Disconnected Contributions to Nucleon Charges

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General Three-point Functions

\[ \text{ME}_\Gamma \sim \langle N | \bar{q}_i \Gamma q_j | N \rangle \]

- Isoscalar operators \((q_i = q_j)\) have disconnected quark loops
Physics with Disconnected Contributions

<table>
<thead>
<tr>
<th>Physics</th>
<th>Observables</th>
<th>Diagrams</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neutron $\beta$-decay</td>
<td>$g_V, g_A, g_S, g_T$</td>
<td>Connected</td>
</tr>
<tr>
<td></td>
<td></td>
<td>No-disconnected</td>
</tr>
<tr>
<td>nEDM</td>
<td></td>
<td>Connected &amp;</td>
</tr>
<tr>
<td>qEDM</td>
<td>$g_T$</td>
<td>Disconnected</td>
</tr>
<tr>
<td>cEDM</td>
<td>$\langle N</td>
<td>\bar{q}\sigma \cdot Gq</td>
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<tr>
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<td></td>
<td>Disconnected</td>
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<tr>
<td>Sigma term</td>
<td>$\langle N</td>
<td>\bar{q}q</td>
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<tr>
<td>Dark-matter search</td>
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<td>Disconnected</td>
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<tr>
<td>TMDs</td>
<td>$\langle N</td>
<td>\bar{q}(0)\Gamma U^b_0 q(b)</td>
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<td></td>
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<td>Disconnected</td>
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</tbody>
</table>
$D^{-1}$: Improvement & Error Reduction Techniques

- Multigrid Solver [Osborn, et al., 2010; Babich, et al., 2010]
  [Plenary talk: Andreas Frommer, Tue 10:15]

- All-Mode Averaging (AMA) for Two-point Correlators
  [Blum, Izubuchi and Shintani, 2013]

- Hopping Parameter Expansion (HPE)
  [Thron, et al., 1998; McNeile and Michael, 2001]

- Truncated Solver Method (TSM) [Bali, Collins and Schäfer, 2007]

- Dilution [Bernardson, et al., 1994; Viehoff, et al., 1998]
Lattice Setup for Comparison Study

- MILC $N_f = 2 + 1 + 1$ HISQ lattice
  - $a = 0.12$ fm
  - $m_\pi = 305$ MeV
  - Geometry $= 24^3 \times 64$
  - Number of confs $= 1013$

- Clover valence quarks (Clover on HISQ)

- HYP smeared gauge links
All-Mode Averaging (AMA) for Two-point Correlators
Improved Estimator of Two-point Function

\[ C^{2pt, \text{imp}} = \frac{1}{N_{\text{LP}}} \sum_{i=1}^{N_{\text{LP}}} C^{2pt}_{\text{LP}}(x_i) + \frac{1}{N_{\text{HP}}} \sum_{j=1}^{N_{\text{HP}}} \left[ C^{2pt}_{\text{HP}}(x_j) - C^{2pt}_{\text{LP}}(x_j) \right] \]

- Exploiting translation symmetry of lattice, 2pt-func is averaged on multiple source positions
- “LP” term is the low-precision estimate, truncated the inverter with a low accuracy stopping criterion (e.g., \( r_{\text{LP}} = 10^{-3} \))
- “HP” (high-precision) correction term (e.g., \( r_{\text{HP}} = 10^{-8} \)) makes the estimator unbiased; Systematic error \( \Rightarrow \) Statistical error
- \( N_{\text{LP}} \gg N_{\text{HP}} \) brings computational gain (e.g., \( N_{\text{LP}} = 60, N_{\text{HP}} = 4 \))
### Gain in Computational Cost

<table>
<thead>
<tr>
<th></th>
<th>$g^\text{dis}_S$</th>
<th>Gain</th>
<th>$g^\text{dis}_A$</th>
<th>Gain</th>
<th>$g^\text{dis}_T$</th>
<th>Gain</th>
<th>$g^\text{dis}$</th>
<th>Gain</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 HP</td>
<td>1.253(192)</td>
<td>1</td>
<td>1.202(83)</td>
<td>1.4</td>
<td>1.203(71)</td>
<td>1.1</td>
<td></td>
<td></td>
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<tr>
<td>28 LP + 4 Crxn</td>
<td>1.202(83)</td>
<td>1.4</td>
<td>1.203(71)</td>
<td>1.1</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>60 LP + 4 Crxn</td>
<td>1.203(71)</td>
<td>1.1</td>
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<td>1.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g^\text{dis}_S$</td>
<td>1.202(83)</td>
<td>1.4</td>
<td>1.203(71)</td>
<td>1.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g^\text{dis}_A$</td>
<td>-0.0828(173)</td>
<td>1</td>
<td>-0.0872(62)</td>
<td>2.0</td>
<td>-0.0872(48)</td>
<td>2.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$g^\text{dis}_T$</td>
<td>-0.0141(82)</td>
<td>1</td>
<td>-0.0122(31)</td>
<td>1.8</td>
<td>-0.0136(23)</td>
<td>2.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cost</td>
<td>1</td>
<td>3.8</td>
<td>6.4</td>
<td>6.4</td>
<td></td>
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</table>

- Nucleon charges at $t_{\text{sep}} = 8a, t_{\text{ins}} = 4a$
- All sources are distributed on 4 timeslices
- Gain = \(\frac{\text{Decrease of } \sigma^2}{\text{Increase of Compt. cost}}\)
- Between “28 LP + 4 Crxn” and “60 LP + 4 Crxn”, cost and error scale in the same way, and the gain stays around 2
- We use “60 LP + 4 Crxn” setup
Hopping Parameter Expansion (HPE)
Hopping Parameter Expansion (Preconditioning)

\[ M = \frac{1}{2\kappa} (1 - \kappa D) \]
\[ M^{-1} = 2\kappa \mathbf{1} + 2\kappa^2 D + \kappa^2 D^2 M^{-1} \]

For disconnected quark loops, we need \( \text{Tr} \left[ M^{-1} \Gamma \right] \)

1. \( \text{Tr} \left[ 2\kappa \Gamma \right]_{\Gamma = \mathbf{1}} = 24\kappa \)
2. \( \text{Tr} \left[ 2\kappa \Gamma \right]_{\Gamma \neq \mathbf{1}} = 0 \)
3. \( \text{Tr} \left[ 2\kappa^2 D \Gamma \right] = 0 \)

- Need to calculate only \( \kappa^2 D^2 M^{-1} \)
- Removed noise from first two orders (1-3) – reduces error
Truncated Solver Method (TSM)
Truncated Solver Method (TSM)

[Bali, Collins and Schäfer, 2007]

\[ M^{-1}_E = \frac{1}{N_{LP}} \sum_{i=1}^{N_{LP}} |s_i\rangle_{LP} \langle \eta_i| + \frac{1}{N_{HP}} \sum_{i=N_{LP}+1}^{N_{LP}+N_{HP}} \left( |s_i\rangle_{HP} - |s_i\rangle_{LP} \right) \langle \eta_i| \]

- LP estimate
- Crxn term

**Stochastic estimate of** \( M^{-1} \)

**Same form as AMA**
- \( C^{2pt} \Rightarrow M^{-1} \)
- Sum over source positions \( \Rightarrow \) Sum over random noise sources

**|\eta_i\rangle** : complex random noise vector satisfying

\[
\frac{1}{N} \sum_{i=1}^{N} |\eta_i\rangle = \mathcal{O} \left( \frac{1}{\sqrt{N}} \right) , \quad \frac{1}{N} \sum_{i=1}^{N} |\eta_i\rangle \langle \eta_i| = 1 + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right)
\]

**|s_i\rangle** : solution vector; \( M |s_i\rangle = |\eta_i\rangle \)
## Random Noise

- **Type of random noises investigated**

<table>
<thead>
<tr>
<th>Type</th>
<th>Noise</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_2 )</td>
<td>( r_r )</td>
<td>( r_r \in {1, -1} )</td>
</tr>
<tr>
<td>( \mathbb{Z}_2 \otimes i\mathbb{Z}_2 )</td>
<td>( r_r + ir_i )</td>
<td>( r_{r,i} \in {\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}} )</td>
</tr>
<tr>
<td>( \mathbb{Z}_4 )</td>
<td>( r_c )</td>
<td>( r_c \in {1, i, -1, -i} )</td>
</tr>
<tr>
<td>Gaussian</td>
<td>( r_r + ir_i )</td>
<td>( r_{r,i} \sim \text{Gaussian} / \sqrt{2} )</td>
</tr>
</tbody>
</table>
Comparing $\sigma_{g_T}^{\text{dis}}$ for Random Noises

- Statistical error in $g_T$ at $t_{\text{sep}} = 8a$, $t_{\text{ins}} = 4a$ (mid point)
- Random noise sources are placed only on 8 timeslices (dilution)
- Gaussian random noise is (marginally) better than others
Required Number of Random Sources

\[ M^{-1} \approx \frac{1}{N_{LP}} \sum_{i=1}^{N_{LP}} |s_i \rangle_{LP} \langle \eta_i | \]

- Total statistical error scales as

\[ \sigma_{tot} = \sigma_{\infty} \sqrt{1 + \frac{X_{TSM}}{N_{LP}}} \]

- \( X_{TSM} \) depends on TSM parameters, \( N_{LP}/N_{HP} \) and \( r_{LP} \)
Required $N_{LP} : g_S$

\[ \sigma = \sigma_\infty \sqrt{1 + \frac{X_{TSM}}{N_{LP}}} \approx 0.71 \sqrt{1 + \frac{39}{N_{LP}}} \]

- Statistical error in $g_S$ at $t_{sep} = 8a$, $t_{ins} = 4a$ (mid point)
- Noise srcs are placed on 44 timeslices that satisfy $|t_{src} - t| \geq 3a$
- Conclusion : $N_{LP} \approx 500$ is enough
Required $N_{LP}$ : $g_A$

\[ \sigma = \sigma_\infty \sqrt{1 + \frac{X_{TSM}}{N_{LP}}} \approx 0.0047 \sqrt{1 + \frac{656}{N_{LP}}} \]

- Statistical error in $g_A$ at $t_{sep} = 8a$, $t_{ins} = 4a$ (mid point)
- Noise srcs are placed on 44 timeslices that satisfy $|t_{src} - t| \geq 3a$
- Conclusion : $N_{LP} \approx 3000$ is enough
Required $N_{LP} : g_T$

\[ \sigma = \sigma_\infty \sqrt{1 + \frac{X_{TSM}}{N_{LP}}} \approx 0.0010 \sqrt{1 + \frac{15140}{N_{LP}}} \]

- Statistical error in $g_T$ at $t_{sep} = 8a$, $t_{ins} = 4a$ (mid point)
- Noise srcs are placed on 44 timeslices that satisfy $|t_{src} - t| \geq 3a$
- Conclusion: $N_{LP} \gtrsim 5000$ is needed
Effect of Hopping Parameter Expansion

![Graph showing the effect of hopping parameter expansion](image)

- **Without HPE**
- **With HPE**

The graph plots $\sigma_{gT}$ against $N_{LP}$, with $\sigma_{gT}$ in units of $10^2$. The data points are shown with error bars for each condition.
Dilution
Dilution Technique

- Divide $\mathcal{R} = \{\text{spacetime} \otimes \text{color} \otimes \text{spin}\}$ into $m$ subspaces $\mathcal{R}_j$

$$\mathcal{R} = \sum_{j=1}^{m} \mathcal{R}_j$$

- Sum $M^{-1}$ evaluated on each subspaces

$$M^{-1} \approx \sum_{j=1}^{m} \left[ \frac{1}{N} \sum_{i=1}^{N} |s_i\rangle \langle \eta_i| \right]$$

- If noise is reduced by more than $\sqrt{m}$, worth doing dilution
Testing Time Dilution (Gaussian Random Source)

- Statistical error of a disconnected quark loop with time Dilution for different number of source timeslices ($N_{tsrc}$)

- e.g., if $N_{tsrc} = 32$, random noises are on 32 timeslices; cover all timeslices with two applications for a $24^3 \times 64$ Lattice

- Total computational cost is fixed ($N_{LP} \times 64/N_{tsrc} = \text{fixed}$)

- Time dilution is not efficient!
Results
Removing Excited States Contamination

Fitting functions include one excited state

\[ C^{2pt}(t_{\text{sep}}) = A_1 e^{-M_0 t_{\text{sep}}} + A_2 e^{-M_1 t_{\text{sep}}} \]

\[ C^{3pt}(t_{\text{sep}}, t_{\text{ins}}) = B_1 e^{-M_0 t_{\text{sep}}} + B_2 e^{-M_1 t_{\text{sep}}} + B_{12} \left[ e^{-M_0 t_{\text{ins}}} e^{-M_1 (t_{\text{sep}} - t_{\text{ins}})} + e^{-M_1 t_{\text{ins}}} e^{-M_0 (t_{\text{sep}} - t_{\text{ins}})} \right] \]

- \( A_1 \) and \( B_1 \) are the results for the ground state
$g_S : t_{\text{sep}}$ and $t_{\text{ins}}$ dependence

Preliminary!!!
$g_A : t_{\text{sep}}$ and $t_{\text{ins}}$ dependence

Preliminary!!!
$g_T$: $t_{\text{sep}}$ and $t_{\text{ins}}$ dependence

Preliminary!!!
Unrenormalized Isoscalar Nucleon Charges
on $a = 0.12 \text{ fm}$ and $m_\pi = 305 \text{ MeV}$ Lattice

Preliminary!!!

<table>
<thead>
<tr>
<th></th>
<th>Disconnected</th>
<th>Connected$^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_S$</td>
<td>1.90(15)</td>
<td>5.79(22)</td>
</tr>
<tr>
<td>$g_A$</td>
<td>$-0.120(12)$</td>
<td>0.632(36)</td>
</tr>
<tr>
<td>$g_T$</td>
<td>$-0.0252(49)$</td>
<td>0.650(24)</td>
</tr>
</tbody>
</table>

$^1$PRD89 094502 (2014)
Renormalized Isoscalar Nucleon Charges

Preliminary!!!

<table>
<thead>
<tr>
<th></th>
<th>PNDME, LANL</th>
<th>Abdel-Rehim, <em>et al.</em> ²</th>
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<tbody>
<tr>
<td></td>
<td>$a = 0.12$ fm, $m_\pi = 305$ MeV</td>
<td>$a = 0.082$ fm, $m_\pi = 375$ MeV</td>
</tr>
<tr>
<td></td>
<td>Clover on HISQ</td>
<td>Twisted Mass</td>
</tr>
<tr>
<td>$g_S$</td>
<td>5.16(24)</td>
<td>1.69(13)</td>
</tr>
<tr>
<td>$g_A$</td>
<td>0.610(37)</td>
<td>−0.116(12)</td>
</tr>
<tr>
<td>$g_T$</td>
<td>0.613(26)</td>
<td>−0.0238(46)</td>
</tr>
</tbody>
</table>

- Renormalization is done using only connected diagrams at 2 GeV

²PRD89 034501 (2014)
Glance at the $a = 0.0888 \text{ fm}$ and $m_\pi = 313 \text{ MeV}$

Preliminary!!!

<table>
<thead>
<tr>
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<th>$0.09 \text{ fm}$</th>
<th>$0.12 \text{ fm}$</th>
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<tbody>
<tr>
<td>$g_S^{\text{dis}}$</td>
<td>$1.43(12)$</td>
<td>$1.69(13)$</td>
</tr>
<tr>
<td>$g_A^{\text{dis}}$</td>
<td>$-0.107(16)$</td>
<td>$-0.116(12)$</td>
</tr>
<tr>
<td>$g_T^{\text{dis}}$</td>
<td>$-0.0114(51)$</td>
<td>$-0.0238(46)$</td>
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</tbody>
</table>

- Renormalization is done using only connected diagrams at $2 \text{ GeV}$
Isoscalar Nucleon Charges (con + disc)

Preliminary!!!

<table>
<thead>
<tr>
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<th>0.09 fm</th>
<th>0.12 fm</th>
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<tbody>
<tr>
<td>$a$</td>
<td></td>
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</tr>
<tr>
<td>$g_S$</td>
<td>7.44(28)</td>
<td>6.91(26)</td>
</tr>
<tr>
<td>$g_T$</td>
<td>0.612(26)</td>
<td>0.594(27)</td>
</tr>
</tbody>
</table>

- Renormalization is done using only connected diagrams at 2 GeV
- $m_\pi \approx 310$ MeV
- Statistical error only
Summary and Outlook

- We calculate disconnected contribution to nucleon charges

- Disconnected contribution is about 30%, 20% and 4% for $g_S$, $g_A$ and $g_T$, respectively

- All-mode averaging, Truncated solver method and Hopping parameter expansion are applied

- Disconnected contribution to the renormalization constants is under investigation

- Techniques can be applied to other physical objects
### Computational Cost

<table>
<thead>
<tr>
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<th>64 LP (31%)</th>
<th>Src/Sink prep.</th>
<th>Inversion</th>
<th>etc.</th>
</tr>
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<tbody>
<tr>
<td>2pt</td>
<td></td>
<td></td>
<td>(18%)</td>
<td>(11%)</td>
</tr>
<tr>
<td>(37%)</td>
<td></td>
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<td>(2%)</td>
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<tr>
<td></td>
<td>4 HP (6%)</td>
<td>Src/Sink prep.</td>
<td>(2%)</td>
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<td></td>
<td></td>
<td>Inversion</td>
<td>(3%)</td>
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<td></td>
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<td>etc.</td>
<td>(1%)</td>
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<td>Disc.</td>
<td>5000 LP estimate (53%)</td>
<td>Inversion</td>
<td>(35%)</td>
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<tr>
<td>(63%)</td>
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<td>Src prep/Trace/HPE</td>
<td>(18%)</td>
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<td>166 Correction (8%)</td>
<td>LP Inversion</td>
<td>(1%)</td>
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<tr>
<td></td>
<td></td>
<td>HP Inversion</td>
<td>(6%)</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>Src prep/Trace/HPE</td>
<td>(1%)</td>
<td></td>
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<tr>
<td>etc.</td>
<td>(2%)</td>
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- **Inverter Improvements:** BiCGStab $\rightarrow$ Multigrid [$x4$; $m_\pi \approx 310$ MeV]
  - HP $\rightarrow$ LP [$x3 \sim x5$; $r_{LP} = 0.001 \sim 0.005$]
64 source positions on a $24^3 \times 64$ Lattice

- On each timeslice $t$, 16 srcs are placed on
  \[(x_1, y_1, z_1, t) \oplus (x_2, y_2, z_2, t)\]
  \[x_1, y_1, z_1 \in \{0, \frac{L}{2}\}, \quad x_2, y_2, z_2 \in \{\frac{L}{4}, \frac{3L}{4}\}\]

- On one of 16 source positions on a timeslice, calculate both HP and LP for correction term
Effect of Hopping Parameter Expansion

![Graph showing the effect of hopping parameter expansion with labels](image-url)

- Gauss
- $Z_2 \otimes iZ_2$
- HPE, $Z_2 \otimes iZ_2$
- HPE, Gauss

**Axes:**
- $N_{LP}$: Number of Longitudinal Parameters
- $\sigma_{\text{dis}} [x10^1]$: Disorder Standard Deviation

**Legend:**
- Green crosses: Gauss
- Orange circles: $Z_2 \otimes iZ_2$
- Red triangles: HPE, $Z_2 \otimes iZ_2$
- Blue squares: HPE, Gauss
Effect of Hopping Parameter Expansion

\[ \sigma_{gA} \text{ dis} \times 10^2 \]

\[ N_{LP} \text{ Gauss, } Z_2 \otimes iZ_2, \text{ HPE, } Z_2 \otimes iZ_2, \text{ HPE, Gauss} \]

\[ 200, 300, 400, 500, 600, 700, 800, 900, 1000 \]
Effect of Hopping Parameter Expansion

\[ \sigma_{gT}^{\text{dis}} \times 10^2 \]

- Gauss
- $Z_2 \otimes iZ_2$
- HPE, $Z_2 \otimes iZ_2$
- HPE, Gauss
Random Noise Type Dependence

\[ \sigma_{\text{gs dis}} \times 10^2 \]

\[ N_{\text{LP}} \]

\[ Z_2 \otimes iZ_2 \]

\[ Z_4 \]

Gauss
Random Noise Type Dependence

\[ \sigma_{gA \text{dis}} \times 10^3 \]

- \( Z_2 \otimes iZ_2 \)
- \( Z_4 \)
- Gauss

\( N_{LP} \)
Random Noise Type Dependence

![Graph showing dependence of \( \sigma_{gT} \) on \( N_{LP} \) for different noise types.](image)

- \( Z_2 \)⊗ i\( Z_2 \)
- \( Z_4 \)
- Gauss

\( \sigma_{gT} \) in \( [x10^3] \)

\( N_{LP} \) ranges from 200 to 1000.