Topologically restricted measurements in lattice sigma-models

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Outline

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(Simplest) example for topological sectors:

- 1d $O(2)$ model (quantum mechanical scalar particle on a circle).

  Functional integral formulation in Euclidean space:

  $$Z = \int D\varphi \, \exp(-S[\varphi]) \ , \ \ \langle \mathcal{O} \rangle = \frac{1}{Z} \int D\varphi \, \mathcal{O}[\varphi] \, \exp(-S[\varphi])$$

  $$\int D\varphi \, : \text{ sum over closed paths } \varphi(t) \in S^1 \ , \quad \varphi(0) = \varphi(T).$$

- Paths occur in disjoint subsets, characterised by the winding number = topological charge $Q = \frac{1}{2\pi} \int_0^T \dot{\varphi} \, dt \in \mathbb{Z}$.

  Continuously deformed paths remain in the same subset = topological sector.

Topological sectors in quantum field theory:

Space with periodic boundary conditions (torus).

- O(N) models in $d = N - 1$ dimensions
  spin $\vec{S}(x) \in \mathbb{R}^N$, $|\vec{S}(x)| = 1$.

  In this talk: 1d $O(2)$ and 2d $O(3)$.

- 2d CP(N-1) models
  $\vec{C}(x) \in \mathbb{C}^N, |\vec{C}(x)| = 1$.

- Gauge theories:

$$\begin{aligned} \mathrm{2d}\ U(1)\ : Q &= \frac{1}{2\pi} \int d^2x\, \epsilon_{\mu\nu} F_{\mu\nu} \in \mathbb{Z} \\ \mathrm{4d}\ SU(2) \text{ and } \mathrm{4d}\ SU(3)\ : Q &= \frac{1}{32\pi^2}\, \mathrm{Tr} \int d^4x\, F_{\mu\nu} \tilde{F}_{\mu\nu} \in \mathbb{Z} \end{aligned}$$

- Under continuous deformations the configurations remain in a fixed topological sector.

Lattice regularization:

- All the configurations can be continuously deformed into one another.
  A priori: No topological sectors.

- Divide lattice field configurations into sectors.
  In the continuum limit $\rightarrow$ Topological sectors.

  Definition of topological charge on the lattice somewhat arbitrary.
  For sigma-models: Geometric definition seems suitable.
  $\rightarrow$ Integer topological charges on periodic lattices.

- Finer lattice spacing:
  $\rightarrow$ Formulation becomes more continuum-like.
  $\rightarrow$ Changing a topological sector is getting harder.
  $\rightarrow$ Continuous deformations have to pass through a statistically suppressed domain (of large Euclidean action).

Monte Carlo simulation of QCD:

- With chiral quarks:
  JLQCD collaboration: Hybrid Monte Carlo trajectory permanently confined to Q = 0 (Fukaya et al., '07) .

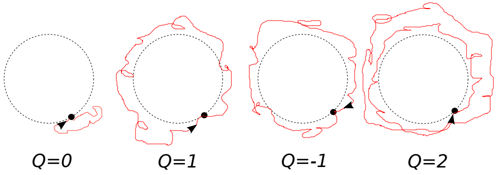
  Non-chiral quarks (e.g. Wilson fermions):
  Usually with a lattice spacing a in the range $0.05\ \mathrm{fm} \lesssim a \lesssim 0.15\ \mathrm{fm}$.
  Problem less severe so far, but will show up on even finer lattices.
- Local updates rarely change the topological sector, in particular Hybrid Monte Carlo algorithm for QCD.
- Monte Carlo history tends to be trapped for a very long time in one topological sector
  $\rightarrow$ Extremely long topological autocorrelation time.
- Here we study the 1d $O(2)$ and 2d $O(3)$ models as toy models.

Non-linear σ models 1d $O(2)$ and 2d $O(3)$:

- 1d $O(2)$ model:
  Angles $\varphi_x \in (-\pi, \pi]$ on sites x of a periodic lattice of size L.
  Standard action:
  $S_{Standard}[\varphi] = \beta \sum_{x=1}^{L} \left(1 - \cos\left(\varphi_{x+1} - \varphi_x\right)\right),$
  Manton action:
  $S_{Manton}[\varphi] = \frac{\beta}{2} \sum_{x=1}^{L} \left(\left(\varphi_{x+1} - \varphi_x\right) \bmod 2\pi\right)^2,$
  Constraint action:
  $$S_{Constraint}[\varphi] = \begin{cases} 0 & |\varphi_{x+1} - \varphi_x| < \delta, \ \forall x \\ \infty & \text{ortherwise} \end{cases}$$
  with constraint angle δ.
- 2d $O(3)$ model:
  Unit vectors $\vec{S}_x \in S^2$ on a periodic lattice of size $V = L \times L$.
  Standard action:
  $S_{Standard}[\vec{S}] = \beta \sum_{x,\mu} (1 - \vec{S}_x \cdot \vec{S}_{x+\hat{\mu}}),$
  Constraint action:
  If any of the relative angles of neighboring vectors is larger than δ,
  $S_{Constraint}[\vec{S}] = \infty$, otherwise $S_{Constraint}[\vec{S}] = 0$.

Topological charge in the 1d $O(2)$ model:

Quantum mechanical scalar particle on a circle with periodic Euclidean time.



Topological charge density q_x of the geometrical topological charge Q

$$q_x = \frac{\Delta\varphi_x}{2\pi}, \;\; \Delta\varphi_x = (\varphi_{x+1} - \varphi_x) \bmod 2\pi \in (-\pi, \pi], \quad Q = \sum_{x=1}^{L} q_x \in \mathbb{Z}.$$

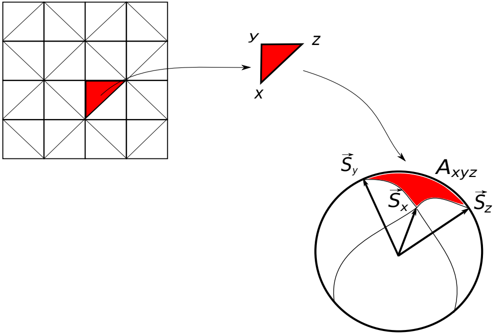
Periodic boundary conditions assure an integer topological charge.

Topological charge in the 2d $O(3)$ model:

Unit vectors $\vec{S}_x \in S^2$ on sites x of a triangulated $L \times L$ lattice.

Topological charge density: $q_{xyz} = A_{xyz}/4\pi$.

Geometrical topological charge: $Q = \sum_{t_{xyz}} q_{xyz} \in \mathbb{Z}$.



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Method to determine the topological susceptibility:

- How can the topological susceptibility χ_t be determined ?

$$\text{if } \langle Q \rangle = 0, \ \chi_t = \frac{\langle Q^2 \rangle}{V}.$$

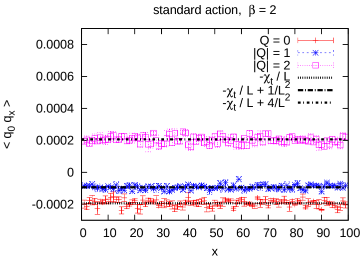
  Measurement of the top. charge density q in one top. sector is sufficient!
  (Aoki/Fukaya/Hashimoto/Onogi '07)

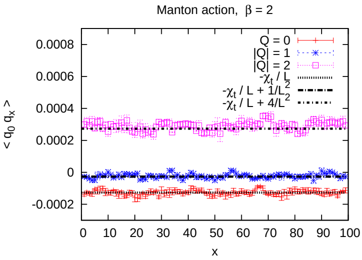
$$\lim_{|x| \gg 1} \langle q_0 q_{|x|} \rangle_{|Q|} \approx -\frac{\chi_t}{V} + \frac{Q^2}{V^2}.$$

- Assumptions:
  Q Gauss distributed
  large $\langle Q^2 \rangle = V\chi_t$
  small $\frac{|Q|}{\langle Q^2 \rangle} \to$ work at small $|Q|$
- All Monte Carlo results presented here obtained using the very efficient Wolff cluster algorithm (Wolff '89).

Correlation function of the topological charge density:

1d $O(2)$ model (at $L = 100, \beta = 2$):



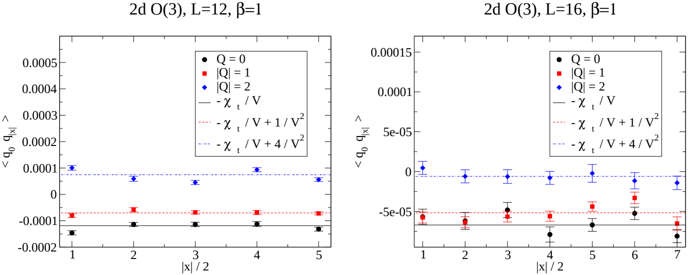


Standard action with theoretical value $\langle Q^2 \rangle = 1.94$.
Manton action with theoretical value $\langle Q^2 \rangle = 1.27$.

The numerical results are consistent with the theoretical values of χ_t.

Correlation function of the topological charge density:

2d $O(3)$ model (at $\beta = 1$):



Left: $V = 12 \times 12$, $\langle Q^2 \rangle = 2.46$.
Right: $V = 16 \times 16$, $\langle Q^2 \rangle = 4.39$.

The numerical results are consistent with the directly measured values of χ_t.
The correlation length is small (≈ 1.3) $\rightarrow$ increasing β and V $\Rightarrow$ *plateau*$\rightarrow$0.

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Topological summation of observables:

Calculation of an observable $\langle \mathcal{O} \rangle$ if only measurements $\langle \mathcal{O} \rangle_{|Q|}$ at fixed topological sectors $|Q|$ are available.

Approximation formula for pion mass in QCD:
(Brower/Chandrasekharan/Negele/Wiese '03)
Generalization:

$$\langle \mathcal{O} \rangle_{|Q|} \approx \langle \mathcal{O} \rangle + \frac{c}{2V\chi_t}\left(1 - \frac{Q^2}{V\chi_t}\right), \quad c = \text{constant.}$$

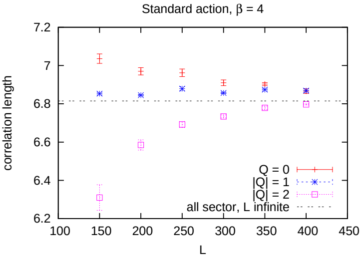
$1/V$ expansion; next order: (Dromard/Wagner '14)

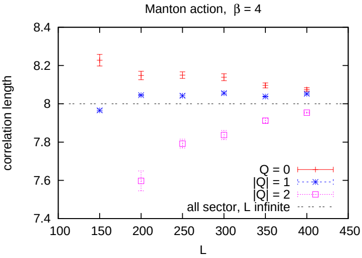
Assumptions as in previous method:
Q Gauss distributed
large $\langle Q^2 \rangle = V\chi_t$
small $\frac{|Q|}{\langle Q^2 \rangle}$ $\rightarrow$ work at small $|Q|$

Measure $\langle \mathcal{O} \rangle_{|Q|}$ for several values of $|Q|$ and V and perform a fit $\rightarrow \langle \mathcal{O} \rangle$, χ_t and c.

Topological summation of observables:

Correlation length in the 1d $O(2)$ model (at $\beta = 4$):

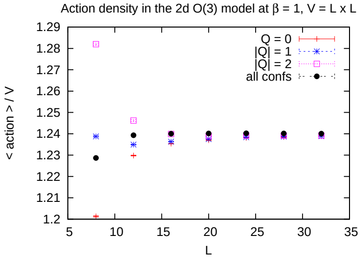




	Standard action			Manton action		
Fitting range for L	250 — 400	300 — 400	Theory	250 — 400	300 — 400	theory
ξ	6.77(5)	6.79(2)	6.815	7.95(5)	7.88(6)	8.000

Topological summation of observables:

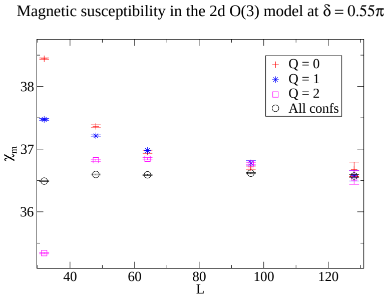
Action density in the 2d $O(3)$ model (at $\beta = 1$ using the standard action):



fitting range for L	$16-24$	$16-28$	$16-32$	directly measured in all sectors at $L=32$
$\langle \text{action} \rangle / V$	1.24038(12)	1.24027(8)	1.24015(5)	1.24008(5)
χ_t	0.0173(6)	0.0169(5)	0.0164(5)	0.01721(4)

Topological summation of observables:

Magnetic susceptibility in the 2d $O(3)$ model
(at $\delta = 0.55\pi$ using the constraint action):



fitting range for L	$48 - 64$	$48 - 96$	directly measured in all sectors at $L = 96$
χ_m	36.56(4)	36.58(3)	36.616(9)
χ_t	0.0026(2)	0.0026(2)	0.002794(2)

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Summary:

- For local update algorithms, Monte Carlo simulations can get confined to one topological sector.
- How can we extract information from measurements $\langle \mathcal{O} \rangle_{|Q|}$ in fixed topological sectors ?
- For very large volume $\to \langle \mathcal{O} \rangle_{|Q|} = \langle \mathcal{O} \rangle$ (the same for all Q).
  In general — for example for QCD simulations — not accessible.
- For the 1d O(2) and 2d O(3) models:
  Determination of χ_t from $\langle q_0 q_{|x|} \rangle_{|Q|}$ works very well if V is not too large.
- The topological summation works well for observables, less reliable for χ_t.
- We used the conditions: $\langle Q^2 \rangle \gtrsim 1.5$, $|Q| \lesssim 2$.
- Application to QCD conceivable, to be explored.
- Next talk by Arthur Dromard: 4d $SU(2)$ Yang-Mills theory.