Renormalization constants for $N_\mathrm{f} = 2+1+1$ twisted mass QCD

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Work in progress with B. Blossier, M. Brinet, P. Guichon, V. Morenas, O. Péne, P. Rodríguez-Quintero.

Bare vs Renormalized

- Lattice formalism is bare QFT
- One computes bare matrix elements of operators at fixed cutoff
- Must renormalize to obtain continuum Physics
- $O_R = Z_O O_b$
- Renormalization can be done perturbatively or non-perturbatively

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Lattice PT

- Lattice PT-notorious for its bad convergence
- MILC collaboration found that m_s was raised by 14% once its renormalization constant (RC) known in 1-loop PT was calculated @ 2-loops.
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- RI-MOM scheme Martinelli et al (1995)
- Work on the calculation of the RCs by many groups many of them belonging to the ETMC

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- focus on local fermion bilinears $O_\Gamma = \bar{\psi}(x)\Gamma\psi(x)$

  Vladikas Les Houches lectures

- Γ can be any Dirac structure and can even potentially contain covariant derivatives (twist-2 operators)
- inserting O_Γ in the fermion 2-pt function
- $G_O = \langle u(x_1) O_\Gamma \bar{d}(x_2)\rangle$
- the amputated Green's function
- $\Lambda_O(p_1, p_2) = S_u^{-1}(P_1) G_O(p_1, p_2) S_d^{-1}(p_2)$
- $\Gamma_O(p) = \frac{1}{12}\mathrm{tr}\left[P_O \Lambda_O(p, p)\right]$
- $\Gamma_O(p)_R = \lim_{a\to 0} Z_q^{-1} Z_O \Gamma_O(p)$
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RI-MOM

- impose that the amputated Green's function in the chiral limit @ a large Euclidean scale $p^2 = \mu^2$ is equal to its tree level value
- $\Gamma_O(p)_R(\mu, g_R, m_R = 0) = \lim_{a \to 0} [Z_q^{-1}(a\mu, g_0) Z_O(a\mu, g_0) \Gamma_O(p, g_0, m)]_{p=\mu^2, m \to 0} = 1$

Window of applicability of RI-MOM

- $\Lambda_{QCD} \ll \mu \ll \frac{\pi}{a}$
- first inequality ensures the possibility of matching with some perturbative scheme $\overline{\mathrm{MS}}$ and protects from Goldstone pole contaminations
- second inequality ensures small cutoff effects

Conversion to $\overline{\text{MS}}$

- make connection with phenomenological calculations and experiments
- the decay width for the dominant channel $H \to \bar{b}b \propto m_b^2$
- one needs the RC for m_b - in the TM framework it is given $1/Z_P$
- need to convert to $\overline{\text{MS}}$ with factors $Z_q^{\overline{\text{MS}}} = C_q^{-1}\, Z_q^{RI'-MOM}$ and $Z_{\mathcal{O}}^{\overline{\text{MS}}} = C_{\mathcal{O}}^{-1}\, Z_{\mathcal{O}}^{RI'-MOM}$
- experiments usually provide results in $\overline{\text{MS}}$ at a reference scale $\mu = 2$ GeV
- evolve $\overline{\text{MS}}$ RCs $Z_{\mathcal{O}}^{\overline{\text{MS}}}$ using the scale dependence predicted by the RG equation van Ritbergen et al (1997), Vermaseren et al (1997), Chetyrkin (1997), Göckeler et al (1998)

$$R_{\mathcal{O}(\mu,\mu_0)} := \frac{Z_{\mathcal{O}(\mu)}}{Z_{\mathcal{O}(\mu_0)}} = \exp\left\{-\int_{\bar{g}(\mu_0^2)}^{\bar{g}(\mu^2)} dg \frac{\gamma(g)}{\beta(g)}\right\}$$

β is the usual QCD-beta function, γ the anomalous dimension of operator $\mathcal{O}$ and $\bar{g}(\mu^2)$ the running coupling

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Simulation setup for $N_\mathrm{f} = 2 + 1 + 1$

- $S = S^{YM}_{Iwa} + S^f_l + S^f_h$
-

$$
\begin{aligned}
S &= S^{YM}_{Iwa} + a^4 \sum_{x,f} \bar{\chi}_f \left(\gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_{0l} + i\mu_l \gamma_5 \tau_3\right) \chi_f \\
&+ a^4 \sum_{x,f} \bar{\chi}_f \left(\gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_{0h} + i\mu_h \gamma_5 \tau_1 + \mu_\delta \tau_3\right) \chi_f
\end{aligned}
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Baron et al (2010)

- polar mass $M = \sqrt{m^2 + \mu^2}$ and twist angle $\omega = \arctan(\mu/m)$
- the quark doublet in the twisted basis is related to the one in the physical basis by the trafo
- $\psi_l = e^{\frac{i}{2}\omega_l \gamma_5 \tau_3} \chi_l$ and $\bar{\psi}_l = \bar{\chi}_l e^{\frac{i}{2}\omega_l \gamma_5 \tau_3}$
- $S_{ph} = a^4 \sum_{x,f} \bar{\psi}_f \left(D_\mathrm{tW} + M\right) \psi_f$
- to achieve the benefits of the TM formulation one needs to work at maximal twist $\omega = \pi/2$ Frezzotti and Rossi (2003-2004)
- automatic $\mathcal{O}(a)$ improvement

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Simulation setup for $N_\mathrm{f} = 4$

- $S = S_{Iwa}^{YM} + a^4 \sum_{x,f} \bar{\chi}_f \left(\gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 + i r_f \mu_f \gamma_5 \right) \chi_f$
- dedicated simulations with $N_\mathrm{f} = 4$ light degenerate quarks to renormalize NP in a mass independent scheme (where RCs are defined in the chiral limit) the $N_\mathrm{f} = 2 + 1 + 1$ ensembles - allow for a reliable chiral extrapolation

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Ensembles

ensemble	κ	am_{PCAC}	$a\mu$ ($a\mu_{sea}$ in bold)	confs #
$\beta = 2.10$ - $32^3.64$				
3p	0.156017	+0.00559(14)	0.0025, **0.0046**, 0.0090, 0.0152, 0.0201, 0.0249, 0.0297	250
3m	0.156209	-0.00585(08)	0.0025, **0.0046**, 0.0090, 0.0152, 0.0201, 0.0249, 0.0297	250
4p	0.155983	+0.00685(12)	0.0039, **0.0064**, 0.0112, 0.0184, 0.0240, 0.0295	210
4m	0.156250	-0.00682(13)	0.0039, **0.0064**, 0.0112, 0.0184, 0.0240, 0.0295	210
5p	0.155949	+0.00823(08)	0.0048, **0.0078**, 0.0119, 0.0190, 0.0242, 0.0293	220
5m	0.156291	-0.00821(11)	0.0048, **0.0078**, 0.0119, 0.0190, 0.0242, 0.0293	220
$\beta = 1.95$ - $24^3.48$				
2p	0.160826	+0.01906(24)	**0.0085**, 0.0150, 0.0203, 0.0252, 0.0298	290
2m	0.161229	-0.02091(16)	**0.0085**, 0.0150, 0.0203, 0.0252, 0.0298	290
3p	0.160826	+0.01632(21)	0.0060, 0.0085, 0.0120, 0.0150, **0.0180**, 0.0203, 0.0252, 0.0298	310
3m	0.161229	-0.01602(20)	0.0060, 0.0085, 0.0120, 0.0150, **0.0180**, 0.0203, 0.0252, 0.0298	310
8p	0.160524	+0.03634(14)	**0.0020**, 0.0085, 0.0150, 0.0203, 0.0252, 0.0298	310
8m	0.161585	-0.03627(11)	**0.0020**, 0.0085, 0.0150, 0.0203, 0.0252, 0.0298	310
$\beta = 1.90$ - $24^3.48$				
1p	0.162876	+0.0275(04)	0.0060, **0.0080**, 0.0120, 0.0170, 0.0210, 0.0260	450
1m	0.163206	-0.0273(02)	0.0060, **0.0080**, 0.0120, 0.0170, 0.0210, 0.0260	450
4p	0.162689	+0.0398(01)	0.0060, **0.0080**, 0.0120, 0.0170, 0.0210, 0.0260	370
4m	0.163476	-0.0390(01)	0.0060, **0.0080**, 0.0120, 0.0170, 0.0210, 0.0260	370

$N_f = 4$ *ensembles used in our analysis*
The lattice spacing values are respectively $a = 0.062$ *fm for* $\beta = 2.10$, $a = 0.078$ *fm for* $\beta = 1.95$ *and* $a = 0.086$ *fm for* $\beta = 1.90$

Pion Pole Contamination

- Correlation functions of the pseudoscalar operator have pion pole contamination
- need to be addressed carefully
- ansatz for the amputated pseudoscalar vertex $\Gamma_P = a_P + b_P m_\pi^2 + \frac{c_P}{m_\pi^2}$
- $\Gamma_P^{sub} = \Gamma_P - \frac{c_P}{m_\pi^2}$

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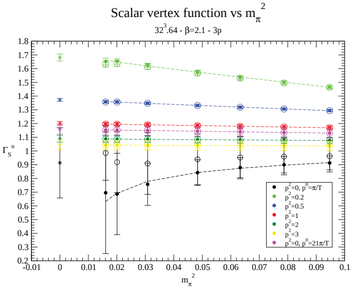
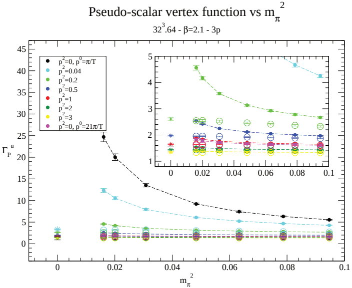
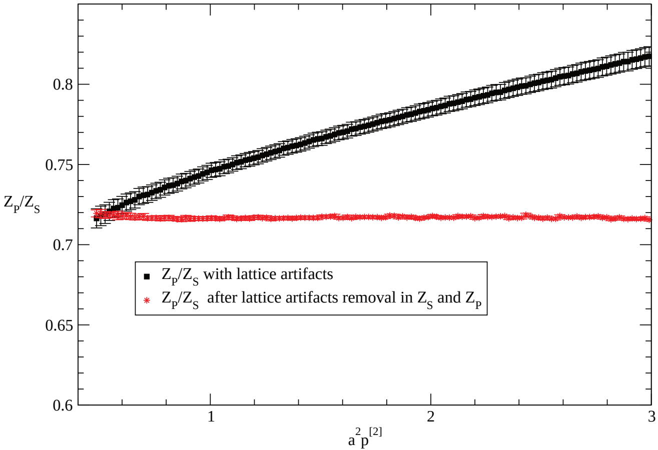
Vertex Functions - The effect of the Goldstone pole subtraction





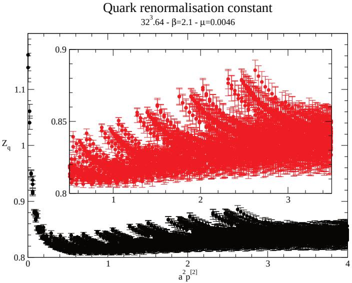
u scalar (LHS) and pseudo-scalar (RHS) vertex functions versus pion mass squared (in lattice unit) for ensemble $3p$ for several values of $a^2\vec{p}^{\,2}$. (Full-) empty circles correspond to (un-)subtracted values while $*$ to the chiral extrapolation, ($a.p^0 = \frac{\pi}{T}$ for all curves except the magenta one, for which $a.p^0 = \frac{21\pi}{T}$).

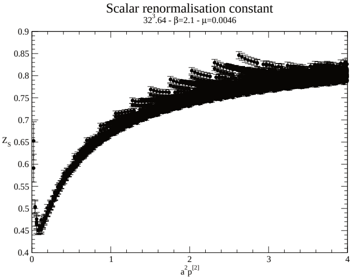
Z_P/Z_S



Z_P/Z_S for ensemble $3mp$ ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3.64$). Lattice artifacts have been removed separately from Z_S and Z_P. The ratio of these two RCs is compatible with a constant over the whole a^2p^2 interval and $Z_P/Z_S = 0.717(3)$.

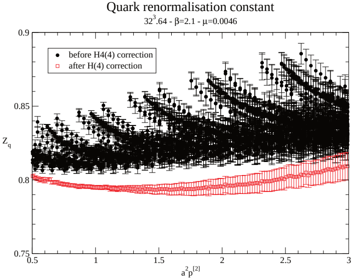
Z_q and Z_S after $H(3)$ corrections

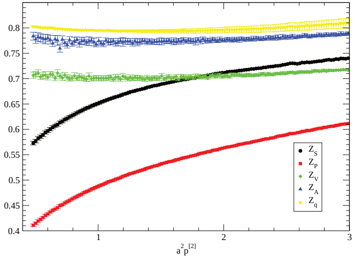




Quark renormalization constant (LHS) and scalar renormalization constant (RHS.) as a function of $a^2 p^{[2]}$. Both exhibit the typical "fishbone" structure induced by the breaking of the $O(4)$ rotational symmetry of the Euclidian space-time by the lattice discretization, into the hypercubic group $H(4)$.

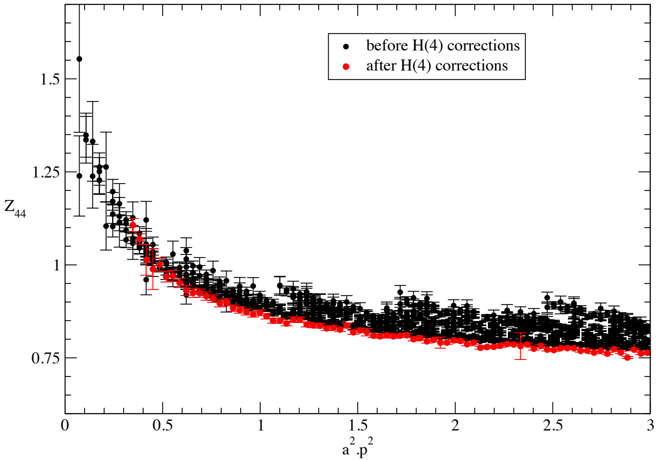
RCs after $H(4)$ corrections





LHS: Effect of hypercubic corrections on quark renormalization constant, as a function of $a^2p^{[2]}$. RHS: renormalization constants as a function of $a^2p^{[2]}$, after removing $H(4)$ artifacts.

Twist-2 operators - Z_{44} -Preliminary



Z_{44} *for ensemble* $1mp$ *(*$\beta = 1.90$, $\mu = 0.0080$, *volume* $24^3.48$*) RC for*

$$O_{44} = \gamma_4 \overset{\leftrightarrow}{D}_4 - \frac{1}{3}\sum_{k=1}^{3} \gamma_k \overset{\leftrightarrow}{D}_k .$$

Correcting for artifacts

- hypercubic artifacts that respect $\mathrm{H}(4)$ but not $\mathrm{O}(4)$
- artifacts that respect $\mathrm{O}(4)$ will be treated NP by introducing corrections to the running
- egalitarian method (does not rely on the selection of diagonal momenta which have small $\mathrm{H}(4)$ artifacts like the method of democratic cuts Boucaud et al (2003), de Soto et al (2007)
- keeps maximum amount of info- allows for the testing of the running of RCs

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Correcting for artifacts

- perform an average over the orbits of $\mathrm{H}(4)$-several orbits correspond to the same value of p^2 e.g. $(1,1,1,1)$ and $(2,0,0,0)$
- we define the $\mathrm{H}(4)$ invariants
- $p^{[4]} = \sum_{\mu=1}^{4} p_\mu^4, \qquad p^{[6]} = \sum_{\mu=1}^{4} p_\mu^6, \qquad p^{[8]} = \sum_{\mu=1}^{4} p_\mu^8$
- any $\mathrm{H}(4)$ invariant polynomial can be written in terms of the four invariants $p^2, p^{[4]}, p^{[6]}, p^{[8]}$
- Expand the RC already averaged over the cubic orbits around $p^{[4]} = 0$
- $Z_{latt}(a^2p^2, a^4p^{[4]}, a^6p^{[6]}, ap_4, a^2\Lambda^2_{QCD}) = Z_{hypcorrected}(a^2p^2, ap_4, a^2\Lambda^2_{QCD}) + R(a^2p^2, a^2\Lambda^2_{QCD})\frac{a^2p^{[4]}}{p^2} + \ldots$
- $R(a^2p^2, a^2\Lambda^2_{QCD}) = \frac{dZ_{latt}(a^2p^2,0,0,0,a^2\Lambda^2_{QCD})}{d\epsilon}|_{\epsilon=0} = c_{a2p4} + c_{a4p4}a^2p^2$

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Running of Z_q

- consider for the running of Z_q Blossier et al (2010)
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\begin{aligned}
Z_q^{hyp-corr}(a^2p^2) &= Z_q^{pert\,RI'}(\mu^2)\, c_{0Z_q}^{RI'}(\frac{p^2}{\mu^2},\alpha(\mu)) \\
&\times \left(1 + \frac{\langle A^2\rangle_{\mu^2}}{32p^2} \frac{c_{2Z_q}^{\overline{\mathsf{MS}}}(\frac{p^2}{\mu^2},\alpha(\mu))}{c_{0Z_q}^{RI'}(\frac{p^2}{\mu^2},\alpha(\mu))} \frac{c_{2Z_q}^{RI'}(\frac{p^2}{\mu^2},\alpha(\mu))}{c_{2Z_q}^{\overline{\mathsf{MS}}}(\frac{p^2}{\mu^2},\alpha(\mu))}\right) \\
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- coefficients $c_{0Z_q}^{RI'}$, $c_{0Z_q}^{RI'}$ and $c_{2Z_q}^{\overline{\mathsf{MS}}}$ known from PT Chetyrkin et al (1999), Chetyrkin (2004), Chetyrkin et al (2009)
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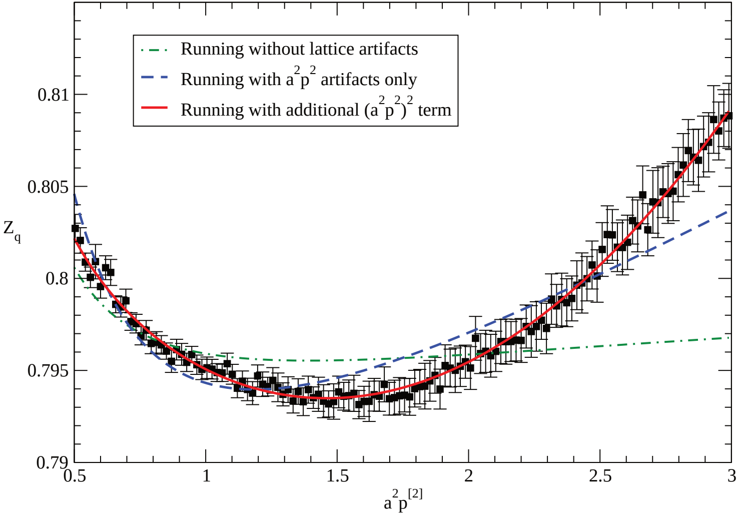
Running of Z_q



Running of Z_q for ensemble $3mp$ ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3.64$) using different fitting formulae.

Running of Z_P and Z_S

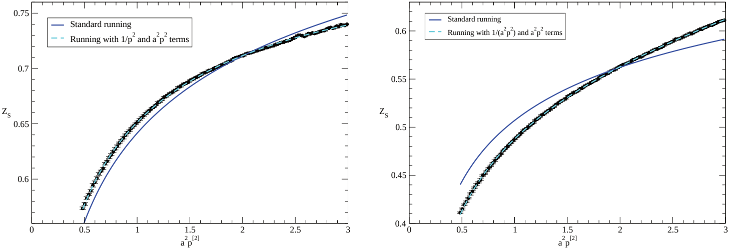
The same study is performed for scalar and pseudo-scalar RCs. Z_S and Z_P have the same running formula, namely:

$$Z_{P/S}(\mu) = Z_{P/S}(\mu_0)\frac{c^{RI'MOM}(\mu)}{c^{RI'MOM}(\mu_0)}$$

$$\begin{aligned} c^{RI'MOM}(\mu) &= x^{\bar{\gamma}_0}\left\{1 + (\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)\,x + \frac{1}{2}\left[(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)^2 + \bar{\gamma}_2 + \bar{\beta}_1{}^2\bar{\gamma}_0 - \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_0\right]\,x^2\right. \\ &+ \left[\frac{1}{6}(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)^3 + \frac{1}{2}(\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_0)(\bar{\gamma}_2 + \bar{\beta}_1{}^2\bar{\gamma}_0 - \bar{\beta}_1\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_0)\right. \\ &+ \left.\left.\frac{1}{3}(\bar{\gamma}_3 - \bar{\beta}_1{}^3\bar{\gamma}_0 + 2\bar{\beta}_1\bar{\beta}_2\bar{\gamma}_0 - \bar{\beta}_3\bar{\gamma}_0 + \bar{\beta}_1{}^2\bar{\gamma}_1 - \bar{\beta}_2\bar{\gamma}_1 - \bar{\beta}_1\bar{\gamma}_2)\right]\,x^3 + \mathcal{O}(x^4)\right\} \end{aligned}$$

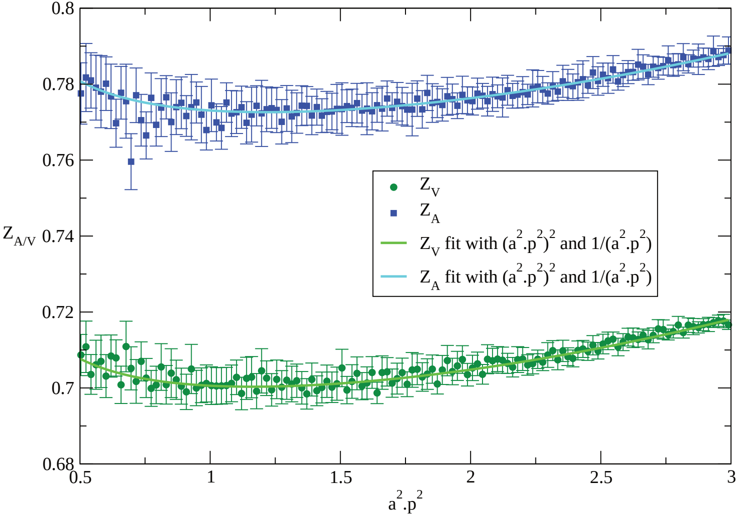
where $x = \alpha/\pi$, $\bar{\gamma}_i = \gamma_i/\beta_0$ and $\bar{\beta}_i = \beta_i/\beta_0$. β_i are the coefficients of the QCD beta-function and they are given at four-loop in Chetyrkin et al (1999) .

Running of Z_S and Z_P



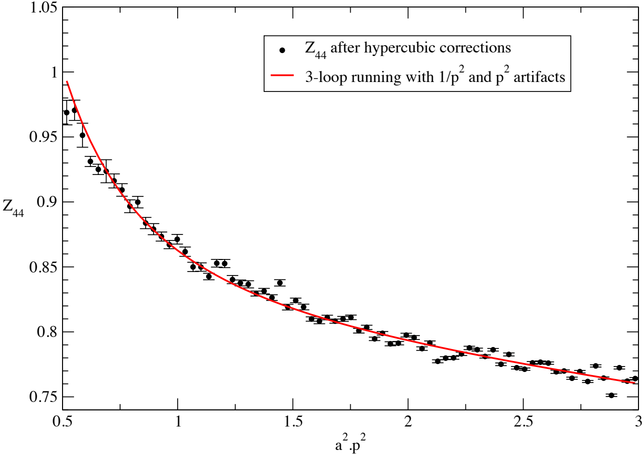
LHS: running of Z_S for ensemble $3mp$ ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3.64$). The standard running formula is represented in solid blue line, the dashed cyan curve includes an $1/a^2p^2$ and an a^2p^2 term. This latter fit leads to $Z_S(10$ GeV$) = 0.869(4)$. RHS: Running of Z_P with the standard running expression Chetyrkin et al (1999) (solid blue curve), and adding an $1/a^2p^2$ and an a^2p^2 terms (dashed cyan curve). The modified running gives $Z_P(10$ GeV$) = 0.623(2)$.

Z_V and Z_A



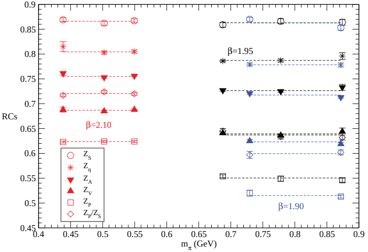
Fits of the residual a^2p^2 dependence of Z_V and Z_A for ensemble $3mp$ ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3.64$)

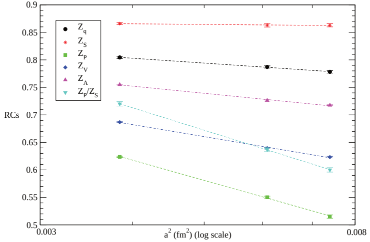
Z_{44} -Preliminary



Correction of the $O(4)$ *artifacts for* Z_{44} *for ensemble* $1mp$ *(*$\beta = 1.90$*,* $\mu = 0.0080$*, volume* $24^3.48$*)*

Chiral extrapolation and lattice spacing dependence





LHS: $N_f = 4$ *local RCs dependence with the pion mass. The straight dashed lines are constant fits for each* β *values. The red points correspond to* $\beta = 2.10$*, the black ones to* $\beta = 1.95$*, and the blue ones to* $\beta = 1.90$*.*
RHS: RCs after chiral extrapolation, vs $\log a^2$*. All RCs follow a linear dependence with* $\log a^2$ *to a very high accuracy.*

Results in $\overline{\text{MS}}$

β	Z_q	Z_S	Z_P	Z_V	Z_A	Z_P/Z_S
1.90	0.767(3)	0.910(3)	0.543(3)	0.623(2)	0.717(1)	0.600(4)
1.95	0.775(2)	0.903(4)	0.576(2)	0.639(2)	0.726(2)	0.637(4)
2.10	0.791(2)	0.887(2)	0.639(1)	0.687(1)	0.755(1)	0.720(4)

converted our RI'-MOM results at 10 GeV to $\overline{\text{MS}}$ values at a reference scale of 2 GeV leads to the final RCs

Conclusions and Outlook

- Provided NP results for the RCs of $N_\mathrm{f} = 2+1+1$ Twisted Mass QCD
- Hypercubic artifacts were taken correctly into account by the "egalitarian" method
- Complete the analysis of twist-2 operators
- Extend the analysis to fermion quadrilinears
- Extend our work to the new ensembles of ETMC with the large volumes $48^3 \times 96$ and masses @ the physical point
- Check the effect of Gribov copies
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for upcoming results ...
Thank you for your attention!