Tree level improvement of the gradient flow

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in collaboration with

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Our goals

- Systematically understand cut-off effects of gradient flow
- Reduce them by tree level improvement
- Come up with optimal simulation parameters

Gradient flow in a (really small) nutshell:

$$\frac{dA_\mu(t)}{dt} = -\frac{\delta S}{\delta A_\mu}\,, \qquad \langle t^2 E(t)\rangle\,, \qquad E = -\frac{1}{2}\mathrm{Tr}\,F_{\mu\nu}F_{\mu\nu}$$

Why care?

- Tuesday 14:55 – Nathan Brown – Gradient Flow Analysis on MILC HISQ Ensembles
- Tuesday 14:35 – Andrea Shindler – Beyond the Standard Model Matrix Elements with the gradient flow
- Tuesday 14:35 – Liam Keegan – TEK twisted gradient flow running coupling
- Wednesday 09:00 – Anna Hasenfratz – Improved gradient flow for step scaling function and scale setting
- Wednesday 09:20 – Jarno Rantaharju – The gradient flow running coupling in SU2 with 8 flavors

- Wednesday 11:10 – Marco Ce – Testing the WittenVeneziano mechanism with the YangMills gradient flow on the lattice
- Thursday 14:55 – Agostino Patella – Energy-momentum tensor on the lattice and Wilson flow
- Thursday 15:15 – Masanori Okawa – String tension from smearing and Wilson flow methods
- Thursday 15:55 – Stefan Sint – How to reduce $O(a^2)$ effects in gradient flow observables
- Friday 10:15 – Alberto Ramos – Wilson flow and renormalization
- Saturday 09:30 – Kitazawa Masakiyo – Measurement of thermodynamics using Gradient Flow

Why care?

Applications include

- Running coupling
- Topology
- Thermodynamics
- Energy momentum tensor, trace anomaly
- Scale setting
- And more ...

In all these projects $\langle t^2 E(t) \rangle$ pops up

What we are after

Continuum:

$$\langle t^2 E(t) \rangle = g^2 \frac{3(N^2-1)}{128\pi^2} \left(1 + O(g^2)\right)$$

Lattice:

$$\langle t^2 E(t) \rangle = g^2 \frac{3(N^2-1)}{128\pi^2} \left(C(a^2/t) + O(g^2)\right)$$

$$C(a^2/t) = 1 + \sum_{m=1}^{\infty} C_{2m} \frac{a^{2m}}{t^m}$$

What we are after

Lattice cut-off coefficients C_{2m} depend on the discretization of 3 ingredients:

- Flow, S_f
- Action, S_g
- Observable, $E = S_e$

In the continuum, all 3 are $F_{\mu\nu}F_{\mu\nu}$. We consider large class of discretizations, 1×1 plaquette, 2×1 plaquette, clover. The 3 ingredients can all be different.

Note

We only consider 2×1 plaquette improvement terms (and clover for observable), c_1 and only consider one observable $t^2 E(t)$.

For full tree-level improvement, improvement of all observables, one needs larger set of terms in action (chairs, etc), c_2, c_3

- In practice, people do use only 2×1 plaquette terms (or clover)
- Calculation is tree level, $g^2 a^2$ will be there anyway, no need to overkill

If interested in general case: $\rightarrow$ Stefan Sint, right after this talk, Alberto Ramos Fri 10:15

Lattice perturbation theory, $c_1 = c$

Action in momentum space

$$S_{\mu\nu}(c) = \delta_{\mu\nu}\left(\hat{p}^2 - a^2 c \sum_\rho \hat{p}_\rho^4 - a^2 c \hat{p}_\mu^2 \hat{p}^2\right) - \hat{p}_\mu \hat{p}_\nu \left(1 - a^2 c (\hat{p}_\mu^2 + \hat{p}_\nu^2)\right)$$

Clover in momentum space

$$K_{\mu\nu} = \left(\delta_{\mu\nu}\tilde{p}^2 - \tilde{p}_\mu \tilde{p}_\nu\right) \cos\left(\frac{ap_\mu}{2}\right) \cos\left(\frac{ap_\nu}{2}\right)$$

where $\hat{p}_\mu = \frac{2}{a}\sin\left(\frac{ap_\mu}{2}\right), \tilde{p}_\mu = \frac{1}{a}\sin(ap_\mu)$

$c = 0$: Wilson plaquette, $c = -1/12$: tree-level improved Symanzik

The 3 ingredients (Flow, Action, Observable) can have different improvement coefficients: c_f, c_g, c_e.

Or if the observable is clover, only 2 parameters: c_f, c_g.

$$\mathcal{S}^f = S(c_f) \qquad \mathcal{S}^g = S(c_g) \qquad \mathcal{S}^e = S(c_e) \text{ or } K$$

We would like to get $C_{2m}(c_f, c_g, c_e)$ or $C_{2m}(c_f, c_g)$ or the full $C(a^2/t)$ similarly as a function of 2 or 3 parameters

Leading order formulae

Gauge fixing term: $\mathcal{G} = \frac{1}{\alpha}\widehat{p}_\mu \widehat{p}_\nu$

$$\frac{dA_\mu(t,p)}{dt} = -\left(\mathcal{S}^f + \mathcal{G}\right) A_\mu(t,p)$$

$$A_\mu(p,t) = \left[e^{-t\left(\mathcal{S}^f + \mathcal{G}\right)}\right]_{\mu\nu} A_\nu(p,0)$$

$$\langle A_\mu(p,0) A_\nu(p,0)\rangle = \left[(\mathcal{S}^g + \mathcal{G})^{-1}\right]_{\mu\nu}$$

$$\langle t^2 E(t)\rangle = g_0^2 t^2 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \mathcal{S}^e_{\mu\nu}(p) \langle A_\mu(p,t) A_\nu(p,t)\rangle$$

Leading order formulae

$$\langle t^2 E(t)\rangle = g_0^2 t^2 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\,\left(e^{-t(\mathcal{S}^f+\mathcal{G})} (\mathcal{S}^g + \mathcal{G})^{-1} e^{-t(\mathcal{S}^f+\mathcal{G})} \mathcal{S}^e \right)$$

Note: these 4×4 matrices don't necessarily commute

Can be evaluated numerically in finite/infinite volume or can be expanded in a^2

Note: with periodic gauge fields zero mode needs to be treated separately (more later)

Expansion in a^2

$$C_2 = 2c_f + \frac{2}{3}c_g - \frac{2}{3}c_e + \frac{1}{8}, \quad \text{with clover} : C_2 = 2c_f + \frac{2}{3}c_g - \frac{1}{24}$$

Similar polynomial expressions for C_4, C_6, C_8.

Notice that we have 3 or 2 free parameters, we can fix them by imposing 3 or 2 conditions

Example 1: $C_2 = C_4 = C_6 = 0$

$$c_f = -0.013993 \qquad c_g = 0.052556 \qquad c_e = 0.198078$$

$O(a^6)$ improvement at tree-level

If you already have the configurations, c_g fixed.

Can set $C_2 = C_4 = 0$

Example 2: $c_g = 0$ fixed $\rightarrow$ $c_f = 0, \;\; c_e = 3/16$

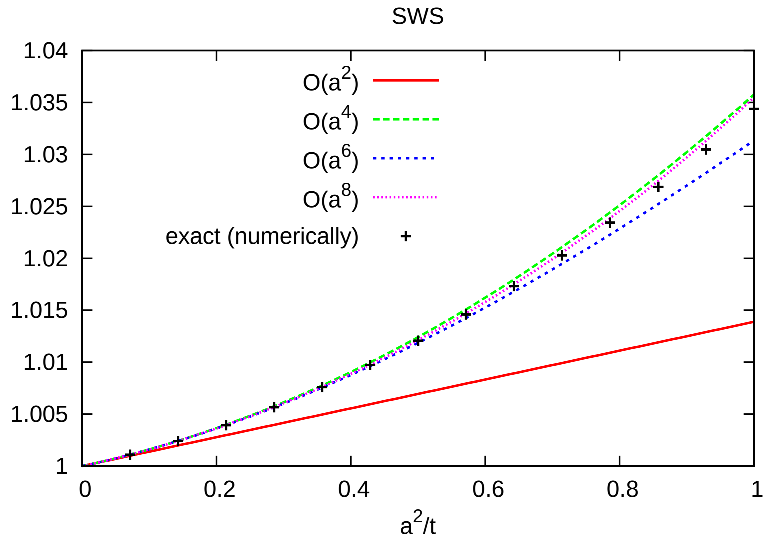
Example 3: $c_g = -1/12$ fixed $\rightarrow c_f = 0.0388441, \;\; c_e = 0.2206988$

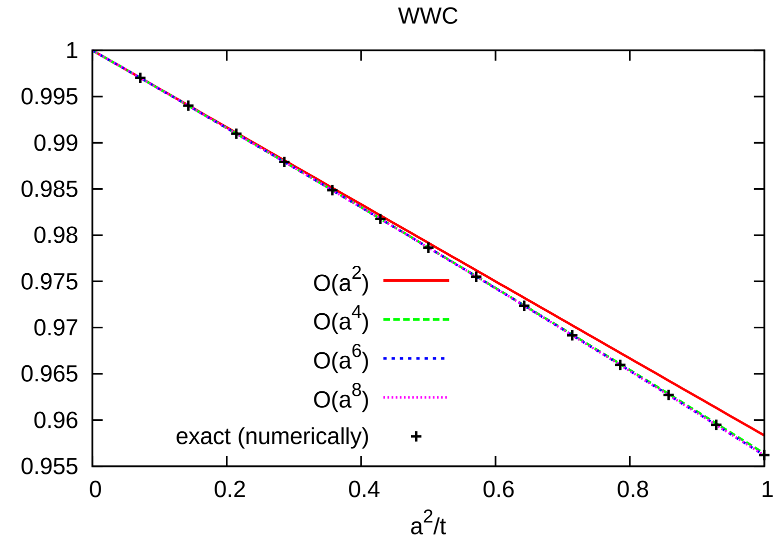
$O(a^4)$ improvement at tree-level

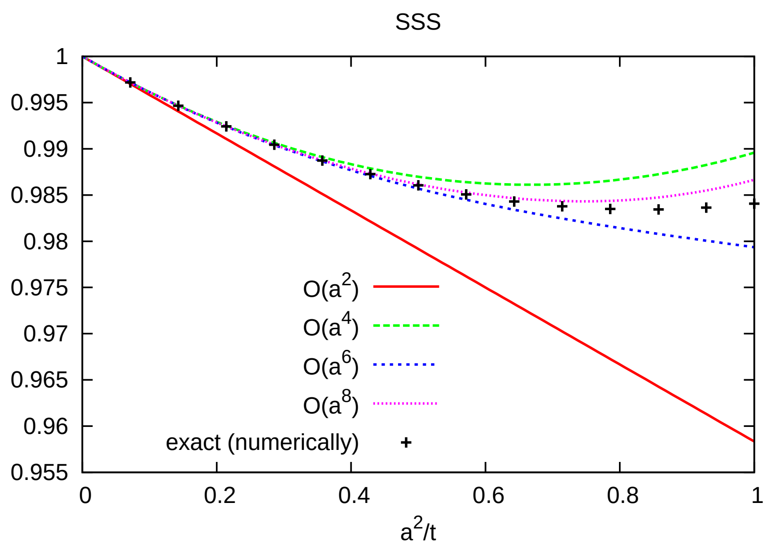
Example 4: with clover, c_g fixed $\rightarrow c_f = \frac{1}{48} - \frac{1}{3} c_g$.

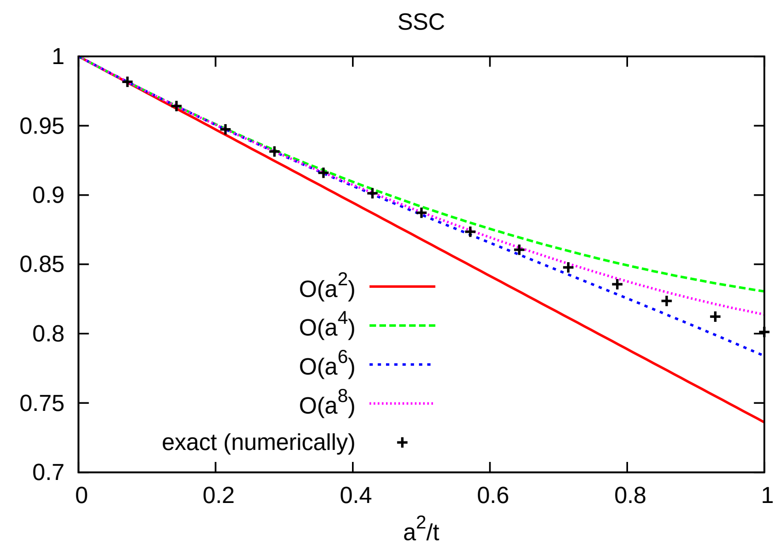
$O(a^2)$ improvement at tree-level

Size of tree-level cut-off effects, $C_{2,4,6,8}$, is thus obtained for all frequently used cases, Wilson-plaquette, tree-level improved Symanzik, clover, and all their combinations.









All the above was about finding optimal simulation/measurement parameters

Another application: improvement of already gathered data with arbitrary simulation/measurement parameters

Improvement of data

If simulation/measurement is already done (with non-optimal parameters):

$$\langle t^2 E(t) \rangle_{imp} = \frac{\langle t^2 E(t) \rangle_{lattice}}{C(a^2/t)} = g^2 \frac{3(N^2-1)}{128\pi^2} \Big(1 + O(g^2)\Big)$$

In this case: evaluate $C(a^2/t)$ in finite L/a volume of the simulation (full a^2-dependence, no expansion)

Continuum limit by construction the same as before

Improvement of data

$$\langle t^2 E(t)\rangle = g_0^2 t^2 \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\, \left(e^{-t(\mathcal{S}^f+\mathcal{G})} (\mathcal{S}^g + \mathcal{G})^{-1} e^{-t(\mathcal{S}^f+\mathcal{G})} \mathcal{S}^e \right)$$

In finite volume: $dp \to \sum_n$ finite lattice sum $p_\mu = 2\pi n_\mu / L$

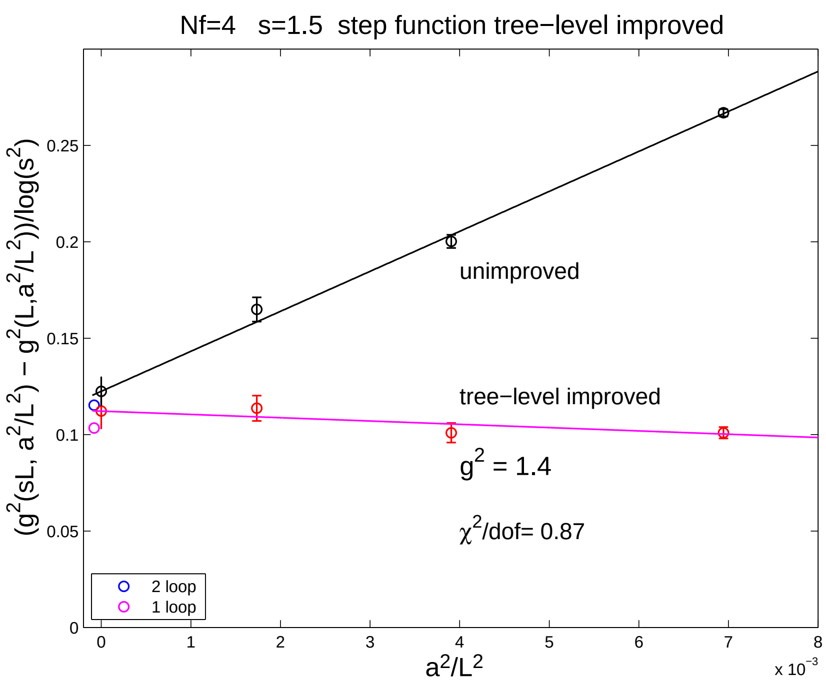
Zero mode (if periodic):

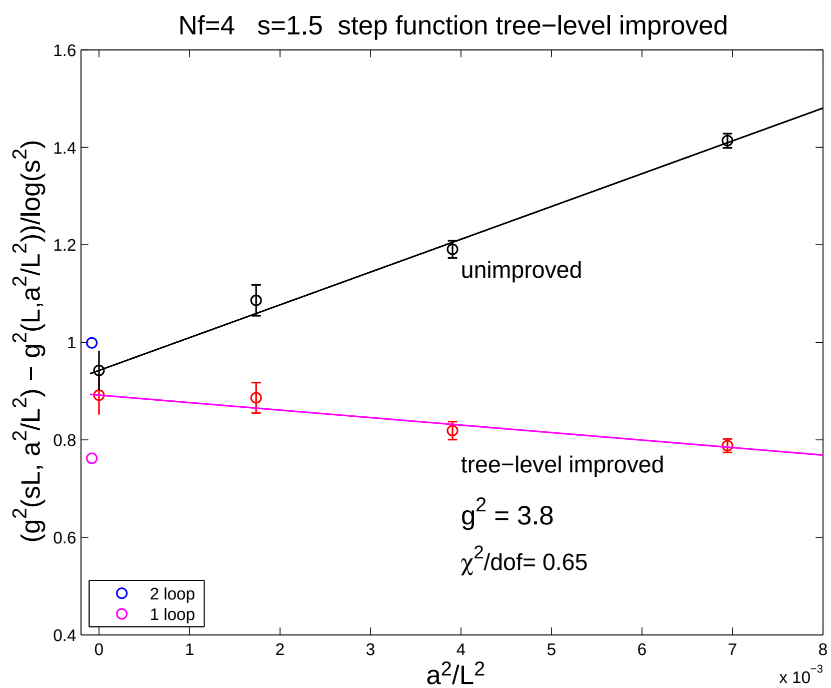
non-Gaussian, can be calculated exactly, lattice = continuum, 1208.1051

Numerical test

We introduced a flow-based finite volume running coupling scheme in 1208.1051

$SU(3)$ with $N_f = 4$ fundamental fermions, $s = 3/2$ step scaling, β-function





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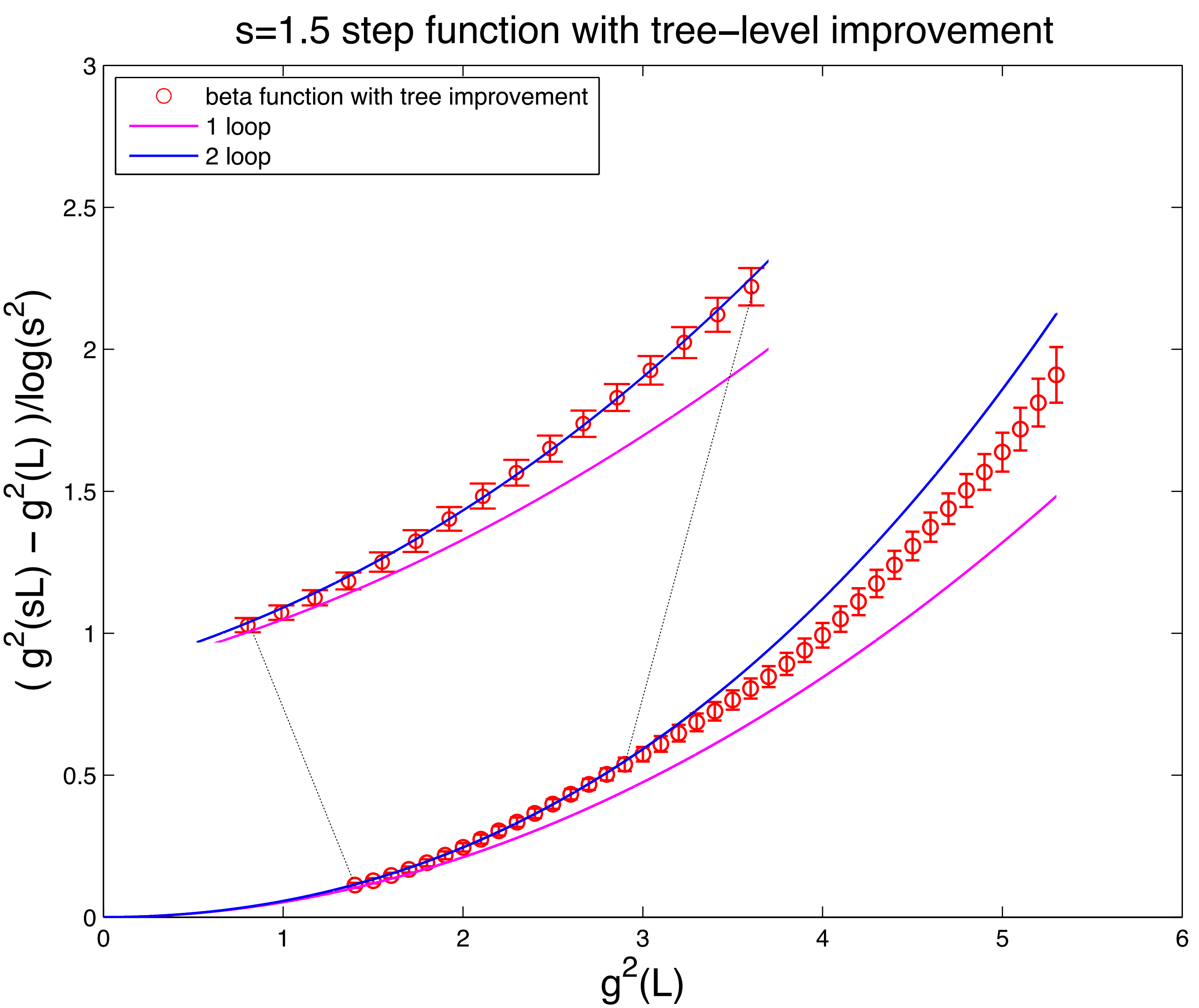

Summary

- Tree level improvement of $\langle t^2 E(t) \rangle$ for a large class of discretizations (frequently used ones among them)

- Application 1: find optimal parameters for simulation/measurement

- Application 2: improve already obtained data with fixed (non-optimal) simulation/measurement

Our continuum extrapolations will be much better in both cases!

Thank you for your attention!