Lattice 2014

Prepotential Formulation of Lattice Gauge Theories

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- Two major obstacles: non-locality and proliferation of loops and string states.
- Not all loop states are mutually independent $\Rightarrow$ Mandelstam constraints.
- The Mandelstam constraints, in turn, are difficult to solve because of their non-locality.
- Becomes a severe problem in the weak-coupling regime (continuum limit) of lattice gauge theory.

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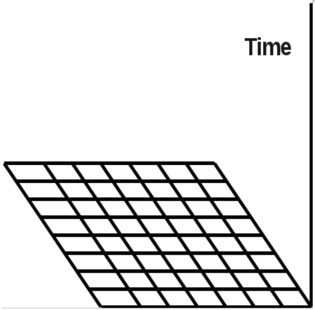
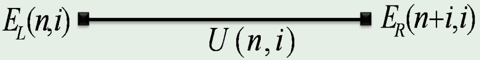
- Prepotentials provide such a platform!
- Prepotentials are seemed to be the most suitable variables for the weak coupling perturbation expansion

Variables

Discrete Space and Continuous time



On a link of the spatial lattice



$$E_R(n+i,i) = -U^\dagger(n,i)E_L(n,i)U(n,i).$$

The Kogut-Susskind Hamiltonian

$SU(2)$ gauge theory

$$H = g^2 \sum_{n,i} \sum_{\mathrm{a}=1}^{3} E^{\mathrm{a}}(n,i) E^{\mathrm{a}}(n,i) + \frac{1}{g^2} \sum_{\Box} Tr \left(1 - U_{\Box} - U_{\Box}^{\dagger}\right)$$

with, $U_{\Box} = U(n,i)U(n+i,j)U^{\dagger}(n+j,i)U^{\dagger}(n,j)$
$\mathrm{a}(=1,2,3) \rightarrow$ **color index.**

Quantization Rules

Canonical variables

$$
\begin{aligned}
[E_L^{\mathrm{a}}(n,i), U^{\alpha}{}_{\beta}(n,i)] &= -\left(\frac{\sigma^{\mathrm{a}}}{2} U(n,i)\right)^{\alpha}{}_{\beta}, \\
\left[E_R^{\mathrm{a}}(n+i,i), U^{\alpha}_{\beta}(n,i)\right] &= \left(U(n,i)\frac{\sigma^{\mathrm{a}}}{2}\right)^{\alpha}{}_{\beta}.
\end{aligned}
$$

Constraints

Gauss Law

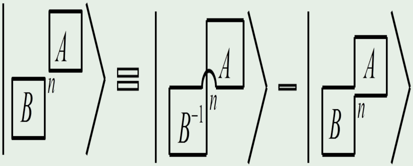
$$G(n) = \sum_{i=1}^{d} \Big(E_L^{\mathrm{a}}(n,i) + E_R^{\mathrm{a}}(n,i) \Big) = 0, \forall n.$$

Electric field constraint

$$E_L^2(n,i) = E_R^2(n+i,i)$$

Wilson loops and Mandelstam Constraints: SU(2)

Involving two loops, each carrying one unit of flux



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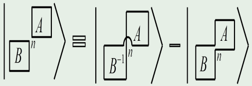
Fundamental Mandelstam identity for SU(2)



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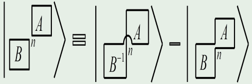

- **In prepotential formulation these fundamental Mandelstam identities becomes local and can be analyzed as well as solved to get Orthonormal Loop states.**

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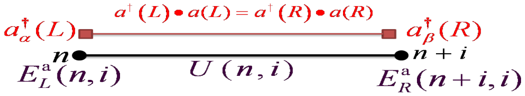
- *Ref: Manu Mathur, Nucl. Phys. B 2007, Phys. Letts. B 2006, J. Phys. A: Math. Gen. 2005, Ramesh Anishetty, MM, IR J. Phys. A 2010, J. Math. Phys 2010, in preparation*

SU(2) Prepotentials



Left electric fields: $$E^a_L(n,i) \equiv a^\dagger(n,i;L)\frac{\sigma^a}{2}a(n,i;L),$$

Right electric fields: $$E^a_R(n+i,i) \equiv a^\dagger(n+i,i;R)\frac{\sigma^a}{2}a(n+i,i;R).$$

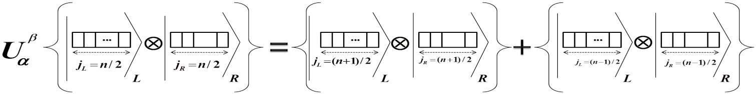
Under SU(2) gauge transformation

$$a^\dagger_\alpha(L) \rightarrow a^\dagger_\beta(L)\left(\Lambda^\dagger_L\right)^\beta{}_\alpha, \qquad a^\dagger_\alpha(R) \rightarrow a^\dagger_\beta(R)\left(\Lambda^\dagger_R\right)^\beta{}_\alpha$$
$$a^\alpha(L) \rightarrow \left(\Lambda_L\right)^\alpha{}_\beta\, a^\beta(L), \qquad a^\alpha(R) \rightarrow \left(\Lambda_R\right)^\alpha{}_\beta\, a^\beta(R).$$

Link Operator

- **From** $SU(2) \otimes U(1)$ **gauge transformations of the prepotentials,**

$$U^{\alpha}{}_{\beta} = \tilde{a}^{\dagger\alpha}(L)\,\eta\, a^{\dagger}_{\beta}(R) + a^{\alpha}(L)\,\theta\,\tilde{a}_{\beta}(R)$$



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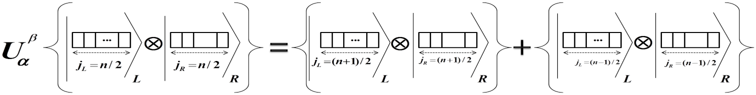


- **Calculating the coefficients from $U^{\dagger}U = UU^{\dagger} = 1$,**

$$U = \underbrace{\frac{1}{\sqrt{\hat{N}+1}}\begin{pmatrix} a^{\dagger}_{2}(L) & a_{1}(L) \\ -a^{\dagger}_{1}(L) & a_{2}(L) \end{pmatrix}}_{U_L}\underbrace{\begin{pmatrix} a^{\dagger}_{1}(R) & a^{\dagger}_{2}(R) \\ a_{2}(R) & -a_{1}(R) \end{pmatrix}\frac{1}{\sqrt{\hat{N}+1}}}_{U_R}$$

Abelian Weaving, Non-abelian Intertwining and Loop States

Link operator: $U^{\alpha}{}_{\beta} = \frac{1}{\sqrt{\hat{n}+1}} \left(\tilde{a}^{\dagger\alpha}(L)\, a^{\dagger}_{\beta}(R) + a^{\alpha}(L)\, \tilde{a}_{\beta}(R) \right) \frac{1}{\sqrt{\hat{n}+1}}$

Four basic gauge invariant operators constructed by $U^{\alpha}{}_{\beta}(n,i) U^{\beta}{}_{\gamma}(n+i,j)$ at site $(n+i)$:

$$a^{\dagger}_{\beta}(i) \frac{1}{\sqrt{\hat{n}_i+1}} \frac{1}{\sqrt{\hat{n}_j+1}} \tilde{a}^{\dagger\beta}(j) = \frac{1}{\sqrt{\hat{n}_i}} \frac{1}{\sqrt{\hat{n}_j+1}} a^{\dagger}(i)\cdot\tilde{a}^{\dagger\beta}(j) \equiv \frac{1}{\sqrt{\hat{n}_i(\hat{n}_j+1)}} k^{ij}_{+} \equiv \hat{\mathcal{O}}^{i+j+}$$

$$a^{\dagger}_{\beta}(i) \frac{1}{\sqrt{\hat{n}_i+1}} \frac{1}{\sqrt{\hat{n}_j+1}} a^{\beta}(j) = \frac{1}{\sqrt{\hat{n}_i}} \frac{1}{\sqrt{\hat{n}_j+1}} a^{\dagger}(i)\cdot a(j) \equiv \frac{1}{\sqrt{\hat{n}_i(\hat{n}_j+1)}} \kappa^{ij} \equiv \hat{\mathcal{O}}^{i+j-}$$

$$\tilde{a}_{\beta}(i) \frac{1}{\sqrt{\hat{n}_i+1}} \frac{1}{\sqrt{\hat{n}_j+1}} \tilde{a}^{\dagger\beta}(j) = \frac{1}{\sqrt{\hat{n}_i+2}} \frac{1}{\sqrt{\hat{n}_j+1}} a(i)\cdot a^{\dagger}(j) \equiv \frac{1}{\sqrt{(\hat{n}_i+2)(\hat{n}_j+1)}} \kappa^{ji} \equiv \hat{\mathcal{O}}^{j+i-}$$

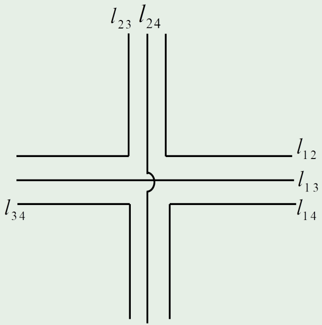
$$\tilde{a}_{\beta}(i) \frac{1}{\sqrt{\hat{n}_i+1}} \frac{1}{\sqrt{\hat{n}_j+1}} a^{\beta}(j) = \frac{1}{\sqrt{\hat{n}_i+2}} \frac{1}{\sqrt{\hat{n}_j+1}} \tilde{a}(i)\cdot a(j) \equiv \frac{1}{\sqrt{(\hat{n}_i+2)(\hat{n}_j+1)}} k^{ji}_{-} \equiv \hat{\mathcal{O}}^{i-j-}$$

for i, j different directions at each site.

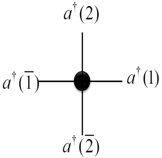
Loop States and Linking Numbers

$$|\{l_{ij}\}\rangle = \prod_{i\neq j} \frac{(k_+)^{l_{ij}}}{l_{ij}!}|0\rangle$$

Linking numbers in 2d

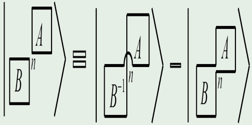


Mandelstam Constraints



$$\left(a^\dagger(1)\cdot\tilde{a}^\dagger(2)\right)\left(a^\dagger(\bar{1})\cdot\tilde{a}^\dagger(\bar{2})\right) \equiv \left(a^\dagger(1)\cdot\tilde{a}^\dagger(\bar{1})\right)\left(a^\dagger(2)\cdot\tilde{a}^\dagger(\bar{2})\right) - \left(a^\dagger(1)\cdot\tilde{a}^\dagger(\bar{2})\right)\left(a^\dagger(2)\cdot\tilde{a}^\dagger(\bar{1})\right)$$

Equivalen to the fundamental Mandelstam identity



Linking Numbers and Constraints

Loop State characterized by 6 linking numbers

$$|l_{12}, l_{1\bar{1}}, l_{1\bar{2}}, l_{2\bar{1}}, l_{2\bar{2}}, l_{\bar{1}\bar{2}}\rangle \equiv |\{l\}\rangle = \frac{\left(k_+^{12}\right)^{l_{12}}}{l_{12}!} \frac{\left(k_+^{1\bar{1}}\right)^{l_{1\bar{1}}}}{l_{1\bar{1}}!} \frac{\left(k_+^{1\bar{2}}\right)^{l_{1\bar{2}}}}{l_{1\bar{2}}!} \frac{\left(k_+^{2\bar{1}}\right)^{l_{2\bar{1}}}}{l_{2\bar{1}}!} \frac{\left(k_+^{2\bar{2}}\right)^{l_{2\bar{2}}}}{l_{2\bar{2}}!} \frac{\left(k_+^{\bar{1}\bar{2}}\right)^{l_{\bar{1}\bar{2}}}}{l_{\bar{1}\bar{2}}!} |0\rangle \qquad (1)$$

with $n_1 = l_{12} + l_{1\bar{1}} + l_{1\bar{2}} \;,\; n_2 = l_{2\bar{1}} + l_{2\bar{2}} + l_{12} \;,\; n_{\bar{1}} = l_{\bar{1}\bar{2}} + l_{1\bar{1}} + l_{2\bar{1}} \;,\; n_{\bar{2}} = l_{1\bar{2}} + l_{2\bar{2}} + l_{\bar{1}\bar{2}}$

One Mandelstam constraint

$$k_+^{12}k_+^{\bar{1}\bar{2}} - k_+^{1\bar{2}}k_+^{2\bar{1}} + k_+^{1\bar{1}}k_+^{2\bar{2}} = 0$$

Two $U(1)$ Gauss Law constraints

$$n_1(x) = n_{\bar{1}}(x + e_1) \;\;\&\;\; n_2(x) = n_{\bar{2}}(x + e_2)$$

Three different class of loop operators

$$\hat{\mathcal{O}}^{i_+ j_+} \equiv \frac{1}{\sqrt{\hat{n}_i(\hat{n}_j+1)}} k_+^{ij}$$

$$\hat{\mathcal{O}}^{i_+ j_-} \equiv \frac{1}{\sqrt{\hat{n}_i(\hat{n}_j+1)}} \kappa^{ij}$$

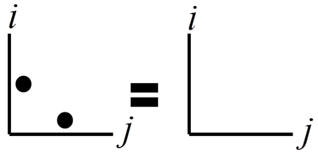
$$\hat{\mathcal{O}}^{i_- j_-} \equiv \frac{1}{\sqrt{(\hat{n}_i+2)(\hat{n}_j+1)}} k_-^{ij}$$

Local Action of Loop Operators on Local Loop states

Action of $\hat{\mathcal{O}}^{i_+j_+}$

$$\hat{\mathcal{O}}^{i_+j_+}|\{l\}\rangle \equiv \frac{1}{\sqrt{\hat{n}_i(\hat{n}_j+1)}} k_+^{ij}|\{l\}\rangle = \frac{(l_{ij}+1)}{\sqrt{\hat{n}_i(\hat{n}_j+1)}}|l_{ij}+1\rangle$$

or pictorially:



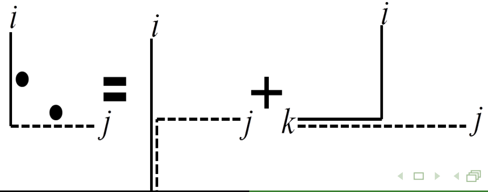
Local Action of Loop Operators on Local Loop states

Action of $\hat{\mathcal{O}}^{i+j-}$

$$\begin{aligned} \hat{\mathcal{O}}^{i+j-}\,|\{l\}\rangle &\equiv \frac{1}{\sqrt{\hat{n}_i(\hat{n}_j+1)}}\,\kappa^{ij}|\{l\}\rangle \\ &= \frac{1}{\sqrt{\hat{n}_i(\hat{n}_j+1)}}\sum_{k\neq i,j}(-1)^{S_{ik}}(l_{<ik>}+1)|l_{jk}-1,l_{<ik>}+1\rangle \end{aligned}$$

where, $<ik>$ **denotes an ordering in these two indices such that the first index is always less than the first one. Here we introduce an ordering convention** $1<2<\bar{1}<\bar{2}$ **and** $S_{ik}=1$ **if** $i>k$ **&** $S_{ik}=0$ **if** $i<k$.

Pictorially,



Local Action of Loop Operators on Local Loop states

Action of $\hat{\mathcal{O}}^{i-j-}$

$$\hat{\mathcal{O}}^{i-j-}|\{l\}\rangle = \frac{1}{\sqrt{(\hat{n}_i+2)(\hat{n}_j+1)}} \Big[(n_i+n_j-l_{ij}+1)|l_{ij}-1\rangle + \sum_{i',j'\{\neq i,j\}} (l_{<i'j'>}+1)(-1)^{S_{i'j'}}|l_{ii'}-1, l_{jj'}-1, l_{<i'j'>}+1\rangle\Big]$$

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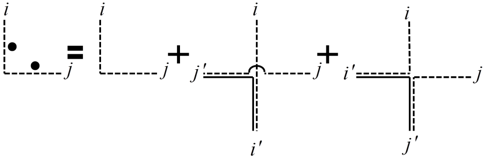
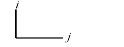
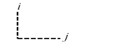
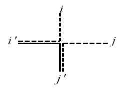


Diagrametic rules

	$C^{i_+ j_+} \mid l_{ij}+1\rangle$
	$C^{i_- j_-} \mid l_{ij}-1\rangle$
	$(C^{i_+ j_-})_k \mid l_{ik}+1, l_{kj}-1\rangle$
	$C^{(i_-)_{i'}(j_-)_{j'}} \mid l_{ii'}-1, l_{jj'}-1, l_{i'j'}+1\rangle$

$$C^{i+j+} = \frac{l_{<ij>}+1}{\sqrt{\hat{n}_i(\hat{n}_j+1)}} \quad , \quad C^{i-j-} = \frac{(\hat{n}_i+\hat{n}_j-l_{<ij>}+3)}{\sqrt{(\hat{n}_i+2)(\hat{n}_j+1)}}$$

$$\left(C^{i+j-}\right)_k = \frac{(-1)^{S_{ik}}(l_{<ik>}+1)}{\sqrt{\hat{n}_i(\hat{n}_j+1)}} \quad , \quad C^{(i-)_{i'}(j-)_{j'}} = \frac{(-1)^{S_{i'j'}}(l_{<i'j'>}+1)}{\sqrt{(\hat{n}_i+2)(\hat{n}_j+1)}}$$

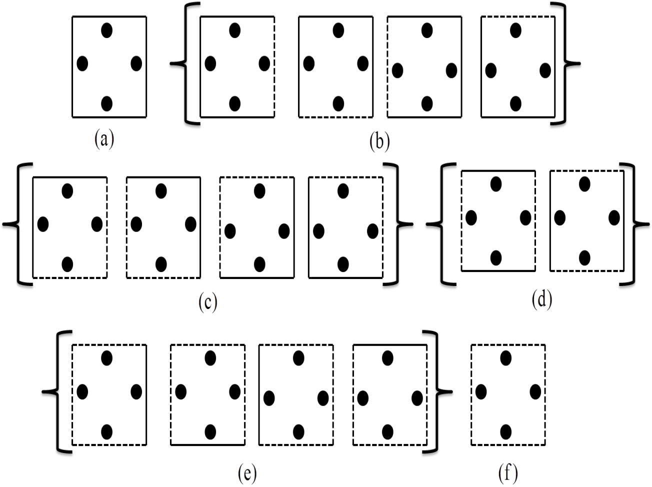
Electric Part:

$$\hat{H}_e = g^2 \sum_{links} E^2_{links} = g^2 \sum_x \left[\frac{n_1(x)}{2}\left(\frac{n_1(x)}{2}+1\right) + \frac{n_2(x)}{2}\left(\frac{n_2(x)}{2}+1\right) \right]$$

Magnetic Part

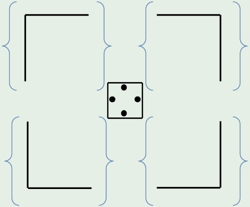
$$\hat{H}_{mag} = \frac{1}{g^2} \sum_{plaquettes} \left(4 - \mathbf{Tr}\, U_{plaquette} - \mathbf{Tr}\, U^{\dagger}_{plaquette}\right)$$

Magnetic Part of the Hamiltonian

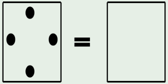


Action of $\hat{H}_{mag}$ on Loop States

Type a: H_{++++}

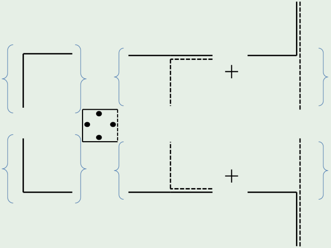


The explicit action: 1 state

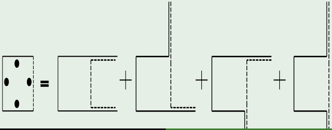


Action of $\hat{H}_{mag}$ on Loop States

Type b: H_{+++-}

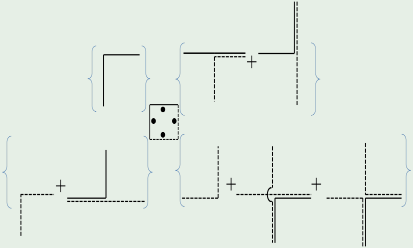


Explicit action $\Rightarrow$ 4 $\times$ 4 states

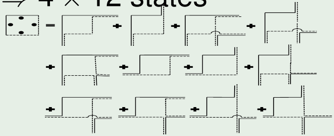


Action of $\hat{H}_{mag}$ on Loop States

Type c: H_{++--}

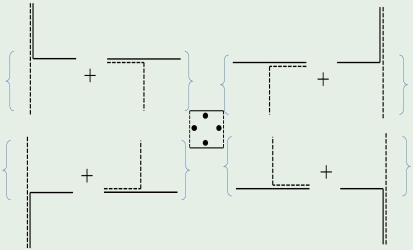


The explicit action $\Rightarrow 4 \times 12$ states



Action of $\hat{H}_{mag}$ on Loop States

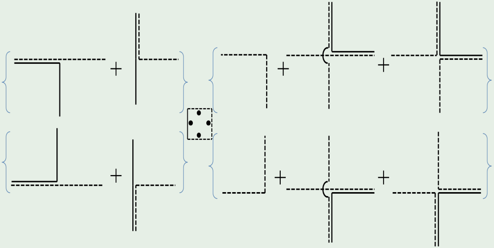
Type d: H_{+-+-}



The explicit action $\Rightarrow 2 \times 16$ states

Action of $\hat{H}_{mag}$ on Loop States

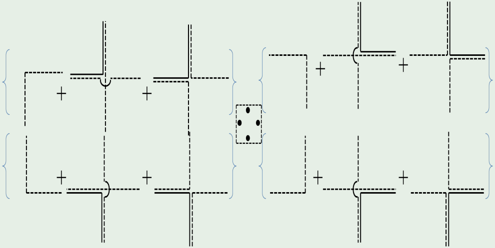
Type e: H_{+---}



The explicit action $\Rightarrow 4 \times 36$ states

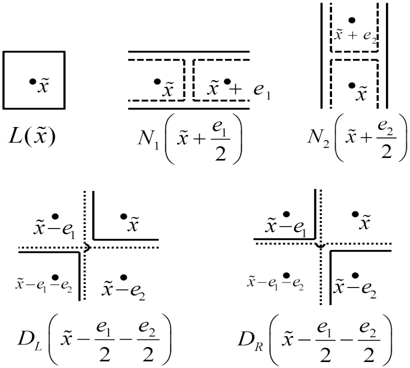
Action of $\hat{H}_{mag}$ on Loop States

Type f: H_{----}



The explicit action $\Rightarrow$ 81 states

Introducing Fusion Variables



Any Loop can be characterized by

$$|L, N_1, N_2, D_L, D_R\rangle \equiv \prod_x |L, N_1, N_2, D_L, D_R\rangle_x$$

Fusion Variables and Linking Numbers

$$
\begin{aligned}
l_{12}(x) &= L(\tilde{x}) - N_2(\tilde{x} - \frac{e_1}{2}) - N_1(\tilde{x} - \frac{e_2}{2}) + D_L(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \\
l_{1\bar{1}}(x) &= N_2(\tilde{x} - \frac{e_1}{2}) + N_2(\tilde{x} - \frac{e_1}{2} - e_2) - D_L(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) - D_R(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \\
l_{14}(x) &= L(\tilde{x} - e_2) - N_2(\tilde{x} - \frac{e_1}{2} - e_2) - N_1(\tilde{x} - \frac{e_2}{2}) + D_R(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \\
l_{23}(x) &= L(\tilde{x} - e_1) - N_2(\tilde{x} - \frac{e_1}{2}) - N_1(\tilde{x} - e_1 - \frac{e_2}{2}) + D_R(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \\
l_{24}(x) &= N_1(\tilde{x} - \frac{e_2}{2}) + N_1(\tilde{x} - e_1 - \frac{e_2}{2}) - D_L(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) - D_R(\tilde{x} - \frac{e_1}{2} - \frac{e_2}{2}) \\
l_{34}(x) &= L(\tilde{x} - e_1 - e_2) - N_2(\tilde{x} - \frac{e_1}{2} - e_2) - N_1(\tilde{x} - e_1 - \frac{e_2}{2}) + D_L(\tilde{x} - \frac{e_1}{2} - - \frac{e_2}{2})
\end{aligned}
$$

The Shift Operators corresponding to Fusion Variables

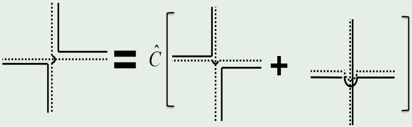
$$
\begin{aligned}
\hat{L}(\tilde{x})\Pi^{\pm}_{L}(\tilde{x})|L,N_1,N_2,D_L,D_R\rangle &= (L(\tilde{x})\pm 1)|L,N_1,N_2,D_L,D_R\rangle \\
\hat{N}_1(\tilde{x}-\frac{e_2}{2})\Pi^{\pm}_{N_1}(\tilde{x}-\frac{e_2}{2})|L,N_1,N_2,D_L,D_R\rangle &= (N_1(\tilde{x}-\frac{e_2}{2})\pm 1)|L,N_1,N_2,D_L,D_R\rangle \\
\hat{N}_2(\tilde{x}-\frac{e_1}{2})\Pi^{\pm}_{N_2}(\tilde{x}-\frac{e_1}{2})|L,N_1,N_2,D_L,D_R\rangle &= (N_2(\tilde{x}-\frac{e_1}{2})\pm 1)|L,N_1,N_2,D_L,D_R\rangle \\
\hat{D}_L(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\Pi^{\pm}_{D_L}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})|L,N_1,N_2,D_L,D_R\rangle &= (D_L(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\pm 1)|L,N_1,N_2,D_L,D_R\rangle \\
\hat{D}_R(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\Pi^{\pm}_{D_R}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})|L,N_1,N_2,D_L,D_R\rangle &= (D_R(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\pm 1)|L,N_1,N_2,D_L,D_R\rangle
\end{aligned}
$$

Fusion Variables and Constraints

U(1) Gauss Law Constraint

Already solved by definition

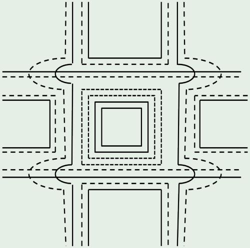
Mandelstam Constraint



$$(l_{12}+1)(l_{\bar{1}\bar{2}}+1)\Pi^{+}_{D_L} = (l_{1\bar{2}}+1)(l_{2\bar{1}}+1)\Pi^{+}_{D_R} + (l_{1\bar{1}}+1)(l_{2\bar{2}}+1)$$

Fusion Variables and Constraints

5 Variables-1Mandelstam Constranit$\Rightarrow$ Another additional Constraint



$$\Pi^-_{D_R}(\tilde{x}-\frac{e_1}{2}-\frac{e_2}{2})\Pi^-_{D_R}(\tilde{x}+\frac{e_1}{2}+\frac{e_2}{2})\Pi^-_{D_L}(\tilde{x}-\frac{e_1}{2}+\frac{e_2}{2})\Pi^-_{D_L}(\tilde{x}+\frac{e_1}{2}-\frac{e_2}{2})$$
$$\Pi^+_{N_1}(\tilde{x}-\frac{e_2}{2})\Pi^+_{N_1}(\tilde{x}+\frac{e_2}{2})\Pi^+_{N_2}(\tilde{x}-\frac{e_1}{2})\Pi^+_{N_2}(\tilde{x}+\frac{e_1}{2})\left(\Pi^+_L(\tilde{x})\right)^2=1$$

Redefinitions:

$$M(x) = \frac{D_L(x) - D_R(x)}{2} \ \& \ \tilde{M}(x) = \frac{D_L(x) + D_R(x)}{2}$$

Ansatz at $g \to 0$ limit

- **All the fusion quantum numbers will attend certain average value all over the lattice, and the real values are small fluctuations about these averages.**

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Ansatz at $g \to 0$ limit

- **All the fusion quantum numbers will attend certain average value all over the lattice, and the real values are small fluctuations about these averages.**
- $$\langle L \rangle \to \infty \ \ \& \ \ \langle M \rangle \to \infty$$
  whereas, the other averages $\langle N_1 \rangle, \langle N_2 \rangle, \langle \tilde{M} \rangle$ remain finite.

Unperturbed Hamiltonian for $g \to 0$ limit

- **Quadratic Hamiltonian: Constructed in terms of fusion variables and associated shift operators.**

Solve the Quadratic Hamiltonian Analytically

- **Solve Mandelstam Constraint and Fusion constraint within the ansatz, left with three variables.**
- **Three degrees of freedom $\Rightarrow$ Three coupled Harmonic Oscillators.**
- **Discrete Spectrum $\Rightarrow$ Mass Gap.**

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Unperturbed Hamiltonian for $g \to 0$ limit

- **Quadratic Hamiltonian: Constructed in terms of fusion variables and associated shift operators.**
- **Independent of g.**
- **Fluctuations from this quadratic Hamiltonian is $\mathcal{O}(g^k)$, for $k = 1, 2, ...$**

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Technical Detail

$$
\begin{array}{ll}
\Rightarrow & l_{12}(x) \to \langle L\rangle + \langle M\rangle\ ,\ l_{1\bar{1}}(x) \to 0\ ,\ l_{1\bar{2}}(x) \to \langle L\rangle - \langle M\rangle \\
& l_{\bar{1}\bar{2}}(x) \to \langle L\rangle + \langle M\rangle\ ,\ l_{2\bar{2}}(x) \to 0\ ,\ l_{2\bar{1}}(x) \to \langle L\rangle - \langle M\rangle \\
\Rightarrow & n_1(x) = 2\langle L\rangle\ \ \&\ \ n_2(x) = 2\langle L\rangle\ \ \forall x
\end{array}
$$

Mandelstam Constraint:

$$
\begin{array}{rcl}
(l_{12}+1)(l_{\bar{1}\bar{2}}+1)\Pi_L^+ & = & (l_{1\bar{2}}+1)(l_{2\bar{1}}+1)\Pi_R^+ + (l_{1\bar{1}}+1)(l_{2\bar{2}}+1) \\
(\langle L\rangle + \langle M\rangle)^2\Pi_L^+ & = & (\langle L\rangle - \langle M\rangle)^2\Pi_L^+ + 0 \\
\Rightarrow (\langle L\rangle + \langle M\rangle)^2\Pi_L^+\Pi_R^- = (\langle L\rangle - \langle M\rangle)^2\Pi_L^+\Pi_R^- & \& & (\langle L\rangle + \langle M\rangle)^2\Pi_L^+\Pi_L^- = (\langle L\rangle - \langle M\rangle)^2\Pi_R^+\Pi_R^- \\
\Rightarrow \Pi_M^+ = \dfrac{(\langle L\rangle - \langle M\rangle)^2}{(\langle L\rangle + \langle M\rangle)^2} & \& & \Pi_M^- = \dfrac{(\langle L\rangle + \langle M\rangle)^2}{(\langle L\rangle - \langle M\rangle)^2}
\end{array}
$$

Technical Detail

Define:

$$\{L_1(x), L_2(x), L_3(x), L_4(x), L_5(x)\} \equiv$$
$$\{L(x+\frac{e_1}{2}+\frac{e_2}{2}), N_1(x+\frac{e_1}{2}), N_2(x+\frac{e_2}{2}), \tilde{M}(x), M(x)\} \Rightarrow \langle L_{1/5}\rangle \to \infty$$

Basic Variables

$$\prod_{i=1}^{5} \exp\left(igQ_i(x)L_i(x)\right)$$

$$L_i(x) \equiv -\frac{i}{g}\frac{\partial}{\partial Q_i(x)}$$
$$\Pi_i^{\pm}(x) \equiv \exp\left(\pm\, igQ_i(x)\right)$$

Technical Detail

Solving the Constraints:

- **Mandelstam Constraint:**
$\Pi_5^+(x) \equiv \exp\left(igQ_5(x)\right) \approx \left(\frac{1/2 - \bar{M})}{(1/2 + \bar{M})}\right)^2$ & $\Pi_5^-(x) \equiv \exp\left(-igQ_5(x)\right) \approx \left(\frac{1/2 + \bar{M})}{(1/2 - \bar{M})}\right)^2$ **implies that, if** $\Pi_5^+(x) < 1 \Rightarrow \Pi_5^-(x) > 1$**, whereas both of them are phase factors. Hence, it must be :**

$$\Pi_5^+(x) = 1 = \Pi_5^-(x) \quad \Rightarrow \quad \bar{M} = 0.$$

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$$q_1 + q_2 + q_3 + 2 * q_4 = 0 \;\; \Rightarrow q_4 = -\frac{1}{2}(q_1 + q_2 + q_3)$$

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- **Left with 3 degrees of freedom** Q_1, Q_2, Q_3**.**
- **Assumption:** $\langle Q_i \rangle = q_i$, **for** $i = 1, 2, 3$ **for the ground state of unperturbed Hamiltonian.**

Technical Detail: The Hamiltonian at $g \to 0$ limit

Kinetic Part: from H_{el}

$$\sim g^2 \left(\frac{-i}{g}\frac{\partial}{\partial q_i}\right)\left(\frac{-i}{g}\frac{\partial}{\partial q_j}\right) \equiv -\frac{\partial^2}{\partial q_i \partial q_j}$$

with $i, j = 1, 2, 3$.

Potential Part: from H_{mag}

$$\begin{aligned}
\frac{1}{g^2} V(q_1, q_2, q_3) \quad &= \quad \frac{1}{g^2}\left[V\Big|_{\bar{q}_1, \bar{q}_2, \bar{q}_3} + \sum_{i=1}^{3} g(q_i - \bar{q}_i)\frac{\partial V}{\partial q_i}\Big|_{\bar{q}_i} \right. \\
&\quad \left. + \sum_{i,j=1}^{3} g^2 (q_i - \bar{q}_i)(q_j - \bar{q}_j)\frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{\bar{q}_i, \bar{q}_j} + \mathcal{O}(g^3) + .. \right] \\
&\approx \quad \frac{1}{g^2}\sum_{i,j=1}^{3} g^2 (q_i - \bar{q}_i)(q_j - \bar{q}_j)\frac{\partial^2 V}{\partial q_i \partial q_j}\Big|_{\bar{q}_i, \bar{q}_j} \quad \text{for } g \to 0
\end{aligned}$$

Existence of Minima of $V(q_1, q_2, q_3)$ at $\bar{q}_1, \bar{q}_2, \bar{q}_3$ and expansion about that point $\Rightarrow$ Nonzero mass gap.

Explicit computation of potential using diagrametic technique

- **The minima do exists.**

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- **Fluctuations about the quadratic Hamiltonian is to be calculated perturbatively.**

Summary

- **Prepotential formulation: Local loop formulation.**

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- Prepotential formulation: Local loop formulation.
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- Prepotential formulation: Local loop formulation.
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- Introducing Fusion variables: Suitable for $g \to 0$ limit.
- Ansatz for the weak coupling limit: choosing relevant loop degrees of freedom.
- Quadratic Hamiltonian, solve analytically.
- Weak coupling unperturbed Hamiltonian is shown to have mass gap analytically.

Further Scopes

- **Refining and improving the ansatz.**

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- **Refining and improving the ansatz.**
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- **Calculation of perturbation expansion.**
- **suggestions and collaborations are welcomed.**

THANK YOU