\Lambda\mu}^{(2)}(x_0 - y_0, P) = \langle 0 | O_{\Lambda\mu}(x_0, P) O_{\Lambda\mu}^\dagger(y_0, -P) | 0 \rangle \]

\[ \int \frac{dP_0}{2\pi} e^{iP_0(x_0 - y_0)} \left\{ \right. \]

\[ \text{generalize to coupled channels} \]

Kim, Sachrajda and Sharpe
for very nice example with coupled channels,
see talk by David Wilson
Monday: 15:35 Resonances in $\pi$-$K$ scattering
see also Dudek, Edwards, Thomas and Wilson
arXiv:1406.4158
The construction of the finite-volume matrix element follows very closely the construction of the two-hadron correlation function

\[ C_{\Lambda f \mu_f; \Lambda \mu}^{(1 \rightarrow 2)}(x_f,0 - y_0; y_0 - x_i,0) = \langle 0 | O_{\Lambda f \mu_f}(x_f,0, P_f) \tilde{J}^{[J,P,|\lambda|]}_{\Lambda \mu}(y_0, Q) \varphi^\dagger(x_i,0, -P_i) | 0 \rangle \]

Interpolating field optimized for two-hadron state in definite irrep.

Interpolating field optimized for one-hadron state

Current subduced onto the \( \Lambda \) irrep. of \( O_h \)

$C^{(1\rightarrow 2)}_{\Lambda f \mu f; \Lambda \mu} (x_{f,0} - y_0; y_0 - x_{i,0}) = \langle 0| O_{\Lambda f \mu f} (x_{f,0}, P_f) \tilde{J}_{\Lambda \mu}^{[J,P,|\lambda|]} (y_0, Q) \varphi^\dagger (x_{i,0}, -P_i) |0\rangle$

$$\int \frac{dP_{f,0}}{2\pi} \frac{dP_{f,0}}{2\pi} e^{iP_{i,0}(x_{f,0} - y_0)} e^{iP_{f,0}(y_0 - x_{i,0})} \left\{ \begin{array}{c} \text{Diagram} \\ \text{+} \end{array} \right\} + \ldots$$

**LO transition amplitude:**

$$\begin{array}{c} \text{Diagram} \\ \text{=} \end{array} \begin{array}{c} \text{Diagram} \\ \text{+} \end{array} \begin{array}{c} \text{Diagram} \\ \text{+} \end{array} \begin{array}{c} \text{Diagram} \\ \text{+} \end{array} + \ldots$$

**Full transition amplitude:** Similar to K-matrix

$$\begin{array}{c} \text{Diagram} \\ \text{=} \end{array} \begin{array}{c} \text{Diagram} \\ \text{+} \end{array} \begin{array}{c} \text{Diagram} \\ \text{P.V.} \end{array} \begin{array}{c} \text{Diagram} \\ \text{P.V.} \end{array} \begin{array}{c} \text{Diagram} \\ \text{P.V.} \end{array} + \ldots$$
\[ C^{(1\to2)}_{\Lambda_f \mu_f; \Lambda_\mu}(x_f,0 - y_0; y_0 - x_i,0) = \langle 0 | \mathcal{O}_{\Lambda_f \mu_f}(x_f,0, P_f) \tilde{\mathcal{J}}^{[J,P;\lambda]}_{\Lambda_\mu}(y_0, Q) \varphi^\dagger(x_i,0, -P_i) | 0 \rangle \]

\[
\int \frac{dP_{f,0}}{2\pi} \frac{dP_{f,0}}{2\pi} e^{iP_{i,0}(x_f,0 - y_0)} e^{iP_{f,0}(y_0 - x_i,0)} \left\{ \begin{array}{c}
\end{array} \right\} + \ldots
\]

\[
\begin{array}{ccc}
\text{Similar to K-matrix}
\end{array}
\]
\[ C^{(1\rightarrow 2)}_{\Lambda_f \mu_f; \Lambda \mu}(x_{f,0} - y_0; y_0 - x_{i,0}) = \langle 0| \mathcal{O}_{\Lambda_f \mu_f}(x_{f,0}, P_f) \tilde{\mathcal{J}}^{[J_\nu, \lambda]}(y_0, Q) \varphi^\dagger(x_{i,0}, -P_i)|0\rangle \]

\[
\int \frac{dP_{f,0}}{2\pi} \frac{dP_{f,0}}{2\pi} e^{iP_{f,0}(x_{f,0} - y_0)} e^{iP_{f,0}(y_0 - x_{i,0})} \left\{ \begin{array}{c}
\text{Power-law volume corrections from on-shell intermediate states} \\
\end{array} \right\}
\]

\[
\begin{array}{c}
V \\
- P \cdot V. = V
\end{array}
\]
3 POINT CORRELATION FUNCTION

\[ C_{\Lambda_f \mu_f; \Lambda \mu}^{(1 \rightarrow 2)}(x_f, 0 - y_0; y_0 - x_i, 0) = \langle 0 | \mathcal{O}_{\Lambda_f \mu_f}(x_f, 0, P_f) \, \mathcal{J}^{[J; P; \lambda]}(y_0, Q) \, \varphi^\dagger(x_i, 0, -P_i) | 0 \rangle \]

\[ \int \frac{dP_{f,0}}{2\pi} \frac{dP_{f,0}}{2\pi} e^{i P_{f,0}(x_{f,0} - y_0)} e^{i P_{f,0}(y_0 - x_{i,0})} \]

\[ = \left( \frac{e^{-(y_0 - x_{i,0})E_{\Lambda_i,0}}}{2E_{\Lambda_i,0}} \right) L^3 \sum_{n_f} e^{-E_{\Lambda_f, n_f}(x_{f,0} - y_0)} \, V^\dagger_{\mathcal{O}, \Lambda_f \mu_f} \, R_{\Lambda_f, n_f} \, \mathcal{H}_{\Lambda_f, n_f; \Lambda \mu} \]

to extract the matrix element of interest - one must take the ratio of the 3-point function to 1- and 2-point correlation functions (using same interpolating operators)

"after a little work" (and simultaneously beer and coffee)

\[ \left| \langle E_{\lambda_f, n_f} P_f | \mathcal{J}(0, Q) | E_{\lambda_i, 0} P_i \rangle \right| = \frac{1}{\sqrt{2E_{\lambda_i,0}}} \sqrt{\mathcal{H}^T_{\lambda_f, n_f} \, R_{\lambda_f, n_f} \, \mathcal{H}_{\lambda_f, n_f}} \]
\[ \langle E_{\lambda_f,n_f} \mathbf{P}_f | \mathcal{J}(0,Q) | E_{\lambda_i,0} \mathbf{P}_i \rangle = \frac{1}{\sqrt{2E_{\lambda_i,0}}} \sqrt{H_{\lambda_f,n_f}^T \mathbf{R}_{\lambda_f,n_f} H_{\lambda_f,n_f}} \]

to get the master formula,

\[ \mathcal{A} = H + K (iP^2/2) H + K (iP^2/2) K (iP^2/2) H + \cdots = \left[ \frac{1}{1 - K (iP^2/2)} \right] H \]

\[ = \left[ \frac{1}{K^{-1} - (iP^2/2)} \right] K^{-1} H = \mathcal{M} K^{-1} H. \]

\[ \mathcal{R}_{\Lambda_f,n_f} = [\mathcal{M}^{-1\dagger} K \mathcal{R} K \mathcal{M}^{-1}]_{\Lambda_f,n_f} \]

\[ \langle E_{\Lambda_f,n_f} \mathbf{P}_f; L | \tilde{\mathcal{J}}_{\Lambda_f}^{[J,P,|\lambda]}(0, \mathbf{P}_f - \mathbf{P}_i) | E_{\Lambda_i,0} \mathbf{P}_i; L \rangle = \frac{1}{\sqrt{2E_{\Lambda_i,0}}} \sqrt{[\mathcal{A}_{\Lambda_f,n_f;\Lambda_f}^\dagger \mathcal{R}_{\Lambda_f,n_f} \mathcal{A}_{\Lambda_f,n_f;\Lambda_f}]} \]

master formula: finite-volume matrix element of a current that

- can inject arbitrary four-momentum and angular momentum
- includes all inelastic coupled channels, “a”
- incorporates partial-wave mixing (from box and/or physics)
3 POINT CORRELATION FUNCTION

\[
\left| \langle E^{\lambda_f,n_f} P_f | \mathcal{J}(0,Q) | E^{\lambda_i,0} P_i \rangle \right| = \frac{1}{\sqrt{2E^{\lambda_i,0}}} \sqrt{\mathcal{H}^T_{\lambda_f,n_f} R_{\lambda_f,n_f} \mathcal{H}_{\lambda_f,n_f}}
\]

to get the master formula,

\[
\mathcal{A} = \mathcal{H} + iK (iP^2/2) \mathcal{H} + K (iP^2/2) K (iP^2/2) \mathcal{H} + \cdots = \left[ \frac{1}{1 - K (iP^2/2)} \right] \mathcal{H}
\]

\[
\mathcal{R}_{\Lambda_f,n_f} = [\mathcal{M}^{-1\dagger} K R K \mathcal{M}^{-1}]_{\Lambda_f,n_f}
\]

Matrix-generalization of Lellouch-Lüscher
3 POINT CORRELATION FUNCTION

\[ \langle E_{\lambda_f,n_f} P_f | J(0,Q) | E_{\lambda_i,0} P_i \rangle = \frac{1}{\sqrt{2E_{\lambda_i,0}}} \sqrt{H^T_{\lambda_f,n_f} R_{\lambda_f,n_f} H_{\lambda_f,n_f}} \]

to get the master formula,

\[ A = H + K (iP^2/2) H + K (iP^2/2) K (iP^2/2) H + \cdots = \left[ \frac{1}{1 - K (iP^2/2)} \right] H \]

\[ R_{\lambda_f,n_f} = [M^{-1\dagger} K R K M^{-1}]_{\lambda_f,n_f} \]

\[ \langle E_{\lambda_f,n_f} P_f; L | \tilde{J}_{\lambda_{\mu}}^{[J,P,|\lambda]}(0,P_f - P_i) | E_{\lambda_i,0} P_i; L \rangle = \frac{1}{\sqrt{2E_{\lambda_i,0}}} \sqrt{A_{\lambda_f,n_f;\lambda_{\mu}}^{\dagger} R_{\lambda_f,n_f} A_{\lambda_f,n_f;\lambda_{\mu}}} \]

See talk by Christian Shultz
Thur. 3:35
to go from these subduced infinite-volume transition amplitudes to back to O(3) symmetric amplitudes
3 POINT CORRELATION FUNCTION

\( K \rightarrow \pi \pi \)

**Lellouch-Lüscher**

\[
\frac{|H_{S,n_f} \cos \delta_S|^2}{|\langle \pi \pi, E_{n_f} P, \Lambda_f \mu_f; L | \tilde{J}_{\Lambda \mu}^{[0,-1,|0|]}(0,0) | K, E_K P; L \rangle|^2} = \frac{16 \pi E_i}{q_{n_f}^* \xi} \frac{E_{n_f}^*}{\partial P_{0,M}} \left( \delta_S + \phi_0^d \right) \bigg|_{P_{0,M} = E_{n_f}}
\]

Finite-Volume Matrix Element

\[ |H_{S,n_f} \cos \delta_S| = |A_{S,n_f}| \]

Infinite-Volume Transition Amplitude

\[ q_{\Lambda,n}^* \cot \phi_{lm}^d = - \frac{4 \pi}{q_{\Lambda,n}^*} c_{lm}^d (q_{\Lambda,n}^*; L) \]

pseudo-phase

\( (\text{Euler})\text{-Reimann-zeta Function} \)

\[ c_{lm}^d (k_j^*; L) = \frac{\sqrt{4 \pi}}{\gamma L^3} \left( \frac{2 \pi}{L} \right)^{l-2} Z_{lm}^d [1; (k_j^* L / 2 \pi)^2], \quad Z_{lm}^d [s; x^2] = \sum_{r \in P_d} \frac{|r|^l Y_{l,m}(r)}{(r^2 - x^2)^{s}} \]
\[ \gamma \pi \rightarrow \pi \pi \]

Lowest energy state is P-wave

\[
\frac{|\mathcal{H}_{f \mu_f, n_f; \Lambda \mu} \cos \delta_P|^2}{|\langle \pi \pi, E_{n_f} P_f, \Lambda_f \mu_f; L | \bar{J}_{\Lambda \mu}^{[1,-1,|\lambda|]}(0, P_f - P_i) | \pi, E_i P_i; L \rangle|^2} = 16\pi E_i \frac{E_{n_f}}{q_{n_f}^*} \xi \sin^2 \delta_P
\]

\[
\times \left[ \csc^2 \delta_P \frac{\partial \delta_P}{\partial P_{0,M}} + \csc^2 \phi_{00} \frac{\partial \phi_{00}}{\partial P_{0,M}} + \sum_{m=0,2} \alpha_{2m, \Lambda_f} \csc^2 \phi_{2m} \frac{\partial \phi_{2m}}{\partial P_{0,M}} \right]_{P_{0,M} = E_{n_f}}
\]

\[
|\mathcal{H}_{f \mu_f, n_f; \Lambda \mu} \cos \delta_P| = |\mathcal{A}_{f \mu_f, n_f; \Lambda \mu}|
\]

See talk by Christian Shultz

Thur. 3:35

Infinite-Volume Transition Amplitude
Comment on recent calculation of $B \to K^* \ell^+ \ell^-$

Horgan, Liu, Meinel, Wingate: PRL 112 (2014)
PRD 89 (2014)

1. Calculation treated $K^*$ as stable - need to use correct FV formalism - includes S-P wave mixing (this is all treated in our paper arXiv:1406.5965)

2. $I=1/2$ $K\pi$ scattering has “quark disconnected” graphs:
   this means the staggered action will give rise to unitarity violating “haripin” interactions in the S-channel graphs, invalidating the Lüscher formalism for understanding the two-hadron spectrum

I believe the hairpin issue makes the calculation practically impossible - at least with our current understanding of scattering with PQ effects
we have extended the Lellouch-Lüscher method to determine a “master formula” describing the mapping between finite-volume matrix element calculations and the corresponding infinite volume transition amplitudes of a current that

• can inject arbitrary four-momentum and angular momentum
• includes all inelastic coupled channels
• incorporates partial-wave mixing (from box and/or physics)

This new formalism is very powerful and makes as few approximations as possible: it is model-independent, non-perturbative and valid below inelastic thresholds
I would like to particularly thank my younger colleagues who “held my hand” as I learned how to think of these problems in this “modern” fashion

Raúl Briceño
Max Hansen

Thank You