Hadron masses from fixed topological simulations: parity partners and SU(2) Yang-Mills results

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Outline

1. Introduction
   - BCNW-equation for the mass
   - Improvement of BCNW-equation

2. Considering parity mixing
   - Parity mixing: $\theta \neq 0$ and fixed topology
   - Consequences on mass extractions

3. Application to SU(2) Yang-Mills theory
   - Test method and parameters
   - Results
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Mass relation

- Relation between mass in topological sector $Q$ and mass of QCD ($\theta = 0$) \(^1\).

\[
M_Q = M(0) + \frac{M^{(2)}(0)}{2\chi_t V} \left( 1 - \frac{Q^2}{\chi_t V} \right) + \mathcal{O} \left( \frac{1}{(\chi_t V)^2} \right)
\]

**Conditions:**

(C1) \(1/\chi_t V \ll 1\) and \(|Q|/\chi_t V \ll 1\): Taylor expansion and saddle point approximation.

(C2) \(\left| M^{(2)}_H(0) t \right|/\chi_t V \ll 1\): Taylor expansion.

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Extracting the mass

- Relation between mass in topological sector \( Q \) and mass of QCD \( (\theta = 0) \)

\[
M_Q = M(0) + \frac{M^{(2)}(0)}{2\chi_t V} \left( 1 - \frac{Q^2}{\chi_t V} \right) + \mathcal{O} \left( \frac{1}{(\chi_t V)^2} \right)
\]

Method to extract the mass from fixed topological simulation.

- Compute \( M_Q, V \) or equivalently \( C_Q, V \) for different physical volumes and topological sectors \( (C_Q, V \text{ two-point function at fixed topology and volume } V) \)
- Fit the BCNW-Equation to those results (3 parameters \( M(0), M^{(2)}(0) \) and \( \chi_t \))
- Extracting \( M(\theta = 0) \) and \( \chi_t \)
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Motivations for improvements up to $1/((\chi_t V)^3)$

**Motivation to improve by** $\mathcal{O}(1/(\chi_t V)^2)$ and $\mathcal{O}(1/(\chi_t V)^3)$

1. Complete $\mathcal{O}(1/(\chi_t V)^2)$ order in BCNW-equation.
2. Increase precision
3. Important when $\chi_t V$ is not to large e.g. $\chi_t V \gtrsim 1$.
4. Helpful to estimate the error of $\mathcal{O}(1/(\chi_t V)^2)$ expansion.

**Literature:** General discussion of n-point functions at fixed topology including also higher orders in $1/V$. ²

**Our contribution:** Expansion of two-point correlation function up to $\mathcal{O}(1/(\chi_t V)^3)$

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Improvement

Improved equation \(^3\)

\[ C_{Q, \nu}(t) = \alpha(0) \exp \left(-M_H(0)t - \frac{1}{\mathcal{E}_2 V} \frac{x_2}{2} - \frac{1}{(\mathcal{E}_2 V)^2} \left( \frac{x_4 - 2(\mathcal{E}_4/\mathcal{E}_2)x_2 - 2x_2^2}{8} - \frac{x_2}{2} Q^2 \right) \right. \]

\[ \left. - \frac{1}{(\mathcal{E}_2 V)^3} \left( \frac{16(\mathcal{E}_4/\mathcal{E}_2)^2 x_2 + x_6 - 3(\mathcal{E}_6/\mathcal{E}_2)x_2 - 8(\mathcal{E}_4/\mathcal{E}_2)x_4 - 12x_2x_4}{48} \right. \right. \]

\[ \left. + \frac{18(\mathcal{E}_4/\mathcal{E}_2)x_2^2 + 8x_2^3}{48} - \frac{x_4 - 3(\mathcal{E}_4/\mathcal{E}_2)x_2 - 2x_2^2}{4} Q^2 \right) \left( \frac{1}{(\mathcal{E}_2 V)^4}, \frac{1}{(\mathcal{E}_2 V)^4} Q^2, \frac{1}{(\mathcal{E}_2 V)^4} Q^4 \right). \]

Improvement cost

- Increasing the number of parameters (3 for BCNW-equations, 8 for \(1/(\chi_t V)^2\), 11 for \(1/(\chi_t V)^3\))

\(^3\)A.D., M. Wagner: arXiv:1404.0247
Problem for limited statistical accuracy

Possibility to benefit of the improvement while keeping the number of parameters small. ⇒ New parameters set to zero

\[ C_{Q,V}(t) = \frac{\alpha(0)}{\sqrt{1 + M_H^{(2)}(0)t/\chi t V}} \exp \left( -M_H(0)t - \frac{1}{\chi t V} \left( \frac{1}{1 + M_H^{(2)}(0)t/\chi t V} - 1 \right) \frac{1}{2} Q^2 \right) \]

- Evidence of the improvement in a toy-model \(^4\) (quantum mechanics on a circle with well potential)

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Parity mixing due to the $\theta$-term

$$S_E(\theta) = S_E - i \theta Q = S_E - i \theta \frac{1}{32\pi^2} \int d^4xF_{\mu\nu}\tilde{F}_{\mu\nu}$$

- The second term violates parity symmetry $P$
- Fixed topology superposition (Fourier transform) of theories with $\theta \neq 0 \Rightarrow$ not $P$ invariant

**Consequences**

- Consider two states, which are parity partners:
  - The heavier state has to be considered as an excitation, while the lighter is the ground state.
- A single correlator is generally not sufficient to determine the mass of an excited state precisely:
  $\Rightarrow$ use a correlation matrix.
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Parity mixing at fixed $Q$

- Consider $O_1$ and $O_2$ with opposite parity.

$$C_Q = \begin{pmatrix} \langle O_1^\dagger(t)O_1(0) \rangle_Q & \langle O_1^\dagger(t)O_2(0) \rangle_Q \\ \langle O_2^\dagger(t)O_1(0) \rangle_Q & \langle O_2^\dagger(t)O_2(0) \rangle_Q \end{pmatrix}$$

If we neglect terms of order $\mathcal{O}\left(\frac{1}{(\chi_t V)^2}\right)$

$$\langle O_1^\dagger(t)O_1(0) \rangle_Q \approx a_{11} e^{-M_{H_1}(0)t} \left(1 - \frac{M_{H_1}^{(2)}(0)t}{2\chi_t V}\right) + \frac{b_{22}}{\chi_t V} e^{-M_{H_2}(0)t}$$

$$\langle O_1^\dagger(t)O_2(0) \rangle_Q \approx \frac{iQa_{12}}{\chi_t V} e^{-M_{H_1}(0)t} + \frac{iQb_{12}}{\chi_t V} e^{-M_{H_2}(0)t}$$

$$\langle O_2^\dagger(t)O_1(0) \rangle_Q \approx \frac{iQa_{21}}{\chi_t V} e^{-M_{H_1}(0)t} + \frac{iQb_{21}}{\chi_t V} e^{-M_{H_2}(0)t}$$

$$\langle O_2^\dagger(t)O_2(0) \rangle_Q \approx a_{22} e^{-M_{H_1}(0)t} + b_{22} e^{-M_{H_2}(0)t} \left(1 - \frac{M_{H_2}^{(2)}(0)t}{2\chi_t V}\right)$$
Masses at fixed $Q$

Now let us assume that $M_{H_1}(\theta = 0) < M_{H_2}(\theta = 0)$:

The previous results yield:

- For $H_1$:
  \[
  \langle O_1^\dagger(t)O_1(0) \rangle_Q = a_{11}e^{(-M_{H_1}(0)t)} \left( 1 - \frac{M_{H_1}^{(2)}(0)t}{2\chi_t V} \right) + \mathcal{O} \left( \frac{e^{-M_{H_2}t}}{\chi_t V}, \frac{1}{(\chi_t V)^2} \right)
  \]
  \( \Rightarrow \) For large $t$, we can use what we have done in previous works.

- For $H_2$:
  \[
  \langle O_2^\dagger(t)O_2(0) \rangle_Q = \frac{a_{22}}{\chi_t V} e^{(-M_{H_1}(0)t)} + \mathcal{O} \left( e^{-M_{H_2}t}, \frac{1}{(\chi_t V)^2} \right)
  \]
  \( \Rightarrow \) Extremely difficult to extract $M_2$, needs a lot of statistics.
  \( \Rightarrow \) Use the correlation matrix $\rightarrow$ fit all four expansions
  \[
  \langle O_i^\dagger(t)O_j(0) \rangle \text{ at once.}
  \]
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Parameters and method

Parameters:

- Action: \( S_E = \frac{1}{4} \int d^4 x F_{\mu \nu} F_{\mu \nu} \), use standard plaquette action
- Observable: static potential \( \psi_{Q\bar{Q}}(R) \) for \( R = 1 \) to 6
- \( \beta = 2.5 \)
- Volumes: \( a^4 V = (aL)^4 \) with \( L \in \{14, 15, 16, 18\} \)
- Number of configurations: 4000 per volume

Method to test the mass extraction from fixed topology:

- Compute \( C_{Q,V}(t) \) for different \( Q \) and \( V \)
- Fit the BCNW-equation or improvement
- Compare results to unfixed topology simulation

Systematic comparison:

1. Using different number of volumes and different volumes
2. Using weaker criterion \( \frac{|Q|}{\chi t V} < 1 \) or stronger one \( \frac{|Q|}{\chi t V} < 0.5 \) to admit topological sectors and volumes in the fit.
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Fitting

\( a^Y q\bar{q} \) for different top.sectors as a function of \( 1/V \) for \( R/a=6 \)
String Tension

\( a\sqrt{q\bar{q}} \) as a function of \( R/a \)

- fixed topology
- unfixed topology \( V/a^4 = 18^4 \)
Topological susceptibility

Examples:
For \( a^4 V = 15^4 \) and \( 16^4 \)

<table>
<thead>
<tr>
<th>eq.</th>
<th>( \gamma_{Q\bar{Q}} ) (a)</th>
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<td>7.0(3.5)</td>
<td>fixed topology, a separate fit for each separation ( r/a )</td>
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<td>10.4(5.2)</td>
<td>7.2(1.5)</td>
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- Large statistical errors
Summary of results for SU(2)

- **Clear Discrepancy** between topological sectors observed → motivates our work (need to have a method to extract masses from fixed topological sectors)

- **Rather precise to determine masses**: \( \left( \frac{|Q|}{\chi_t V} < 1 \right) \)
  
  - Improvements when using more volumes (still works reasonably well with two volumes)
  - Reducing statistical errors using larger number of points for the fit (more topological sectors, more volumes)

- **Topological susceptibility**: large statistical error!
  
  - Improved by increasing the number of topological sectors or number of volumes
  - Not possible to determine \( \chi_t \) with high precision.

- Improved equation is slightly better or as good as BCNW-equation
Outlook

1. Apply to QCD
2. Apply it for the heaviest parity partner

Thank you for your attention!