# Quark number susceptibilities from fugacity expansion at finite chemical potential

Hans-Peter Schadler

In cooperation with
Christof Gattringer

University of Graz
Institute of Physics, Division of Theoretical Physics

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# (Lattice) QCD at finite chemical potential

- Temperature driven phase transition of QCD is well understood (ab initio info from lattice methods), but problem for finite $\mu$.
- Different approaches: Reweighting, Taylor expansion, ... .
- Here: **Fugacity expansion**.
- Fugacity expansion has different properties than Taylor expansion (Laurent v.s. Taylor series and finite sum for finite $V$).
- Recently it was shown that it can have better convergence properties than a Taylor expansion ($Z_3$ model).
  E. Grünwald, Y. Delgado Mercado, C. Gattringer, PoS Lattice 2013, [arXiv:1310.6520 [hep-lat]].
  E. Grünwald, Y. Delgado Mercado, C. Gattringer, (2014), [arXiv:1403.2086 [hep-lat]].
- OOH: **Numerically hard** (calculation of expansion coefficients).
- OTOH: Interesting **observables easily accessible**.

# The fugacity expansion - 1

The **grand canonical determinant** with **real chemical potential** $\mu$ can be written as the **fugacity series** (exact for $q_{\text{cut}} = 6\,V$)

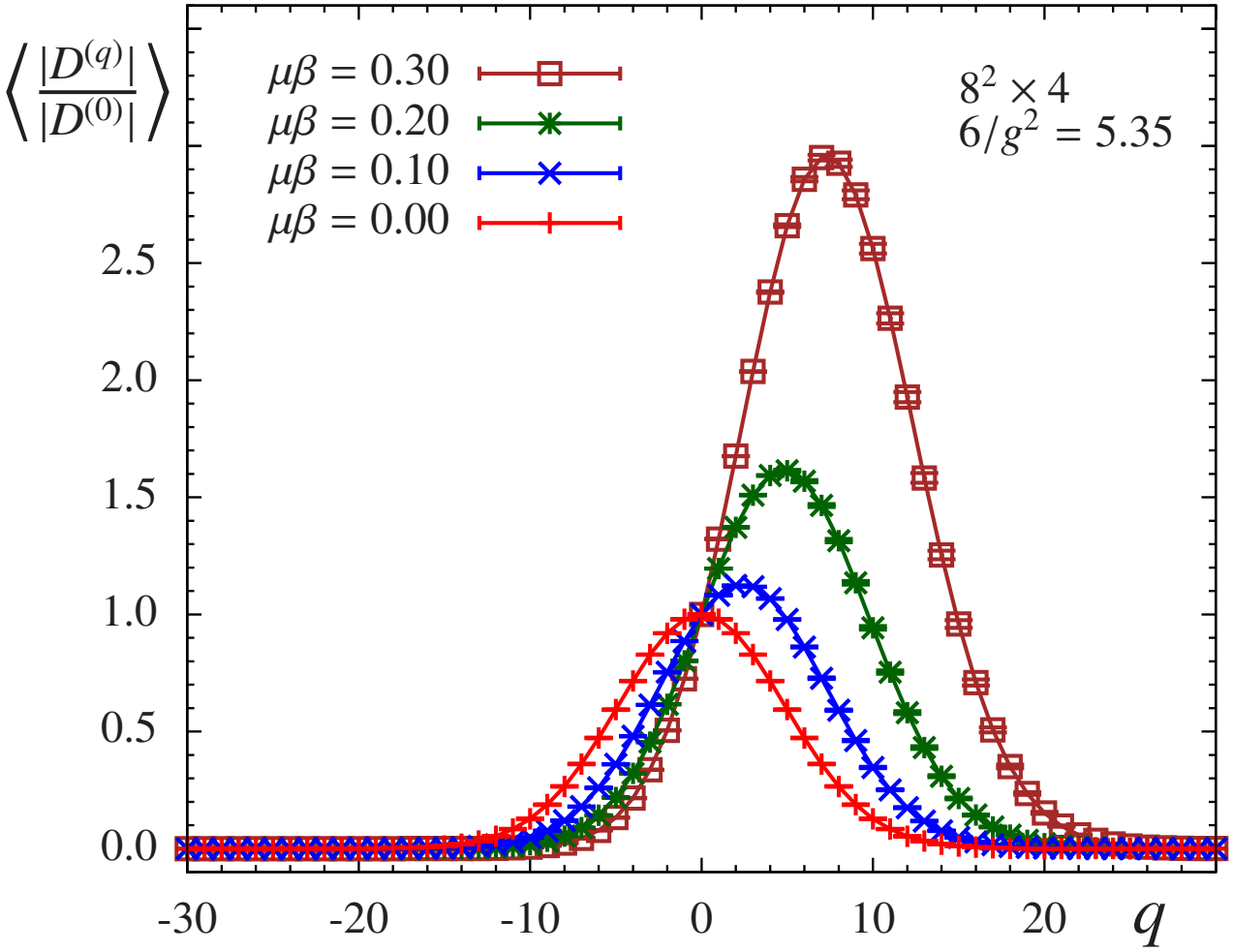
$$\det[D(\mu)] = \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu\beta q}\, D^{(q)} \,.$$

$D^{(q)}$: **Canonical determinants** with net quark number $q$,

$$D^{(q)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi\, e^{-iq\phi} \det[D(\mu\beta = i\phi)] \,.$$

Fourier integral is done numerically $\Rightarrow q_{\text{cut}} \ll 6\,V$.

# The fugacity expansion - 2




$$\det[D(\mu)] = \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu\beta q} D^{(q)}$$

# The fugacity expansion - 3

Important to have $D^{(q)}$ at high precision.

$\Downarrow$

Calculation of $\det[D(\mu\beta = i\phi)]$ for many values of $\phi$.

$\Downarrow$

**Expensive!**

Dimensional reduction: J. Danzer, C. Gattringer, Phys. Rev. D 78 (2008) 114506 [arXiv:0809.2736 [hep-lat]].
Use a domain decomposition of the Dirac operator $D(\mu)$ to obtain

$$\det[D(\mu)] = A\, W(\mu)\ ,$$

with a $\mu$-independent factor $A$ and

$$W(\mu) = \det[K_0 - e^{\mu\beta} K - e^{-\mu\beta} K^\dagger]\ .$$

$K_0, K$ are dense matrices living on a single time slice
($\dim K = N_s^3 \times 1 \times 3 \times 4$).

# Observables related to quark number - 1

**Grand canonical partition sum** written using fugacity series:

$$Z_\mu = \int D[U]\, e^{-S_g[U]} \det[D(\mu)]^2$$
$$= \int D[U]\, e^{-S_g[U]} \left( \sum_{q=-q_{\text{cut}}}^{q_{\text{cut}}} e^{\mu\beta q}\, D^{(q)} \right)^2 .$$

Observables related to quark numbers are **derivatives w.r.t $\mu$**, i.e.,

$$\chi_n^q \propto \frac{\partial^n \ln Z_\mu}{\partial(\mu\beta)^n} ,$$

and they take a **simple form in the fugacity approach**.

# Observables related to quark number - 2

**Moments of $D^{(q)}$:**

$$M^n = \sum_{q=-q_{\rm cut}}^{q_{\rm cut}} e^{\mu\beta q}\, q^n \, \frac{D^{(q)}}{\det[D(\mu=0)]} \, .$$

**Quark number density:**

$$\frac{\chi_1^q}{T^3} = \frac{n_q}{T^3} = 2\,\frac{\beta^3}{V}\,\frac{\langle M^0 M^1\rangle_0}{\langle (M^0)^2\rangle_0} \, .$$

**Quark number susceptibility:**

$$\frac{\chi_2^q}{T^2} = 2\,\frac{\beta^3}{V}\left[\frac{\langle (M^1)^2\rangle_0 + \langle M^0 M^2\rangle_0}{\langle (M^0)^2\rangle_0} - 2\left(\frac{\langle M^0 M^1\rangle_0}{\langle (M^0)^2\rangle_0}\right)^2\right] \, .$$

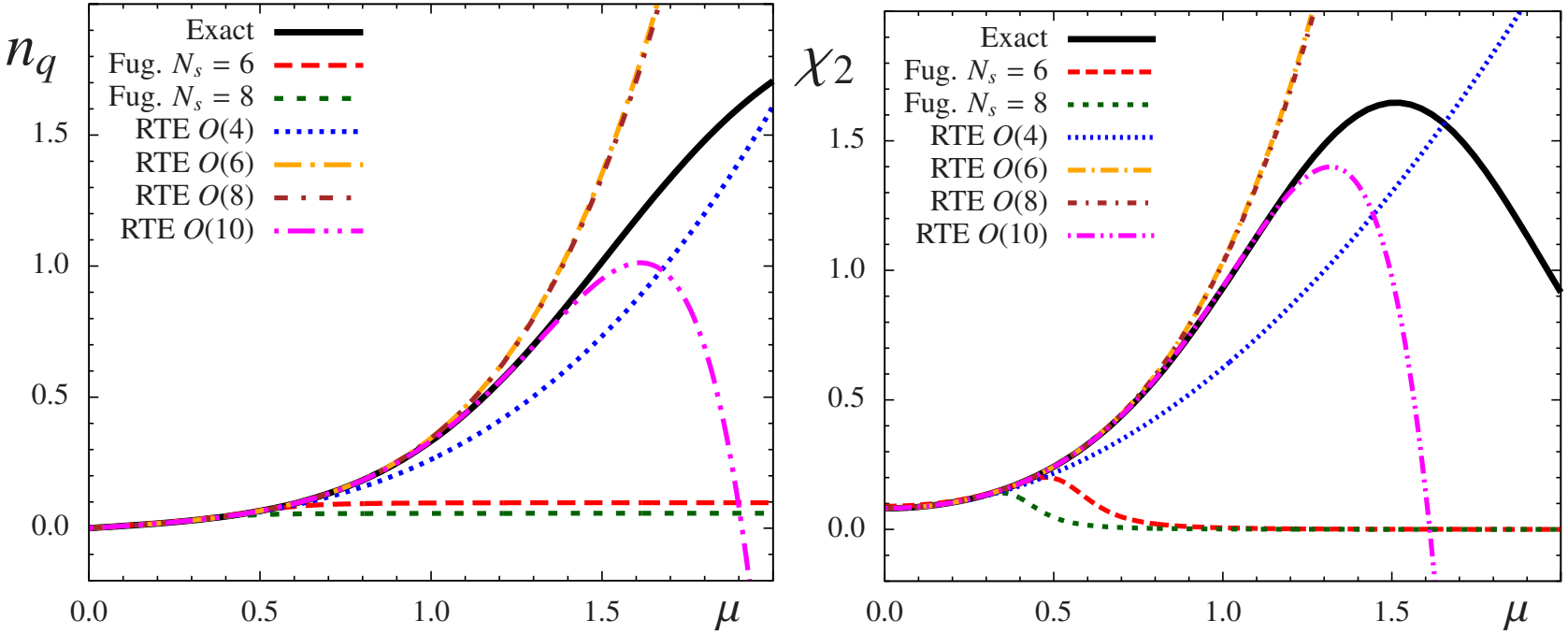
**+ higher derivatives (3rd and 4th) and ratios.**

$\langle \ldots \rangle_0$: expectation value evaluated on configurations with $\mu = 0$.

# Comparison with Taylor expansion - free theory

**Regular Taylor expansion:**

(free case for Wilson fermions) (preliminary)

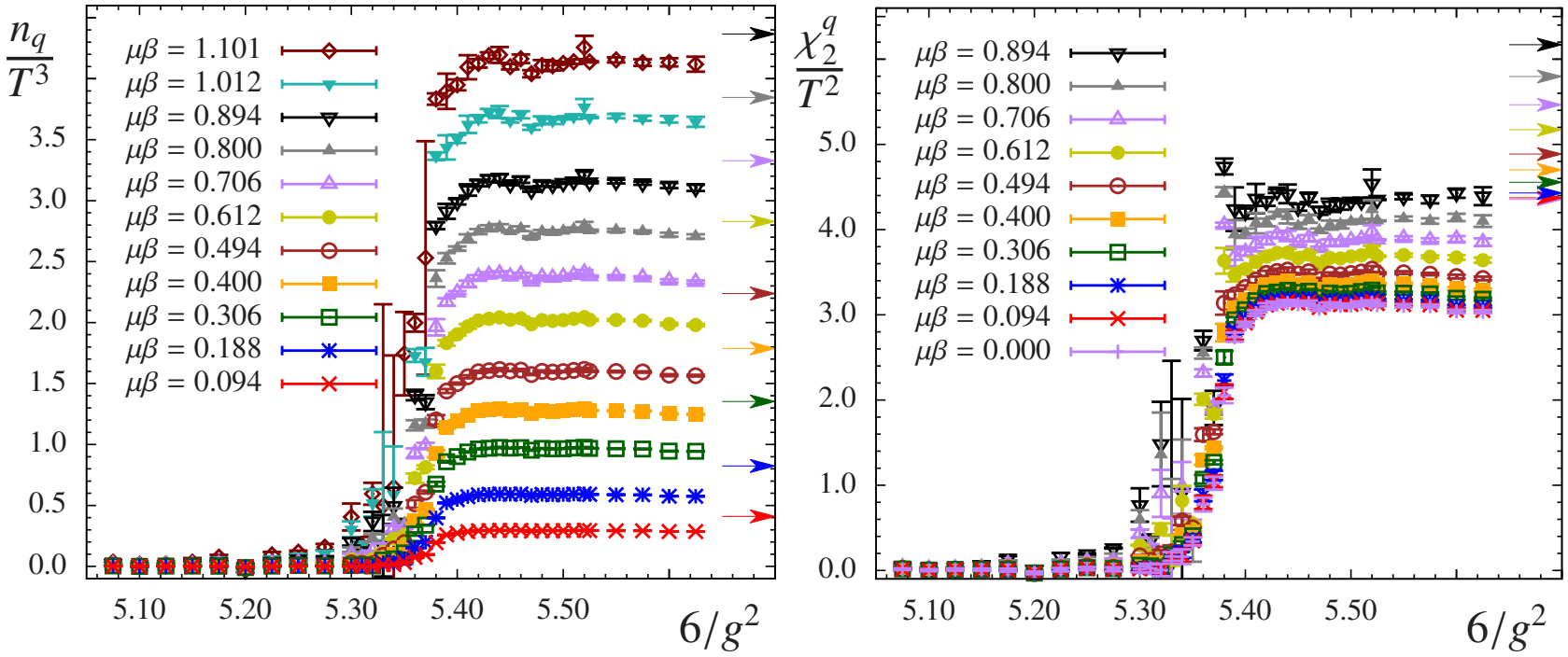



Quark number density (l.h.s.) and susceptibility (r.h.s.) as a function of $\mu$.

Taylor from M. Wilfling, C. Gattringer, PoS Lattice 2013, [arXiv:1311.7436 [hep-lat]].

# Wilson: Quark number density & susceptibility

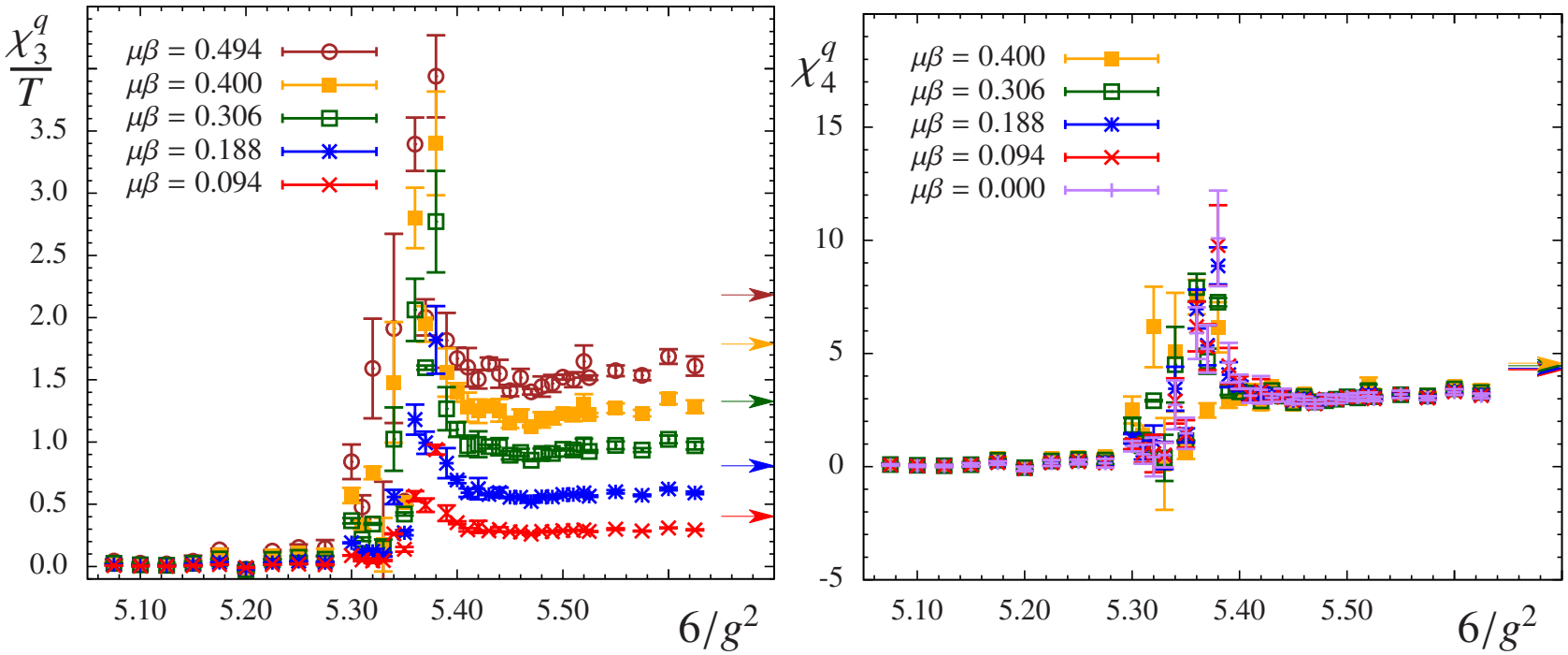
$(12^3 \times 6,\ \kappa = 0.162)$




Quark number density (l.h.s.) and susceptibility (r.h.s.) as a function of $\frac{6}{g^2}$.

# Wilson: Higher susceptibilities

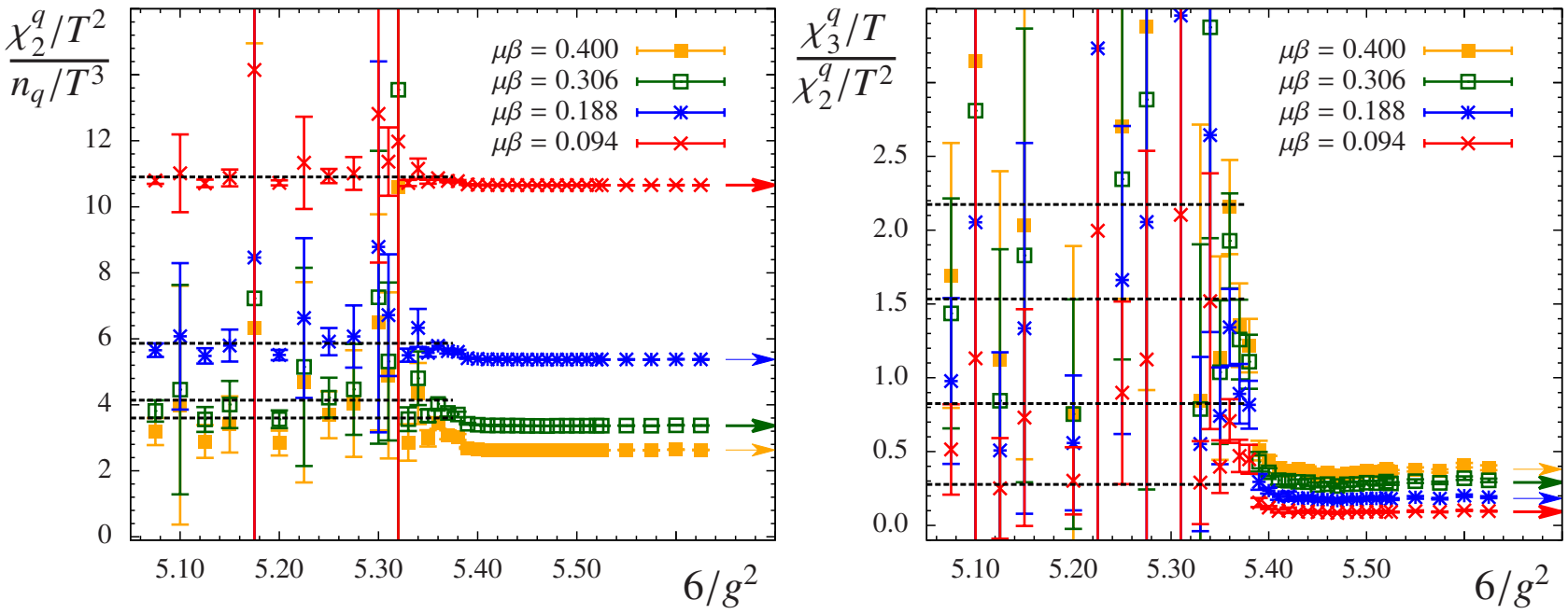
$(12^3 \times 6,\ \kappa = 0.162)$




3rd derivative (l.h.s.) and 4th derivative (r.h.s.) as a function of $\frac{6}{g^2}$.

# Wilson: Ratios of derivatives

$(12^3 \times 6,\ \kappa = 0.162)$

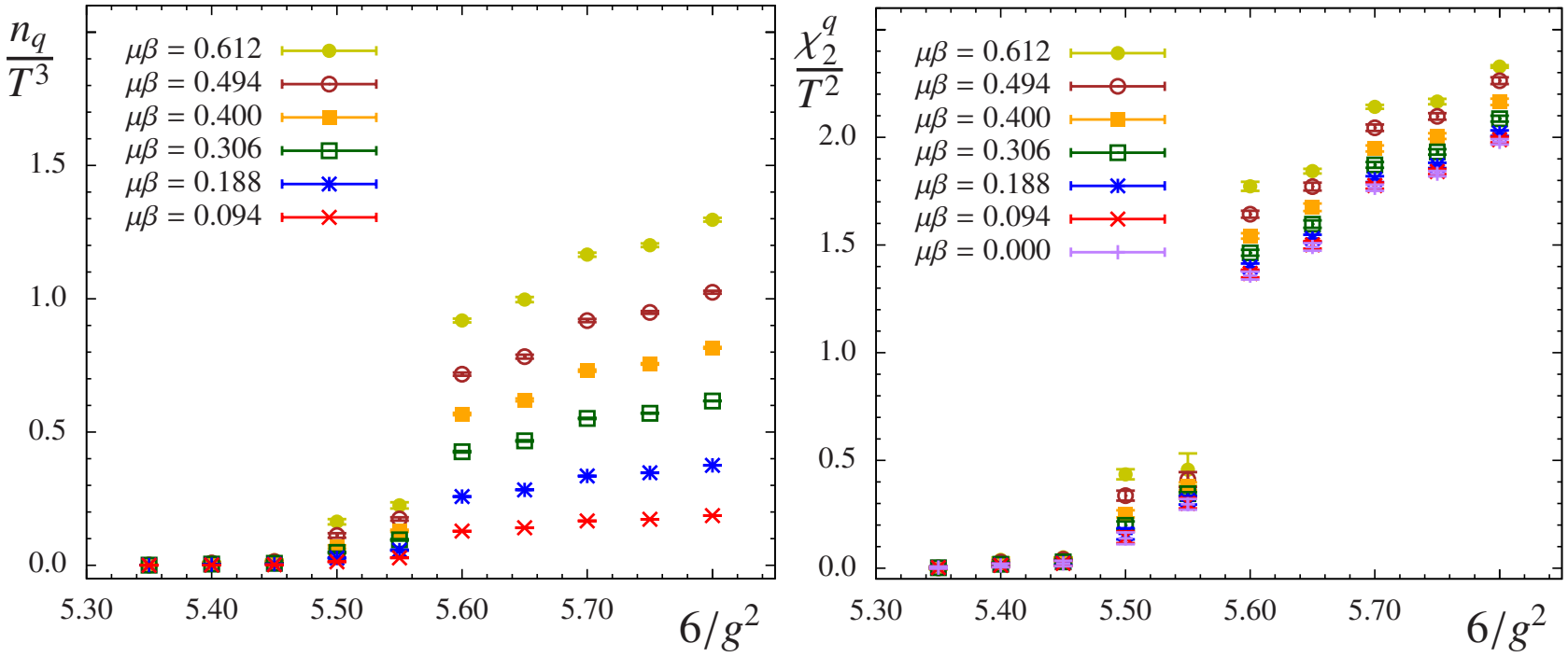



HRG (dashed lines): $\dfrac{\chi_2^q/T^2}{n_q/T^3} = 3\,\mathrm{sech}(3\mu\beta)$, $\dfrac{\chi_3^q/T}{\chi_2^q/T^2} = 3\tanh(3\mu\beta)$.

B. Friman, F. Karsch, K. Redlich, V. Skokov, Eur.Phys.J. C71 (2011) 1694

# Staggered: Quark number density & susceptibility

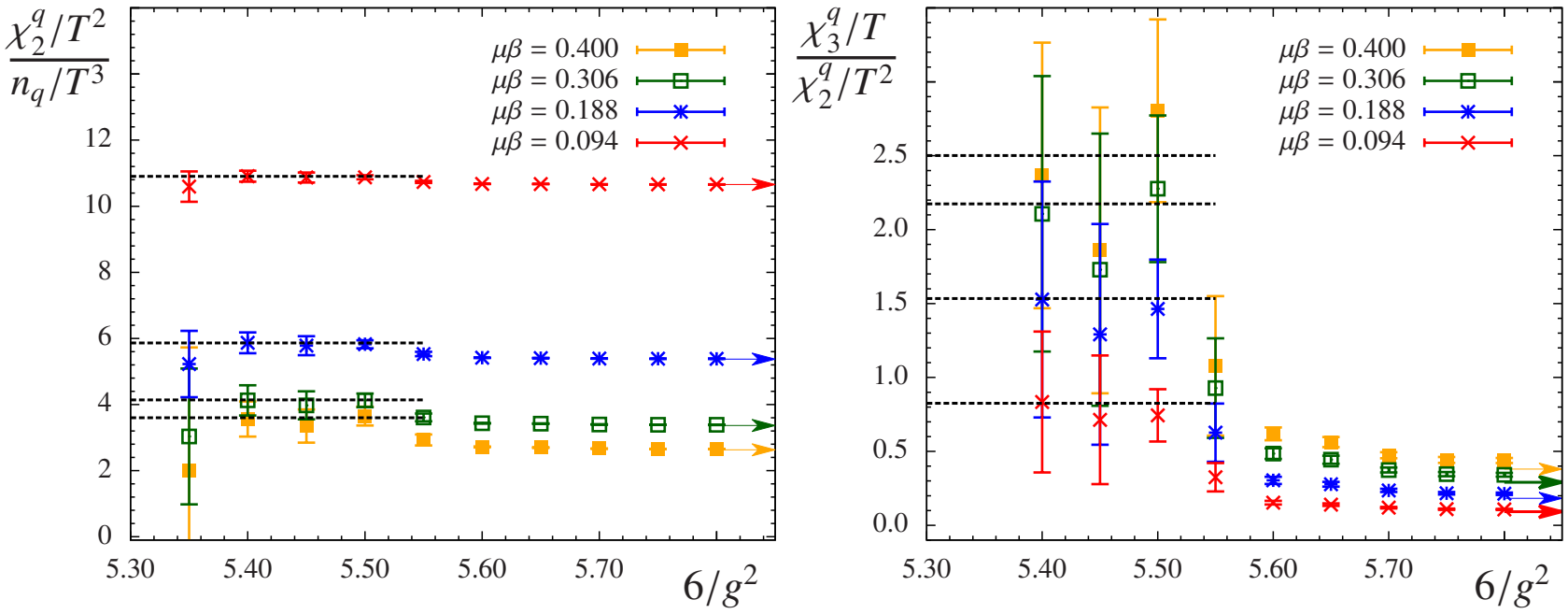
$(16^3 \times 6,\ m = 0.1)$




Quark number density (l.h.s.) and susceptibility (r.h.s.) as a function of $\frac{6}{g^2}$.

# Staggered: Ratios of derivatives

$(16^3 \times 6,\ m = 0.1)$




Note: HRG and free results are the same as in Wilson case.

**Summary:**

- Fugacity expansion: (finite) Laurent series in $e^{\mu\beta}$.
- Observables related to $n_q \Rightarrow$ moments of $D^{(q)}$.
- Used to continue from $\mu = 0$ to small chemical potential.
- Calculation numerically challenging $\Rightarrow$ accuracy.
- Ratios of susceptibilities are very robust (cf. Wilson/staggered).
- Conf. reg.: Reproduces HRG; deconf. reg.: Reaches free case.

**Outlook:**

- Comparison of full QCD results with other expansion techniques.
- Improve staggered resolution.

Thank you for your attention!

# Backup slides

# Full QCD Results - Lattice parameters

Two flavor degenerate **Wilson fermions** & Wilson gauge action:

- Lattices $N_s^3 \times N_t$: $8^3 \times 4$, $10^3 \times 4$, $12^3 \times 4$, $\mathbf{12^3} \times \mathbf{6}$ $\quad (\beta = N_t = 1/T)$
- Inverse coupling: $5.00 \leq \frac{6}{g^2} \leq 5.80$
- Lattice spacing: $0.343\,\text{fm} \geq a \geq 0.100\,\text{fm}$ (+ finer)
- Temperature: $100\,\text{MeV} \leq T \leq 320\,\text{MeV}$ (+ higher)
- $\kappa = 0.158$, $0.160$, $\mathbf{0.162}$ (pion mass: $m_\pi = 700\,\text{MeV} - 950\,\text{MeV}$)

Two flavor degenerate **staggered fermions** & Wilson gauge action:

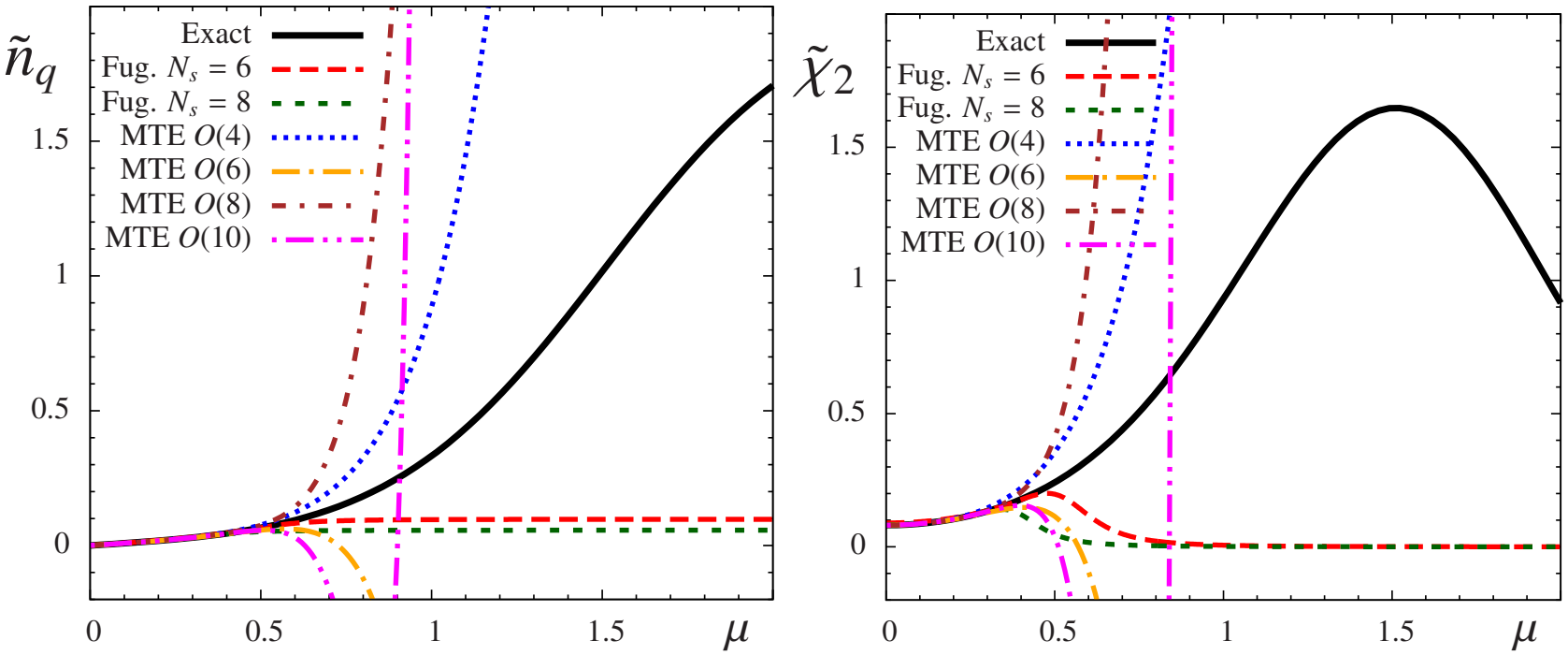
- Lattices $N_s^3 \times N_t$: $8^3 \times 4$, $\mathbf{16^3} \times \mathbf{6}$, $16^3 \times 8$ $\quad (\beta = N_t = 1/T)$
- Inverse coupling: $5.30 \leq \frac{6}{g^2} \leq 6.00$
- $m = \mathbf{0.10}$, $0.05$, $0.01$

Configurations generated in-house using MILC code.
(`http://www.physics.utah.edu/~detar/milc/`)

# Comparison with modified Taylor exp.

**Modified Taylor expansion:**

(free case for Wilson fermions) (preliminary)

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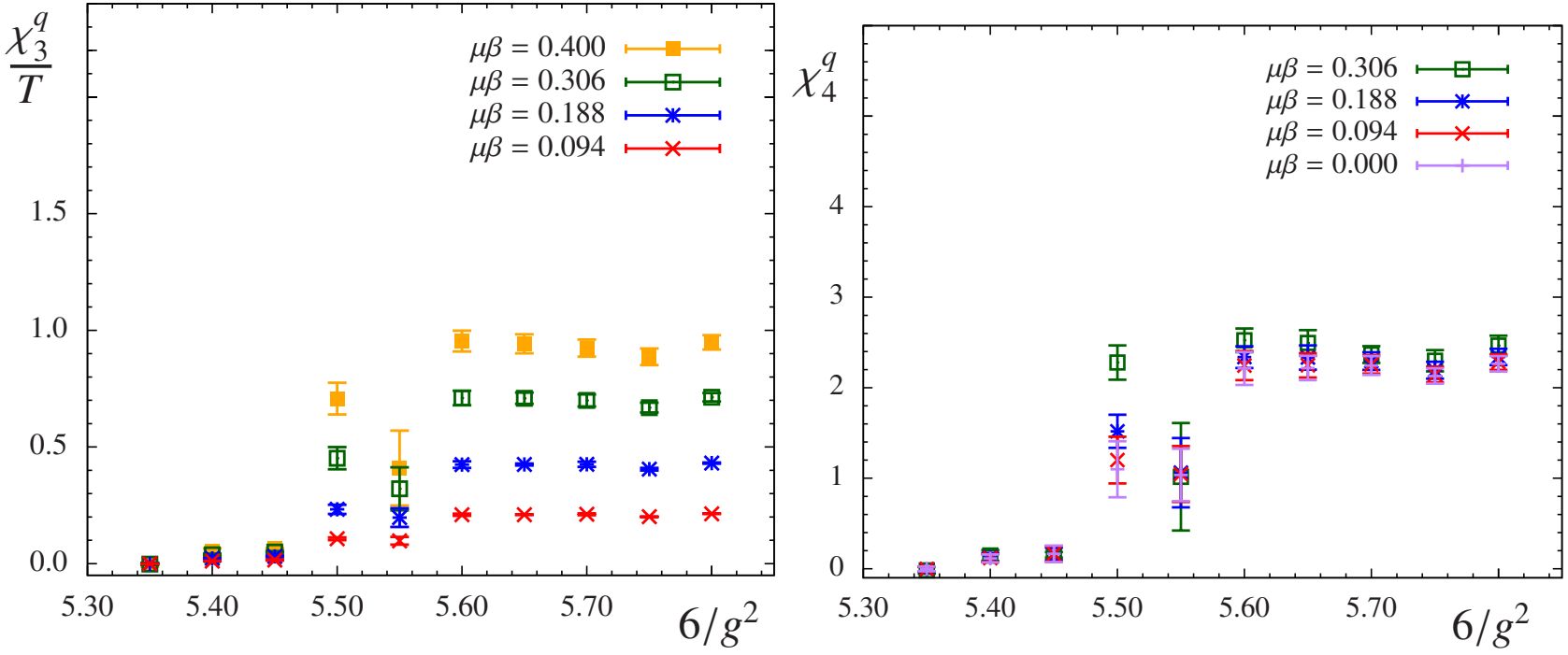


3rd derivative (l.h.s.) and 4th derivative (r.h.s.) as a function of $\frac{6}{g^2}$.