Renormalization constants for $N_f = 2 + 1 + 1$ twisted mass QCD

Savvas Zafeiropoulos

Laboratoire de Physique Corpusculaire
Université Blaise Pascal, CNRS/IN2P3

23-28 June 2014
Lattice 2014
Columbia University

Work in progress with B. Blossier, M. Brinet, P. Guichon, V. Morenas, O. Péne, P. Rodríguez-Quintero.
Lattice formalism is bare QFT
- One computes bare matrix elements of operators at fixed cutoff
- Must renormalize to obtain continuum Physics
- $O_R = Z_O O_b$
- Renormalization can be done perturbatively or non-perturbatively
Lattice formalism is bare QFT

One computes bare matrix elements of operators at fixed cutoff

Must renormalize to obtain continuum Physics

\[ O_R = Z_O O_b \]

Renormalization can be done perturbatively or non-perturbatively
Lattice formalism is bare QFT

One computes bare matrix elements of operators at fixed cutoff

Must renormalize to obtain continuum Physics

$O_R = Z_O O_b$

Renormalization can be done perturbatively or non-perturbatively
Bare vs Renormalized

- Lattice formalism is bare QFT
- One computes bare matrix elements of operators at fixed cutoff
- Must renormalize to obtain continuum Physics
- \( O_R = Z_O O_b \)
- Renormalization can be done perturbatively or non-perturbatively
- Lattice PT—notorious for its bad convergence

- MILC collaboration found that $m_s$ was raised by 14% once its renormalization constant (RC) known in 1-loop PT was calculated @ 2-loops.

- Göckeler et al found that $m_s$ was raised by 24% once its RC known in 1-loop PT was calculated non-perturbatively.
Lattice PT

- Lattice PT - notorious for its bad convergence
- MILC collaboration found that $m_s$ was raised by 14% once its renormalization constant (RC) known in 1-loop PT was calculated @ 2-loops.
- Göckeler et al found that $m_s$ was raised by 24% once its RC known in 1-loop PT was calculated non-perturbatively.
Lattice PT

- Lattice PT—notorious for its bad convergence
- MILC collaboration found that $m_s$ was raised by 14% once its renormalization constant (RC) known in 1-loop PT was calculated @ 2-loops.
- Göckeler et al found that $m_s$ was raised by 24% once its RC known in 1-loop PT was calculated non-perturbatively.
Non-Perturbative Renormalization


- Work on the calculation of the RCs by many groups many of them belonging to the ETMC


Work on the calculation of the RCs by many groups many of them belonging to the ETMC


focus on local fermion bilinears $O_\Gamma = \bar{\psi}(x)\Gamma\psi(x)$

Vladikas Les Houches lectures

\(\Gamma\) can be any Dirac structure and can even potentially contain covariant derivatives (twist-2 operators)

inserting $O_\Gamma$ in the fermion 2-pt function

\[ G_O = \langle u(x_1)O_\Gamma\bar{d}(x_2) \rangle \]

the amputated Green’s function

\[ \Lambda_O(p_1, p_2) = S_u^{-1}(P_1)G_O(p_1, p_2)S_d^{-1}(p_2) \]

\[ \Gamma_O(p) = \frac{1}{12} \text{tr} [P_O\Lambda_O(p, p)] \]

\[ \Gamma_O(p)_R = \lim_{a \to 0} Z_q^{-1} Z_O \Gamma_O(p) \]

\[ Z_q(\mu^2 = p^2) = -\frac{i}{12p^2} \text{tr} [S_{bare}^{-1}(p)\hat{p}] \]
**RI-MOM**

- Focus on local fermion bilinears $O_{\Gamma} = \bar{\psi}(x) \Gamma \psi(x)$

  *Vladikas Les Houches lectures*

- $\Gamma$ can be any Dirac structure and can even potentially contain covariant derivatives (twist-2 operators)

- Inserting $O_{\Gamma}$ in the fermion 2-pt function

- $G_O = \langle u(x_1) O_{\Gamma} \bar{d}(x_2) \rangle$

- The amputated Green’s function

- $\Lambda_O(p_1, p_2) = S_u^{-1}(P_1) G_O(p_1, p_2) S_d^{-1}(p_2)$

- $\Gamma_O(p) = \frac{1}{12} \text{tr} [P_O \Lambda_O(p, p)]$

- $\Gamma_O(p)_R = \lim_{a \to 0} Z_q^{-1} Z_O \Gamma_O(p)$

- $Z_q(\mu^2 = p^2) = -\frac{i}{12p^2} \text{tr} [S_{bare}^{-1}(p) \not{p}]$
focus on local fermion bilinears $O_\Gamma = \bar{\psi}(x)\Gamma\psi(x)$

Vladikas Les Houches lectures

$\Gamma$ can be any Dirac structure and can even potentially contain covariant derivatives (twist-2 operators)

inserting $O_\Gamma$ in the fermion 2-pt function

$G_O = \langle u(x_1)O_\Gamma\bar{d}(x_2) \rangle$

the amputated Green’s function

$\Lambda_O(p_1, p_2) = S_u^{-1}(P_1)G_O(p_1, p_2)S_d^{-1}(p_2)$

$\Gamma_O(p) = \frac{1}{12} \text{tr} [P_O \Lambda_O(p, p)]$

$\Gamma_O(p)_R = \lim_{a \to 0} Z_q^{-1}Z_O\Gamma_O(p)$

$Z_q(\mu^2 = p^2) = -\frac{i}{12p^2} \text{tr} [S_{bare}^{-1}(p)p]$
focus on local fermion bilinears $O_{\Gamma} = \bar{\psi}(x)\Gamma\psi(x)$

Vladikas Les Houches lectures

$\Gamma$ can be any Dirac structure and can even potentially contain covariant derivatives (twist-2 operators)

inserting $O_{\Gamma}$ in the fermion 2-pt function

$G_O = \langle u(x_1)O_{\Gamma}\bar{d}(x_2) \rangle$

the amputated Green’s function

$\Lambda_O(p_1, p_2) = S_u^{-1}(P_1)G_O(p_1, p_2)S_d^{-1}(p_2)$

$\Gamma_O(p) = \frac{1}{12} \text{tr} [P_O\Lambda_O(p, p)]$

$\Gamma_O(p)_R = \lim_{a \to 0} Z_q^{-1}Z_O\Gamma_O(p)$

$Z_q(\mu^2 = p^2) = -\frac{i}{12p^2} \text{tr} [S_{bare}^{-1}(p)\bar{p}]$
focus on local fermion bilinears \( O_\Gamma = \bar{\psi}(x)\Gamma\psi(x) \)

Vladikas Les Houches lectures

\( \Gamma \) can be any Dirac structure and can even potentially contain covariant derivatives (twist-2 operators)

inserting \( O_\Gamma \) in the fermion 2-pt function

\[ G_O = \langle u(x_1)O_\Gamma \bar{d}(x_2) \rangle \]

the amputated Green’s function

\[ \Lambda_O(p_1, p_2) = S_u^{-1}(P_1)G_O(p_1, p_2)S_d^{-1}(p_2) \]

\[ \Gamma_O(p) = \frac{1}{12} \text{tr} [P_O \Lambda_O(p, p)] \]

\[ \Gamma_O(p)_R = \lim_{a \to 0} Z_q^{-1} Z_O \Gamma_O(p) \]

\[ Z_q(\mu^2 = p^2) = -\frac{i}{12p^2} \text{tr} \left[ S_{bare}^{-1}(p)\slashed{p} \right] \]
focus on local fermion bilinears $O_\Gamma = \bar{\psi}(x)\Gamma\psi(x)$

Vladikas Les Houches lectures

$\Gamma$ can be any Dirac structure and can even potentially contain covariant derivatives (twist-2 operators)

inserting $O_\Gamma$ in the fermion 2-pt function

$G_O = \langle u(x_1)O_\Gamma\bar{d}(x_2) \rangle$

the amputated Green’s function

$\Lambda_O(p_1, p_2) = S_u^{-1}(P_1)G_O(p_1, p_2)S_d^{-1}(p_2)$

$\Gamma_O(p) = \frac{1}{12} \text{tr} [P_O\Lambda_O(p, p)]$

$\Gamma_O(p)_R = \lim_{\alpha \to 0} Z_q^{-1}Z_O\Gamma_O(p)$

$Z_q(\mu^2 = p^2) = -\frac{i}{12p^2} \text{tr} [S_{bare}^{-1}(p)\slashed{p}]$
impose that the amputated Green’s function in the chiral limit
@ a large Euclidean scale $p^2 = \mu^2$ is equal to its tree level value

$\Gamma_O(p)_{R}(\mu, g_R, m_R = 0) = \lim_{a \rightarrow 0} [Z^{-1}_q(a\mu, g_0) Z_O(a\mu, g_0) \Gamma_O(p, g_0, m)]_{p=\mu^2, m \rightarrow 0} = 1$
\( \Lambda_{QCD} \ll \mu \ll \frac{\pi}{a} \)

- first inequality ensures the possibility of matching with some perturbative scheme \( \overline{MS} \) and protects from Goldstone pole contaminations
- second inequality ensures small cutoff effects
Conversion to $\overline{\text{MS}}$

- make connection with phenomenological calculations and experiments
- the decay width for the dominant channel $H \rightarrow \bar{b}b \propto m_b^2$
- one needs the RC for $m_b$ - in the TM framework it is given $1/Z_P$
- need to convert to $\overline{\text{MS}}$ with factors $Z_{\overline{\text{MS}}}^{q} = C_q^{-1} Z_q^{RI' - MOM}$
  and $Z_{\overline{\text{MS}}}^{O} = C_O^{-1} Z_O^{RI' - MOM}$
- experiments usually provide results in $\overline{\text{MS}}$ at a reference scale $\mu = 2$ GeV
- evolve $\overline{\text{MS}}$ RCs $Z_{\overline{\text{MS}}}^{O}$ using the scale dependence predicted by the RG equation

$$R_{O}(\mu, \mu_0) := \frac{Z_{O}(\mu)}{Z_{O}(\mu_0)} = \exp \left\{ - \int_{\bar{g}(\mu_0^2)}^{\bar{g}(\mu^2)} dg \frac{\gamma(g)}{\beta(g)} \right\}$$

$\beta$ is the usual QCD-beta function, $\gamma$ the anomalous dimension of operator $O$ and $\bar{g}(\mu^2)$ the running coupling
Conversion to $\overline{\text{MS}}$

- make connection with phenomenological calculations and experiments
- the decay width for the dominant channel $H \to \bar{b}b \propto m_b^2$
- one needs the RC for $m_b$ - in the TM framework it is given $1/Z_P$
- need to convert to $\overline{\text{MS}}$ with factors $Z^{\overline{\text{MS}}}_q = C_q^{-1} Z^{\text{RI'}-\text{MOM}}_q$
- and $Z^{\overline{\text{MS}}}_O = C_O^{-1} Z^{\text{RI'}-\text{MOM}}_O$
- experiments usually provide results in $\overline{\text{MS}}$ at a reference scale $\mu = 2$ GeV
- evolve $\overline{\text{MS}}$ RCs $Z^{\overline{\text{MS}}}_O$ using the scale dependence predicted by the RG equation
  
  $R_O(\mu, \mu_0) := \frac{Z_O(\mu)}{Z_O(\mu_0)} = \exp \left\{ - \int_{\bar{g}(\mu_0^2)}^{\bar{g}(\mu^2)} dg \frac{\gamma(g)}{\beta(g)} \right\}$

  $\beta$ is the usual QCD-beta function, $\gamma$ the anomalous dimension of operator $O$ and $\bar{g}(\mu^2)$ the running coupling
Simulation setup for $N_f = 2 + 1 + 1$

- $S = S^{YM}_{Iwa} + S^f_I + S^f_h$

\[
\begin{align*}
S &= S^{YM}_{Iwa} + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0l + i \mu l \gamma_5 \tau_3 \right) \chi_f \\
&\quad + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0h + i \mu h \gamma_5 \tau_1 + \mu_5 \delta \tau_3 \right) \chi_f
\end{align*}
\]

Baron et al (2010)

- polar mass $M = \sqrt{m^2 + \mu^2}$ and twist angle $\omega = \arctan(\mu/m)$
- the quark doublet in the twisted basis is related to the one in the physical basis by the trafo
  - $\psi_l = e^{i \frac{\omega l \gamma_5 \tau_3}{2}} \chi_l$ and $\bar{\psi}_l = \bar{\chi}_l e^{i \frac{\omega l \gamma_5 \tau_3}{2}}$
  - $S_{ph} = a^4 \sum_{x,f} \bar{\psi}_f \left( D_{tw} + M \right) \psi_f$
- to achieve the benefits of the TM formulation one needs to work at maximal twist $\omega = \pi/2$ Frezzotti and Rossi (2003-2004)
- automatic $O(a)$ improvement

Savvas Zafeiropoulos

RCs for $N_f = 2 + 1 + 1$ twisted mass QCD
Simulation setup for $N_f = 2 + 1 + 1$

- $S = S_{Iwa}^Y + S_l^f + S_h^f$

- $S = S_{Iwa}^Y + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 + i\mu_1\gamma_5\tau_3 \right) \chi_f + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 + i\mu_h\gamma_5\tau_1 + \mu_\delta\tau_3 \right) \chi_f$

Baron et al (2010)

- polar mass $M = \sqrt{m^2 + \mu^2}$ and twist angle $\omega = \arctan(\mu/m)$

- the quark doublet in the twisted basis is related to the one in the physical basis by the trafo

- $\psi_l = e^{\frac{i}{2} \omega_l \gamma_5 \tau_3} \chi_l$ and $\bar{\psi}_l = \bar{\chi}_l e^{\frac{i}{2} \omega_l \gamma_5 \tau_3}$

- $S_{ph} = a^4 \sum_{x,f} \bar{\psi}_f \left( D_{tw} + M \right) \psi_f$

- to achieve the benefits of the TM formulation one needs to work at maximal twist $\omega = \pi/2$ Frezzotti and Rossi (2003-2004)

- automatic $O(a)$ improvement
Simulation setup for $N_f = 2 + 1 + 1$

- $S = S_I^{YM} + S_I^f + S_h^f$

- $S = S_I^{YM} + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 l + i \mu_\perp \gamma_5 \tau_3 \right) \chi_f$

$$+ a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 h + i \mu_\parallel \gamma_5 \tau_1 + \mu_\delta \tau_3 \right) \chi_f$$

Baron et al (2010)

- polar mass $M = \sqrt{m^2 + \mu^2}$ and twist angle $\omega = \arctan(\mu/m)$

- the quark doublet in the twisted basis is related to the one in the physical basis by the trafo

- $\psi_l = e^{i 2 \omega \gamma_5 \tau_3} \chi_l$ and $\bar{\psi}_l = \bar{\chi}_l e^{i 2 \omega \gamma_5 \tau_3}$

- $S_{ph} = a^4 \sum_{x,f} \bar{\psi}_f (D_{tW} + M) \psi_f$

- to achieve the benefits of the TM formulation one needs to work at maximal twist $\omega = \pi/2$ Frezzotti and Rossi (2003-2004)

- automatic $O(a)$ improvement
Simulation setup for $N_f = 2 + 1 + 1$

- $S = S_{Iwa}^M + S_f^I + S_{h}^f$

$$
S = S_{Iwa}^M + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 + i \mu_l \gamma_5 \tau_3 \right) \chi_f \\
+ a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 + i \mu_h \gamma_5 \tau_1 + \mu \delta \tau_3 \right) \chi_f
$$

Baron et al (2010)

- polar mass $M = \sqrt{m^2 + \mu^2}$ and twist angle $\omega = \arctan(\mu/m)$
- the quark doublet in the twisted basis is related to the one in the physical basis by the trafo
  - $\psi_l = e^{i \frac{\omega}{2} \gamma_5 \tau_3} \chi_l$ and $\bar{\psi}_l = \bar{\chi}_l e^{i \frac{\omega}{2} \gamma_5 \tau_3}$
  - $S_{ph} = a^4 \sum_{x,f} \bar{\psi}_f \left( D_{tW} + M \right) \psi_f$

- to achieve the benefits of the TM formulation one needs to work at maximal twist $\omega = \pi/2$ Frezzotti and Rossi (2003-2004)
- automatic $O(a)$ improvement
Simulation setup for $N_f = 2 + 1 + 1$

- $S = S_{Iwa}^Y + S_{l}^f + S_{h}^f$

- $S = S_{Iwa}^Y + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 l + i \mu_l \gamma_5 \tau_3 \right) \chi_f$

  + $a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 h + i \mu_h \gamma_5 \tau_1 + \mu_\delta \tau_3 \right) \chi_f$

Baron et al (2010)

- polar mass $M = \sqrt{m^2 + \mu^2}$ and twist angle $\omega = \arctan(\mu/m)$

- the quark doublet in the twisted basis is related to the one in the physical basis by the trafo

  - $\psi_l = e^{i \omega l \gamma_5 \tau_3} \chi_l$ and $\bar{\psi}_l = \bar{\chi}_l e^{i \omega l \gamma_5 \tau_3}$

  - $S_{ph} = a^4 \sum_{x,f} \bar{\psi}_f \left( D_{tW} + M \right) \psi_f$

- to achieve the benefits of the TM formulation one needs to work at maximal twist $\omega = \pi/2$ Frezzotti and Rossi (2003-2004)

- automatic $O(a)$ improvement
Simulation setup for $N_f = 4$

$S = S_{Iwa}^{YM} + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 + i r_f \mu_f \gamma_5 \right) \chi_f$

- dedicated simulations with $N_f = 4$ light degenerate quarks to renormalize NP in a mass independent scheme (where RCs are defined in the chiral limit) the $N_f = 2 + 1 + 1$ ensembles - allow for a reliable chiral extrapolation
Simulation setup for $N_f = 4$

$S = S_{Iwa}^{YM} + a^4 \sum_{x,f} \bar{\chi}_f \left( \gamma \cdot \nabla - \frac{a}{2} \nabla \cdot \nabla + m_0 + i r_f \mu_f \gamma_5 \right) \chi_f$

- dedicated simulations with $N_f = 4$ light degenerate quarks to renormalize NP in a mass independent scheme (where RCs are defined in the chiral limit) the $N_f = 2 + 1 + 1$ ensembles - allow for a reliable chiral extrapolation
Ensembles

The lattice spacing values are respectively $a = 0.062$ fm for $\beta = 2.10$, $a = 0.078$ fm for $\beta = 1.95$ and $a = 0.086$ fm for $\beta = 1.90$.

$N_f = 4$ ensembles used in our analysis.

<table>
<thead>
<tr>
<th>ensemble</th>
<th>$\kappa$</th>
<th>$am_{PCAC}$</th>
<th>$a\mu$ ($a\mu_{sea}$ in bold)</th>
<th>confs #</th>
</tr>
</thead>
<tbody>
<tr>
<td>3p</td>
<td>0.156017</td>
<td>+0.00559(14)</td>
<td>0.0025, <strong>0.0046</strong>, 0.0090, 0.0152, 0.0201, 0.0249, 0.0297</td>
<td>250</td>
</tr>
<tr>
<td>3m</td>
<td>0.156209</td>
<td>-0.00585(08)</td>
<td>0.0025, <strong>0.0046</strong>, 0.0090, 0.0152, 0.0201, 0.0249, 0.0297</td>
<td>250</td>
</tr>
<tr>
<td>4p</td>
<td>0.155983</td>
<td>+0.00685(12)</td>
<td>0.0039, <strong>0.0064</strong>, 0.0112, 0.0184, 0.0240, 0.0295</td>
<td>210</td>
</tr>
<tr>
<td>4m</td>
<td>0.156250</td>
<td>-0.00682(13)</td>
<td>0.0039, <strong>0.0064</strong>, 0.0112, 0.0184, 0.0240, 0.0295</td>
<td>210</td>
</tr>
<tr>
<td>5p</td>
<td>0.155949</td>
<td>+0.00823(08)</td>
<td>0.0048, <strong>0.0078</strong>, 0.0119, 0.0190, 0.0242, 0.0293</td>
<td>220</td>
</tr>
<tr>
<td>5m</td>
<td>0.156291</td>
<td>-0.00821(11)</td>
<td>0.0048, <strong>0.0078</strong>, 0.0119, 0.0190, 0.0242, 0.0293</td>
<td>220</td>
</tr>
</tbody>
</table>

$\beta = 2.10 - 32^3.64$

<table>
<thead>
<tr>
<th>ensemble</th>
<th>$\kappa$</th>
<th>$am_{PCAC}$</th>
<th>$a\mu$ ($a\mu_{sea}$ in bold)</th>
<th>confs #</th>
</tr>
</thead>
<tbody>
<tr>
<td>2p</td>
<td>0.160826</td>
<td>+0.01906(24)</td>
<td><strong>0.0085</strong>, 0.0150, 0.0203, 0.0252, 0.0298</td>
<td>290</td>
</tr>
<tr>
<td>2m</td>
<td>0.161229</td>
<td>-0.02091(16)</td>
<td><strong>0.0085</strong>, 0.0150, 0.0203, 0.0252, 0.0298</td>
<td>290</td>
</tr>
<tr>
<td>3p</td>
<td>0.160826</td>
<td>+0.01632(21)</td>
<td>0.0060, 0.0085, 0.0120, 0.0150, <strong>0.0180</strong>, 0.0203, 0.0252, 0.0298</td>
<td>310</td>
</tr>
<tr>
<td>3m</td>
<td>0.161229</td>
<td>-0.01602(20)</td>
<td>0.0060, 0.0085, 0.0120, 0.0150, <strong>0.0180</strong>, 0.0203, 0.0252, 0.0298</td>
<td>310</td>
</tr>
<tr>
<td>8p</td>
<td>0.160524</td>
<td>+0.03634(14)</td>
<td><strong>0.0020</strong>, 0.0085, 0.0150, 0.0203, 0.0252, 0.0298</td>
<td>310</td>
</tr>
<tr>
<td>8m</td>
<td>0.161585</td>
<td>-0.03627(11)</td>
<td><strong>0.0020</strong>, 0.0085, 0.0150, 0.0203, 0.0252, 0.0298</td>
<td>310</td>
</tr>
</tbody>
</table>

$\beta = 1.95 - 24^3.48$

<table>
<thead>
<tr>
<th>ensemble</th>
<th>$\kappa$</th>
<th>$am_{PCAC}$</th>
<th>$a\mu$ ($a\mu_{sea}$ in bold)</th>
<th>confs #</th>
</tr>
</thead>
<tbody>
<tr>
<td>1p</td>
<td>0.162876</td>
<td>+0.0275(04)</td>
<td>0.0060, <strong>0.0080</strong>, 0.0120, 0.0170, 0.0210, 0.0260</td>
<td>450</td>
</tr>
<tr>
<td>1m</td>
<td>0.163206</td>
<td>-0.0273(02)</td>
<td>0.0060, <strong>0.0080</strong>, 0.0120, 0.0170, 0.0210, 0.0260</td>
<td>450</td>
</tr>
<tr>
<td>4p</td>
<td>0.162689</td>
<td>+0.0398(01)</td>
<td>0.0060, <strong>0.0080</strong>, 0.0120, 0.0170, 0.0210, 0.0260</td>
<td>370</td>
</tr>
<tr>
<td>4m</td>
<td>0.163476</td>
<td>0.0390(01)</td>
<td>0.0060, <strong>0.0080</strong>, 0.0120, 0.0170, 0.0210, 0.0260</td>
<td>370</td>
</tr>
</tbody>
</table>
- Correlation functions of the pseudoscalar operator have pion pole contamination
- need to be addressed carefully
- ansatz for the amputated pseudoscalar vertex
  \[ \Gamma_P = a_P + b_P m_\pi^2 + \frac{c_P}{m_\pi^2} \]
- \[ \Gamma_{P}^{\text{sub}} = \Gamma_P - \frac{c_P}{m_\pi^2} \]
Correlation functions of the pseudoscalar operator have pion pole contamination need to be addressed carefully.

Ansatz for the amputated pseudoscalar vertex:
\[ \Gamma_P = a_P + b_P m^2_\pi + \frac{c_P}{m^2_\pi} \]

\[ \Gamma_{sub}^P = \Gamma_P - \frac{c_P}{m^2_\pi} \]
Correlation functions of the pseudoscalar operator have pion pole contamination need to be addressed carefully.

ansatz for the amputated pseudoscalar vertex

\[ \Gamma_P = a_P + b_P m^2_\pi + \frac{c_P}{m^2_\pi} \]

\[ \Gamma_{P}^{sub} = \Gamma_P - \frac{c_P}{m^2_\pi} \]
Correlation functions of the pseudoscalar operator have pion pole contamination

need to be addressed carefully

ansatz for the amputated pseudoscalar vertex

\[ \Gamma_P = a_P + b_P m_{\pi}^2 + \frac{c_P}{m_{\pi}^2} \]

\[ \Gamma_{P}^{sub} = \Gamma_P - \frac{c_P}{m_{\pi}^2} \]
u scalar (LHS) and pseudo-scalar (RHS) vertex functions versus pion mass squared (in lattice unit) for ensemble 3p for several values of $a^2 \vec{p}^2$. (Full-) empty circles correspond to (un-)subtracted values while * to the chiral extrapolation, ($a.p^0 = \frac{\pi}{T}$ for all curves except the magenta one, for which $a.p^0 = \frac{21\pi}{T}$).
$Z_P/Z_S$ for ensemble $3mp$ ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3 \times 64$).

Lattice artifacts have been removed separately from $Z_S$ and $Z_P$. The ratio of these two RCs is compatible with a constant over the whole $\alpha^2 p^2$ interval and $Z_P/Z_S = 0.717(3)$. 
Quark renormalization constant (LHS) and scalar renormalization constant (RHS.) as a function of $a^2p^{[2]}$. Both exhibit the typical “fishbone” structure induced by the breaking of the $O(4)$ rotational symmetry of the Euclidian space-time by the lattice discretization, into the hypercubic group $H(4)$. 
LHS: Effect of hypercubic corrections on quark renormalization constant, as a function of $a^2 p^{[2]}$. RHS: renormalization constants as a function of $a^2 p^{[2]}$, after removing $H(4)$ artifacts.
Twist-2 operators - $Z_{44}$ - Preliminary

$Z_{44}$ for ensemble $1mp$ ($\beta = 1.90$, $\mu = 0.0080$, volume $24^3 .48$) RC for

$$O_{44} = \gamma_4 \leftrightarrow D_4 - \frac{1}{3} \sum_{k=1}^{3} \gamma_k \leftrightarrow D_k.$$
Correcting for artifacts

- hypercubic artifacts that respect $H(4)$ but not $O(4)$
- artifacts that respect $O(4)$ will be treated NP by introducing corrections to the running
  - egalitarian method (does not rely on the selection of diagonal momenta which have small $H(4)$ artifacts like the method of democratic cuts Boucaud et al (2003), de Soto et al (2007))
  - keeps maximum amount of info- allows for the testing of the running of RCs
Correcting for artifacts

- hypercubic artifacts that respect $H(4)$ but not $O(4)$
- artifacts that respect $O(4)$ will be treated NP by introducing corrections to the running
- egalitarian method (does not rely on the selection of diagonal momenta which have small $H(4)$ artifacts like the method of democratic cuts) Boucaud et al (2003), de Soto et al (2007)
- keeps maximum amount of info- allows for the testing of the running of RCs
Correcting for artifacts

- hypercubic artifacts that respect $H(4)$ but not $O(4)$
- artifacts that respect $O(4)$ will be treated NP by introducing corrections to the running
- egalitarian method (does not rely on the selection of diagonal momenta which have small $H(4)$ artifacts like the method of democratic cuts Boucaud et al (2003), de Soto et al (2007))
- keeps maximum amount of info- allows for the testing of the running of RCs
Correcting for artifacts

- perform an average over the orbits of $H(4)$—several orbits correspond to the same value of $p^2$ e.g. $(1, 1, 1, 1)$ and $(2, 0, 0, 0)$
- we define the $H(4)$ invariants

$$p^{[4]} = \sum_{\mu=1}^{4} p_{\mu}^4, \quad p^{[6]} = \sum_{\mu=1}^{4} p_{\mu}^6, \quad p^{[8]} = \sum_{\mu=1}^{4} p_{\mu}^8$$

- any $H(4)$ invariant polynomial can be written in terms of the four invariants $p^2, p^{[4]}, p^{[6]}, p^{[8]}$
- Expand the RC already averaged over the cubic orbits around $p^{[4]} = 0$

$$Z_{latt}(a^2 p^2, a^4 p^{[4]}, a^6 p^{[6]}, a p_4, a^2 \Lambda_{QCD}^2) =\
Z_{hypcorrected}(a^2 p^2, a p_4, a^2 \Lambda_{QCD}^2) + R(a^2 p^2, a^2 \Lambda_{QCD}^2) \frac{a^2 p^{[4]}}{p^2} + \
\ldots$$

$$R(a^2 p^2, a^2 \Lambda_{QCD}^2) = \frac{dZ_{latt}(a^2 p^2, 0, 0, 0, a^2 \Lambda_{QCD}^2)}{d\epsilon} |_{\epsilon=0} = c_{a^2 p_4} + c_{a^4 p_4} a^2 p^2$$
Correcting for artifacts

- perform an average over the orbits of $H(4)$—several orbits correspond to the same value of $p^2$ e.g. $(1, 1, 1, 1)$ and $(2, 0, 0, 0)$
- we define the $H(4)$ invariants

$$p^{[4]} = \sum_{\mu=1}^{4} p_\mu^4, \quad p^{[6]} = \sum_{\mu=1}^{4} p_\mu^6, \quad p^{[8]} = \sum_{\mu=1}^{4} p_\mu^8$$

- any $H(4)$ invariant polynomial can be written in terms of the four invariants $p^2, p^{[4]}, p^{[6]}, p^{[8]}$
- Expand the RC already averaged over the cubic orbits around $p^{[4]} = 0$

$$Z_{latt}(a^2 p^2, a^4 p^{[4]}, a^6 p^{[6]}, a p_4, a^2 \Lambda^2_{QCD}) =$$

$$Z_{hypcorrected}(a^2 p^2, a p_4, a^2 \Lambda^2_{QCD}) + R(a^2 p^2, a^2 \Lambda^2_{QCD}) \frac{a^2 p^{[4]}}{p^2} + \ldots$$

$$R(a^2 p^2, a^2 \Lambda^2_{QCD}) = \left. \frac{dZ_{latt}(a^2 p^2, 0, 0, 0, a^2 \Lambda^2_{QCD})}{d\epsilon} \right|_{\epsilon=0} = c_{a2p4} + c_{a4p4} a^2 p^2$$
Correcting for artifacts

- perform an average over the orbits of $H(4)$-several orbits correspond to the same value of $p^2$ e.g. $(1, 1, 1, 1)$ and $(2, 0, 0, 0)$
- we define the $H(4)$ invariants

$$p^{[4]} = \sum_{\mu=1}^{4} p_\mu^4, \quad p^{[6]} = \sum_{\mu=1}^{4} p_\mu^6, \quad p^{[8]} = \sum_{\mu=1}^{4} p_\mu^8$$

- any $H(4)$ invariant polynomial can be written in terms of the four invariants $p^2, p^{[4]}, p^{[6]}, p^{[8]}$

- Expand the RC already averaged over the cubic orbits around $p^{[4]} = 0$

$$Z_{\text{latt}}(a^2 p^2, a^4 p^{[4]}, a^6 p^{[6]}, a p_4, a^2 \Lambda_{QCD}^2) =$$

$$Z_{\text{hypcorrected}}(a^2 p^2, a p_4, a^2 \Lambda_{QCD}^2) + R(a^2 p^2, a^2 \Lambda_{QCD}^2) \frac{a^2 p^{[4]}}{p^2} + \ldots$$

$$R(a^2 p^2, a^2 \Lambda_{QCD}^2) = \frac{dZ_{\text{latt}}(a^2 p^2, 0, 0, 0, a^2 \Lambda_{QCD}^2)}{d\epsilon} \bigg|_{\epsilon=0} = c_{a2p4} + c_{a4p4} a^2 p^2$$
consider for the running of $Z_q$ \cite{Blossier:2010nq}

\[
Z_q^{\text{hyp-corr}}(a^2p^2) = Z_q^{\text{pert } RI'}(\mu^2) c_{Rlq}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu)) \\
\times \left( 1 + \frac{\langle A^2 \rangle_{\mu^2}}{32p^2} \frac{\overline{MS}}{c_{Zq}^{RI'}}(\frac{p^2}{\mu^2}, \alpha(\mu)) \frac{c_{Rlq}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))}{c_{Zq}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))} \right) \\
+ c_{a2p2} a^2 p^2 + c_{a4p4} (a^2 p^2)^2
\]

coefficients $c_{Rlq}^{RI'}$, $c_{Rlq}^{RI'}$ and $\overline{MS}$ known from PT \cite{Chetyrkin:1999pq, Chetyrkin:2004mf, Chetyrkin:2009fv}

the running formula contains lattice artifact terms $\propto a^2 p^2$ and $\propto (a^2 p^2)^2$, not yet removed.

need to determine, $Z_q^{\text{pert } RI'}(\mu^2)$, $\langle A^2 \rangle_{\mu^2}$, $c_{a2p2}$ and $c_{a4p4}$
consider for the running of $Z_q$ \cite{Blossier:2010za}

\[
Z_q^{hyp-corr}(a^2 p^2) = Z_q^{pert RI'}(\mu^2) c_{0Zq}^{RI'} \left( \frac{p^2}{\mu^2}, \alpha(\mu) \right) \\
\times \left( 1 + \frac{\langle A^2 \rangle_{\mu^2}}{32 p^2} \frac{\overline{MS}}{c_{2Zq}} \left( \frac{p^2}{\mu^2}, \alpha(\mu) \right) \frac{c_{RI'}(\mu^2, \alpha(\mu))}{c_{0Zq}^{RI'}(\mu^2, \alpha(\mu))} \overline{MS} \right)
\]

+ $c_{a2p2} a^2 p^2 + c_{a4p4} (a^2 p^2)^2$

coefficients $c_{0Zq}^{RI'}$, $c_{0Zq}^{RI'}$ and $\overline{MS}$ known from PT \cite{Chetyrkin:1999ys, Chetyrkin:2004mf, Chetyrkin:2009se}

the running formula contains lattice artifact terms $\propto a^2 p^2$ and $\propto (a^2 p^2)^2$, not yet removed.

need to determine, $Z_q^{pert RI'}(\mu^2)$, $\langle A^2 \rangle_{\mu^2}$, $c_{a2p2}$ and $c_{a4p4}$
consider for the running of $Z_q$ Blossier et al (2010)

$$Z_q^{hyp-corr}(a^2p^2) = Z_q^{pert RI'}(\mu^2)c_{0Z_q}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))$$

$$\times \left( 1 + \frac{\langle A^2 \rangle_{\mu^2}}{32p^2} \frac{\overline{MS}_{2Z_q}(\frac{p^2}{\mu^2}, \alpha(\mu))}{c_{0Z_q}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))} \frac{c_{2Z_q}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))}{c_{2Z_q}^{MS}(\frac{p^2}{\mu^2}, \alpha(\mu))} \right)$$

$$+ c_{a2p2}a^2p^2 + c_{a4p4}(a^2p^2)^2$$

coefficients $c_{0Z_q}^{RI'}$, $c_{0Z_q}^{RI'}$ and $\overline{MS}_{2Z_q}$ known from PT Chetyrkin et al (1999), Chetyrkin (2004), Chetyrkin et al (2009)

the running formula contains lattice artifact terms $\propto a^2p^2$ and $\propto (a^2p^2)^2$, not yet removed.

need to determine, $Z_q^{pert RI'}(\mu^2)$, $\langle A^2 \rangle_{\mu^2}$, $c_{a2p2}$ and $c_{a4p4}$
consider for the running of $Z_q$ \cite{Blossier2010}

$$Z_q^{hyp-corr}(a^2p^2) = Z_q^{pert RI'}(\mu^2) c_{0Zq}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))$$

$$\times \left(1 + \frac{\langle A^2 \rangle_\mu^2}{32p^2} \frac{\overline{MS}_{2Zq}(\frac{p^2}{\mu^2}, \alpha(\mu))}{c_{2Zq}^{RI'}(\frac{p^2}{\mu^2}, \alpha(\mu))} \right)$$

$$+ c_{a2p2} a^2 p^2 + c_{a4p4} (a^2 p^2)^2$$

coefficients $c_{0Zq}^{RI'}$, $c_{0Zq}^{RI'}$ and $\overline{MS}_{2Zq}$ known from PT \cite{Chetyrkin1999, Chetyrkin2004, Chetyrkin2009}

the running formula contains lattice artifact terms $\propto a^2 p^2$ and $\propto (a^2 p^2)^2$, not yet removed.

need to determine, $Z_q^{pert RI'}(\mu^2)$, $\langle A^2 \rangle_\mu^2$, $c_{a2p2}$ and $c_{a4p4}$
Running of \( Z_q \) for ensemble 3\( mp \) (\( \beta = 2.10, \mu = 0.0046, \text{ volume } 32^3.64 \)) using different fitting formulae.
The same study is performed for scalar and pseudo-scalar RCs. $Z_S$ and $Z_P$ have the same running formula, namely:

$$Z_{P/S}(\mu) = Z_{P/S}(\mu_0) \frac{c^{RI'MOM}(\mu)}{c^{RI'MOM}(\mu_0)}$$

$$c^{RI'MOM}(\mu) = x \tilde{\gamma}_0 \left\{ 1 + (\tilde{\gamma}_1 - \tilde{\beta}_1 \tilde{\gamma}_0) x + \frac{1}{2} \left[ (\tilde{\gamma}_1 - \tilde{\beta}_1 \tilde{\gamma}_0)^2 + \tilde{\gamma}_2 + \tilde{\beta}_1^2 \tilde{\gamma}_0 - \tilde{\beta}_1 \tilde{\gamma}_1 - \tilde{\beta}_2 \tilde{\gamma}_0 \right] x^2 
+ \left[ \frac{1}{6} (\tilde{\gamma}_1 - \tilde{\beta}_1 \tilde{\gamma}_0)^3 + \frac{1}{2} (\tilde{\gamma}_1 - \tilde{\beta}_1 \tilde{\gamma}_0)(\tilde{\gamma}_2 + \tilde{\beta}_1^2 \tilde{\gamma}_0 - \tilde{\beta}_1 \tilde{\gamma}_1 - \tilde{\beta}_2 \tilde{\gamma}_0) 
+ \frac{1}{3} (\tilde{\gamma}_3 - \tilde{\beta}_1^3 \tilde{\gamma}_0 + 2 \tilde{\beta}_1 \tilde{\beta}_2 \tilde{\gamma}_0 - \tilde{\beta}_3 \tilde{\gamma}_0 + \tilde{\beta}_1^2 \tilde{\gamma}_1 - \tilde{\beta}_2 \tilde{\gamma}_1 - \tilde{\beta}_1 \tilde{\gamma}_2) \right] x^3 + O(x^4) \right\}$$

where $x = \alpha/\pi$, $\tilde{\gamma}_i = \gamma_i/\beta_0$ and $\tilde{\beta}_i = \beta_i/\beta_0$. $\beta_i$ are the coefficients of the QCD beta-function and they are given at four-loop in Chetyrkin et al (1999).
Running of $Z_S$ and $Z_P$

**LHS:** running of $Z_S$ for ensemble $3mp$ ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3 \times 64$). The standard running formula is represented in solid blue line, the dashed cyan curve includes an $1/a^2 p^2$ and an $a^2 p^2$ term. This latter fit leads to $Z_S(10 \text{ GeV}) = 0.869(4)$.

**RHS:** Running of $Z_P$ with the standard running expression Chetyrkin et al (1999) (solid blue curve), and adding an $1/a^2 p^2$ and an $a^2 p^2$ terms (dashed cyan curve). The modified running gives $Z_P(10 \text{ GeV}) = 0.623(2)$. 
$Z_V$ and $Z_A$

Fits of the residual $a^2p^2$ dependence of $Z_V$ and $Z_A$ for ensemble 3mp ($\beta = 2.10$, $\mu = 0.0046$, volume $32^3.64$)
Correction of the $O(4)$ artifacts for $Z_{44}$ for ensemble 1mp ($\beta = 1.90$, $\mu = 0.0080$, volume $24^3 \cdot 48$)
LHS: $N_f = 4$ local RCs dependence with the pion mass. The straight dashed lines are constant fits for each $\beta$ values. The red points correspond to $\beta = 2.10$, the black ones to $\beta = 1.95$, and the blue ones to $\beta = 1.90$.

RHS: RCs after chiral extrapolation, vs $\log a^2$. All RCs follow a linear dependence with $\log a^2$ to a very high accuracy.
converted our RI’-MOM results at 10 GeV to MS values at a reference scale of 2 GeV leads to the final RCs

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$Z_q$</th>
<th>$Z_S$</th>
<th>$Z_P$</th>
<th>$Z_V$</th>
<th>$Z_A$</th>
<th>$Z_P/Z_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.90</td>
<td>0.767(3)</td>
<td>0.910(3)</td>
<td>0.543(3)</td>
<td>0.623(2)</td>
<td>0.717(1)</td>
<td>0.600(4)</td>
</tr>
<tr>
<td>1.95</td>
<td>0.775(2)</td>
<td>0.903(4)</td>
<td>0.576(2)</td>
<td>0.639(2)</td>
<td>0.726(2)</td>
<td>0.637(4)</td>
</tr>
<tr>
<td>2.10</td>
<td>0.791(2)</td>
<td>0.887(2)</td>
<td>0.639(1)</td>
<td>0.687(1)</td>
<td>0.755(1)</td>
<td>0.720(4)</td>
</tr>
</tbody>
</table>
Conclusions and Outlook

- Provided NP results for the RCs of $N_f = 2 + 1 + 1$ Twisted Mass QCD
- Hypercubic artifacts were taken correctly into account by the "egalitarian" method
- Complete the analysis of twist-2 operators
- Extend the analysis to fermion quadrilinears
- Extend our work to the new ensembles of ETMC with the large volumes $48^3 \times 96$ and masses @ the physical point
- Check the effect of Gribov copies
- Perform the analysis using the RI-SMOM scheme
Conclusions and Outlook

- Provided NP results for the RCs of $N_f = 2 + 1 + 1$ Twisted Mass QCD
- Hypercubic artifacts were taken correctly into account by the "egalitarian" method
- Complete the analysis of twist-2 operators
- Extend the analysis to fermion quadrilinears
- Extend our work to the new ensembles of ETMC with the large volumes $48^3 \times 96$ and masses @ the physical point
- Check the effect of Gribov copies
- Perform the analysis using the RI-SMOM scheme
Conclusions and Outlook

- Provided NP results for the RCs of $N_f = 2 + 1 + 1$ Twisted Mass QCD
- Hypercubic artifacts were taken correctly into account by the ”egalitarian” method
- Complete the analysis of twist-2 operators
  - Extend the analysis to fermion quadrilinears
  - Extend our work to the new ensembles of ETMC with the large volumes $48^3 \times 96$ and masses @ the physical point
- Check the effect of Gribov copies
- Perform the analysis using the RI-SMOM scheme
Conclusions and Outlook

- Provided NP results for the RCs of $N_f = 2 + 1 + 1$ Twisted Mass QCD
- Hypercubic artifacts were taken correctly into account by the "egalitarian" method
- Complete the analysis of twist-2 operators
- Extend the analysis to fermion quadrilinears
  - Extend our work to the new ensembles of ETMC with the large volumes $48^3 \times 96$ and masses @ the physical point
  - Check the effect of Gribov copies
  - Perform the analysis using the RI-SMOM scheme
Conclusions and Outlook

- Provided NP results for the RCs of $N_f = 2 + 1 + 1$ Twisted Mass QCD
- Hypercubic artifacts were taken correctly into account by the "egalitarian" method
- Complete the analysis of twist-2 operators
- Extend the analysis to fermion quadrilinears
- Extend our work to the new ensembles of ETMC with the large volumes $48^3 \times 96$ and masses @ the physical point
- Check the effect of Gribov copies
- Perform the analysis using the RI-SMOM scheme
Conclusions and Outlook

- Provided NP results for the RCs of $N_f = 2 + 1 + 1$ Twisted Mass QCD
- Hypercubic artifacts were taken correctly into account by the "egalitarian" method
- Complete the analysis of twist-2 operators
- Extend the analysis to fermion quadrilinears
- Extend our work to the new ensembles of ETMC with the large volumes $48^3 \times 96$ and masses @ the physical point
- Check the effect of Gribov copies
- Perform the analysis using the RI-SMOM scheme
Conclusions and Outlook

- Provided NP results for the RCs of $N_f = 2 + 1 + 1$ Twisted Mass QCD
- Hypercubic artifacts were taken correctly into account by the "egalitarian" method
- Complete the analysis of twist-2 operators
- Extend the analysis to fermion quadrilinears
- Extend our work to the new ensembles of ETMC with the large volumes $48^3 \times 96$ and masses @ the physical point
- Check the effect of Gribov copies
- Perform the analysis using the RI-SMOM scheme
Stay Tuned!

for upcoming results . . .
Thank you for your attention!