Radial Conformal Lattice Field Theory: Spherical Manifolds

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

Richard Brower Boston University

Lattice 2014 June 25

*RCB, G. Fleming, H. Neuberger, M. Cheng

Radial Quantization: *Early History*

- S. Fubini, A. Hanson and R. Jackiw PRD 7, 1732 (1972)

  Abstract: A field theory is quantized covariantly on Lorentz-invariant surfaces. Dilatations replace time translations as dynamical equations of motion. .... The Virasoro algebra of the dual resonance model is derived in a wide class of 2-dimensional Euclidean field theories.

- J. Cardy J. Math. Gen 18 757 (1985).

  Abstract: The relationship between the correlation length and critical exponents in finite width strips in two dimensions is generalised to cylindrical geometries of arbitrary dimensionality d. For d > 2 these correspond however, to curved spaces. The result is verified for the spherical model

Radial Quantization

Evolution: $H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$

$$ds^2 = dx^\mu dx_\mu = e^{2\tau}[d\tau^2 + d\Omega^2]$$

Can drop Weyl factor!

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$

$$"time" \quad \tau = log(r) \ , \ "mass" \quad \Delta = d/2 - 1 + \eta$$

$$D \to x_\mu \partial_\mu = r\partial_r = \frac{\partial}{\partial \tau}$$

Motivation

- (near) Conformal Field Theories, interesting for
  - BSM composite Higgs
  - AdS/CFT weak-strong duality
  - Model building & Critical Phenomena in general

- Lattices on $\mathbb{R}^d$ are problematic:
  - scales are exponentially divergent.

- Linear Hypercubic vs Exponential Radial Lattice

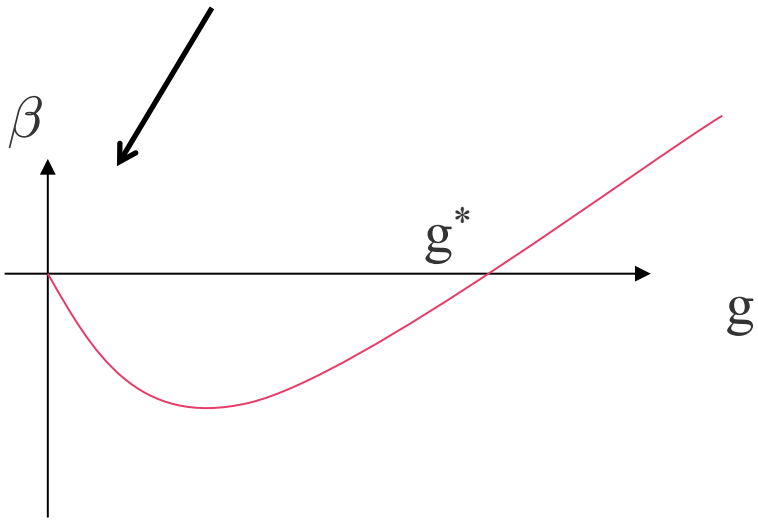
$$a < \Delta r < L \quad \text{vs} \quad a < \Delta log(r) < L$$

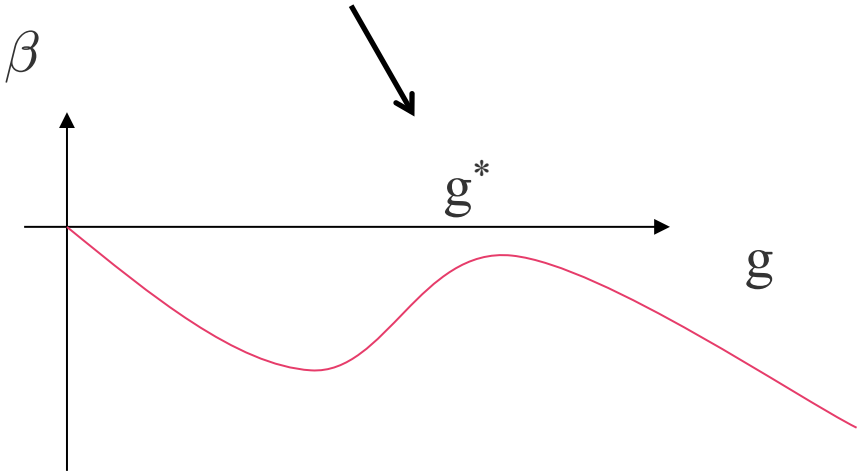
An IR fixed point can emerge already in the two-loop β function as you increase the number N_f of fermions.

$$\beta(g) = b_0 g^3 + b_1 g^5 + \cdots$$

$b_0 < 0$ for $N_f < 11N_c/2 = 16.5$

$b_1 > 0$ for $N_f > 153/14$





Conformal Window:
$16.5 > N_f > N^*_f$

Near conformal
$N_f < N^*_f$, close to N^*_f

CFT are highly constrained

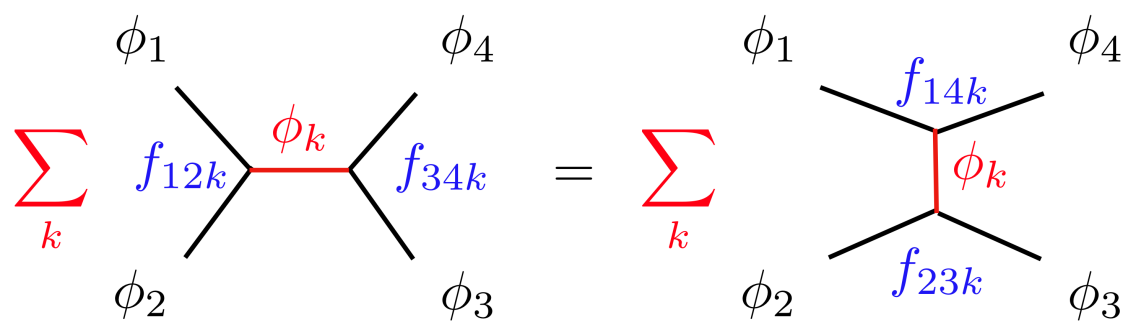
(i.e. Data: spectra + couplings to conformal blocks)

Exact 2 and 3 correlators

$$\langle \phi(x_1)\phi(x_2)\rangle = C\frac{1}{|x_1 - x_2|^{2\Delta}}$$

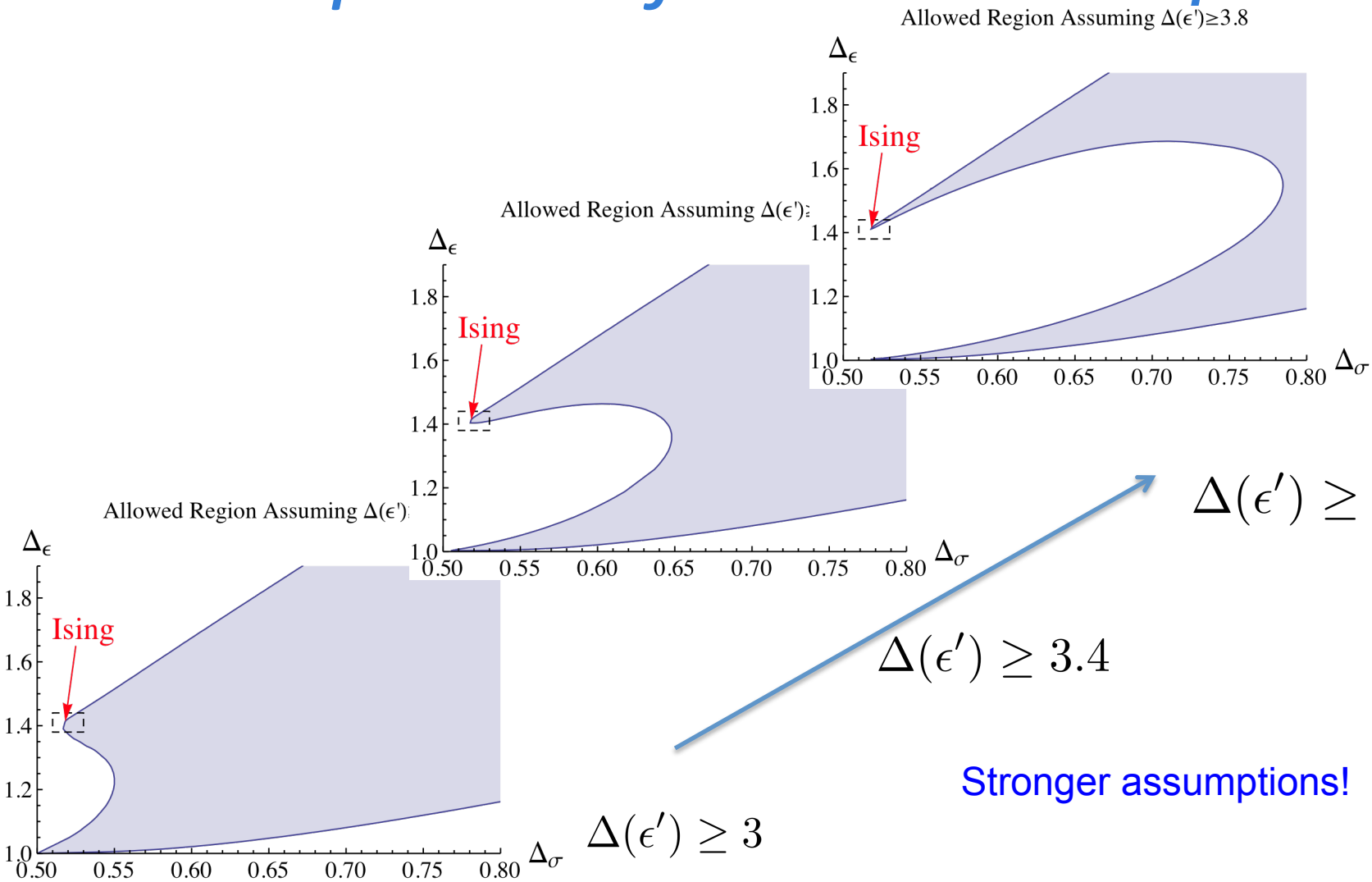
$$\mathcal{O}_i(x_1)\mathcal{O}_j(x_2) = \sum_k \frac{f_{ijk}}{|x_1 - x_2|^{\Delta_i+\Delta_j-\Delta_k}}\mathcal{O}_k(0)$$

Only "tree" diagrams!
"partial waves" exp: sum over conformal blocks



CFT Bootstrap: OPE & factorization completely fixed the theory

*Inequalities from Bootstrap**



Stronger assumptions!

- "Solving the 3D Ising Model with the Conformal Bootstrap" (El-Showk, Paulos, Poland, Rychkov, Simmons-Duffin and Vichi) arXiv:1203.6064v1v [hep-th] (2012)

Narrative

I. First attempt

-- *Lattice Radial Quantization: 3D Ising*

R.C.B., G.T. Fleming and H. Neuberger, Phys. Lett. B 721 (2013)

II. What worked and what failed.

III. Finite Elements Methods (FEM) to the rescue ?

IV. Future hopes and dreams

Exact CFT: Power Law Correlator

Conformal correlator: $\langle \phi(x_1)\phi(x_2)\rangle = C\dfrac{1}{|x_1 - x_2|^{2\Delta}}$

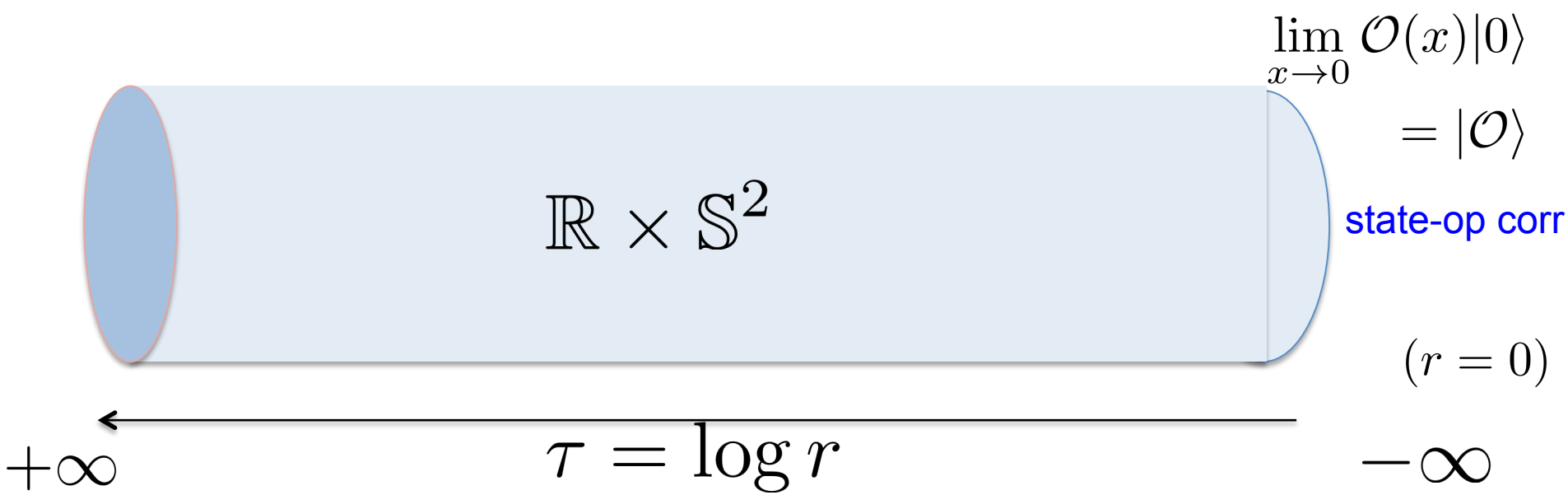
$$\begin{aligned} r_1^{\Delta} r_2^{\Delta} \langle \phi(\tau_1, \Omega_1)\phi(\tau_2, \Omega_2)\rangle &= C\frac{1}{[r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]^{\Delta}} \\ &\simeq Ce^{-(\log(r_2) - \log(r_1)\Delta} \\ &= Ce^{-\tau\Delta} \end{aligned}$$

With $|x_1 - x_2|^2 = r_1 r_2 [r_2/r_1 + r_1/r_2 - 2\cos(\theta_{12})]$

as $\tau = \log(r_2) - \log(r_1) \to \infty$

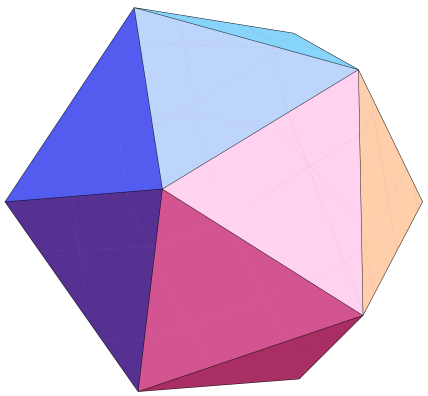
3-d Ising at Wilson-Fisher FP

$$Z_{Ising} = \sum_{\sigma(x,t)=\pm 1} e^{\beta \sum_{t,\langle x,y\rangle} \sigma(t,x)\sigma(t,y) + \beta \sum_{t,x} \sigma(t+1,x)\sigma(t,x)}$$

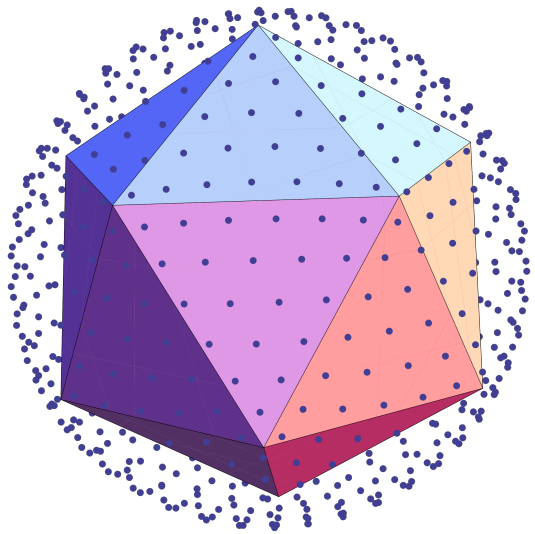


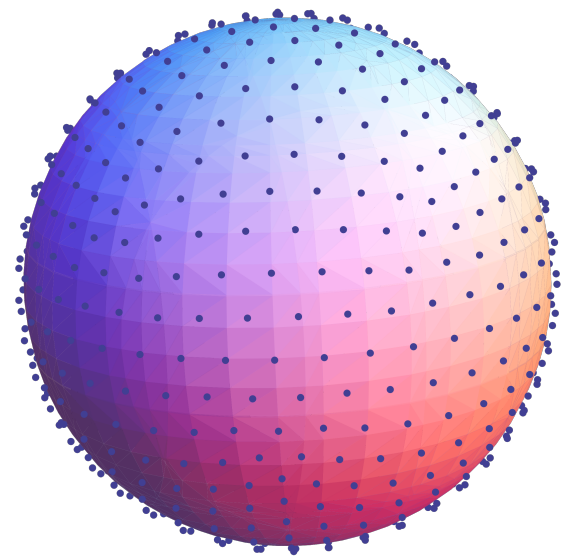
Order s Refined Triangulated Icosahedron

$s = 1$



$s = 8$

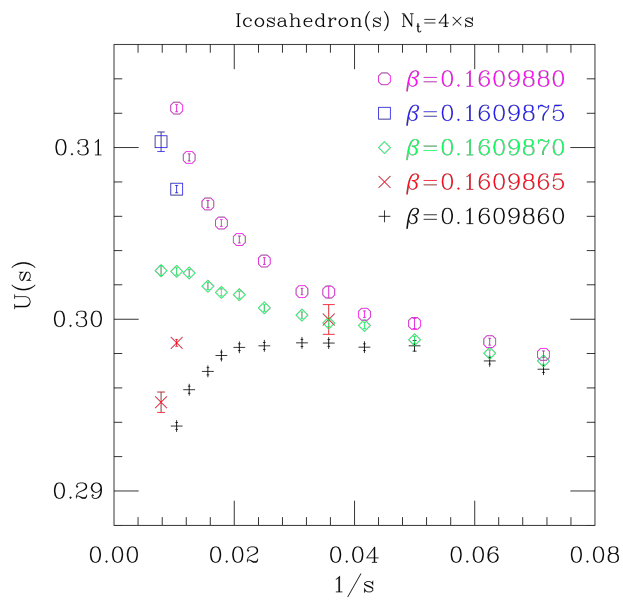




I = 0 (A),1 (T1) , 2 (H) are irreducible 120 Iscosahedral subgroup of O(3)

Fitting to Finite scaling

$$U[(\beta-\beta_{cr})L^{1/\nu},(\lambda-\lambda_{cr})L^{-\omega},...]\simeq$$
$$U^*(x)+O(L^{-\omega})\simeq U^*(0)+a_1(\beta-\beta_{cr})L^{1/\nu}+c(\lambda)L^{-\omega}+\cdots$$



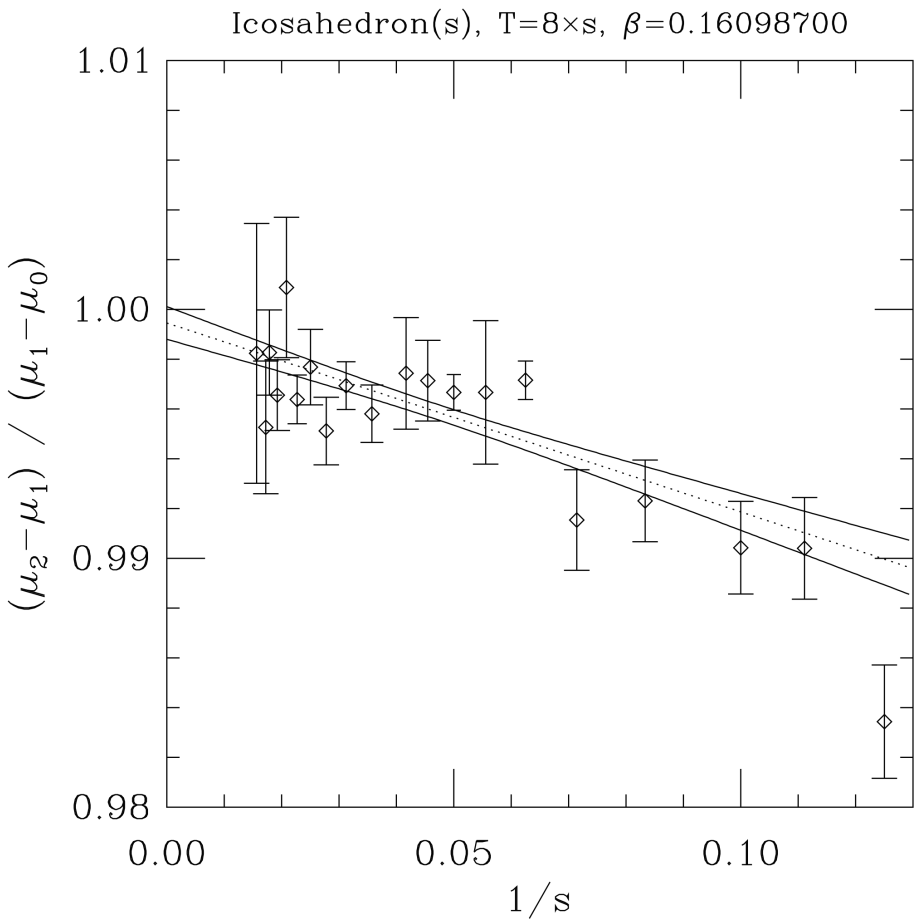
$$U(\beta,s=L)=1-\frac{\langle M^4\rangle}{3\langle M^2\rangle^2}$$

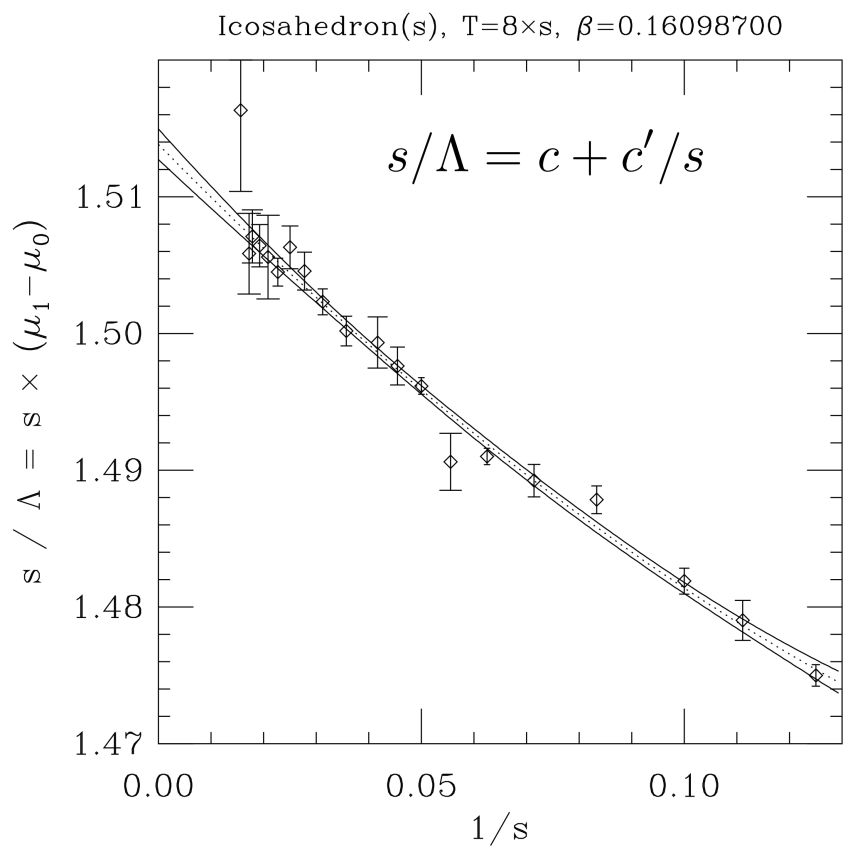
$$\beta_{cross}\simeq\beta_{cr}+c_1L^{-1/\nu-\omega}$$

$$\beta_{crit}=0.16098703(3)$$

Double Scaling: $x=(\beta-\beta_{cr})L^{1/\nu}$

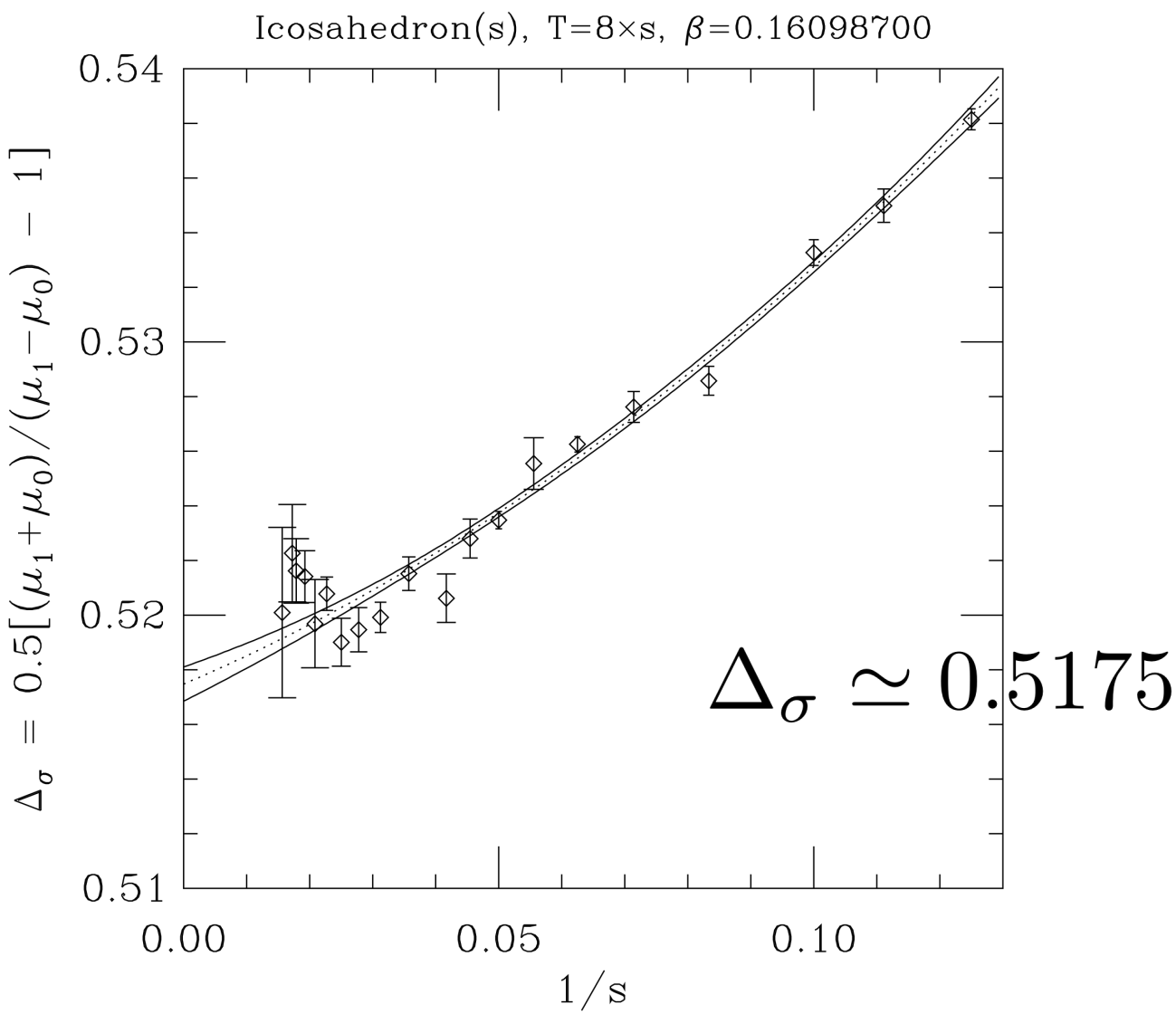
Determine "speed of light" via Descendant Relation & rescale "log(r)"





$$c = 1.5105(7)$$

Current Fit:



$$\Delta_\sigma \simeq 0.5175$$

Improved cluster Estimator

Swendsen-Wang: Real space

$$g(x-y) = \langle s_x s_y \rangle \simeq \frac{1}{N_{config}} \sum_{i=1}^{N_{config}} \sum_{C_i} \Delta_{C_i}(x) \Delta_{C_i}(y)$$

$$\Delta_C(x) = \quad 1 \text{ if } x \in C \text{ else } 0$$

Wolff single cluster

$$\widetilde{g}_{lm}(k) \simeq \frac{1}{N_{config}} \sum_{i=1}^{N_{config}} \frac{1}{|C|} | \sum_{t,x \in C} e^{i2\pi kt/L_t} Y_{lm}(\Omega_x)|^2$$

Note: All to All O(V) improved estimator in Momentum space *

*C. Ruge, P. Zhu and F. Wagner Physica A (1994) 431:

Numerical Test

- Equal spacing test of descendants:

$$\frac{\mu_2 - \mu_1}{\mu_1 - \mu_0} = 0.999(1)$$

- "Speed of light" $c = 1.5105(7)$

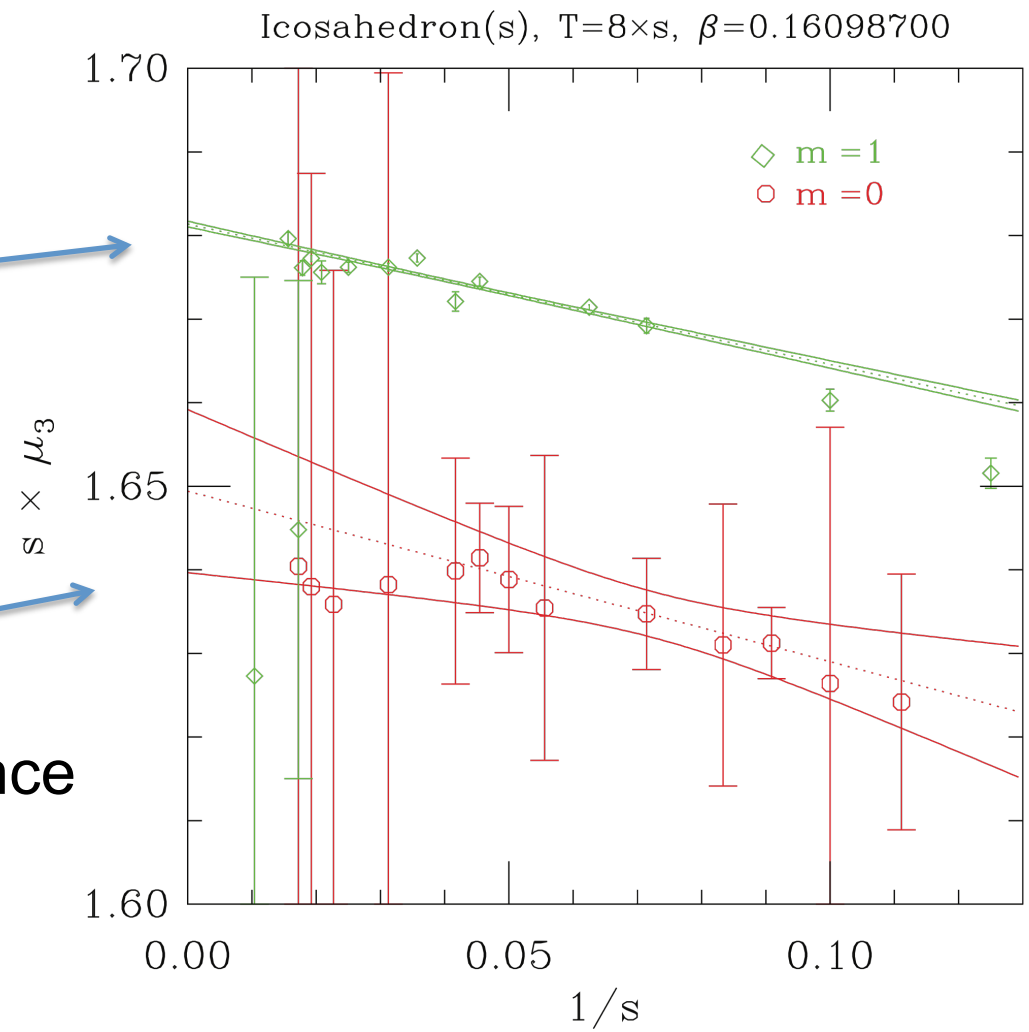
- But critical point $\beta_{crit} = 0.16098703(3)$

- Current anomalous dimensions (more soon)
  - from Binder: $\omega + 1/\nu = 2.51(11)$
  - from corr: $\Delta_\sigma = 1/2 + \eta/2 = 0.5175(6)$
  - Simulation are on going to reduce errors

Wrong Theory?
Failure to recover O(4,1) of l = 3?

G rep

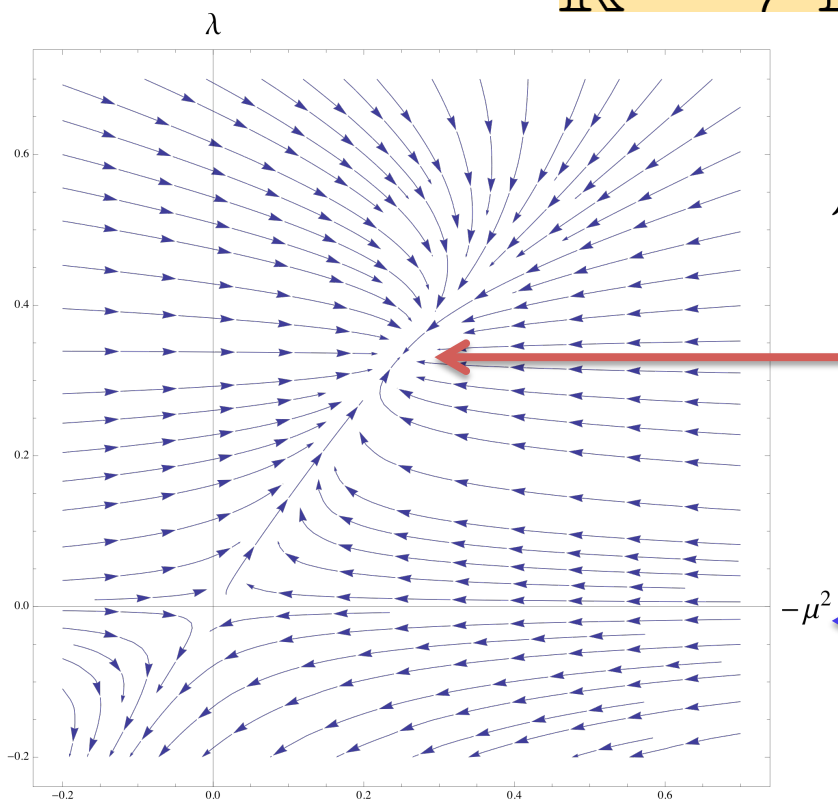
T2 rep



Apparent lack of convergence to a single O(3) irreducible representation for l = 3

Replace Ising Model by phi 4th

$$\mathbb{R}^d \to \mathbb{R} \times \mathbb{S}^{d-1}$$



$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu/2\lambda)^2$$

Wilson-Fisher FP

Gaussian FP

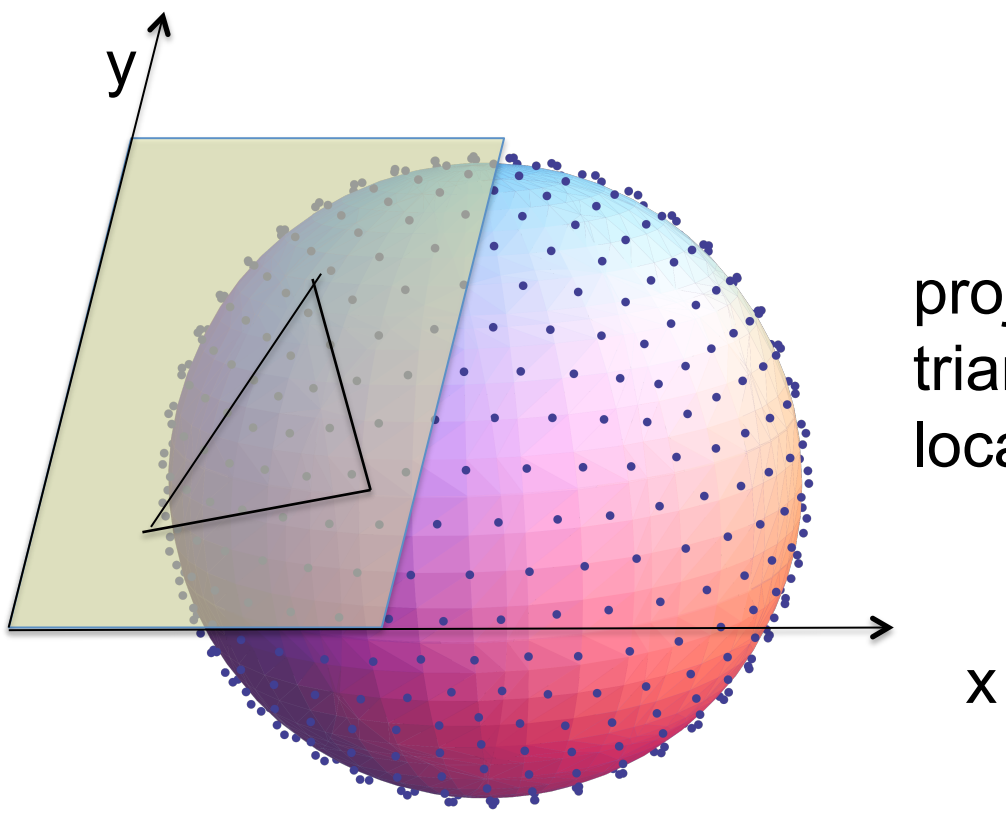
$$\beta_g = \epsilon g - \frac{3}{16\pi^2} g^2 + O(g^3, \epsilon g^2, \mu^4, \mu^2 g;)$$

$$\beta_{\mu^2} = 2\mu^2 + ag + \frac{9}{16\pi^2} g\mu^2 + O(\mu^4)$$

$$\lambda = 4g/4!$$

Finite element Discrete Lagrangian on triangulated sphere.

$$Z = \int \mathcal{D}\phi e^{-\frac{1}{2}K_{ij}(\phi_i - \phi_j)^2 - \lambda\omega_i(\phi^2 - \mu/2\lambda)^2}$$



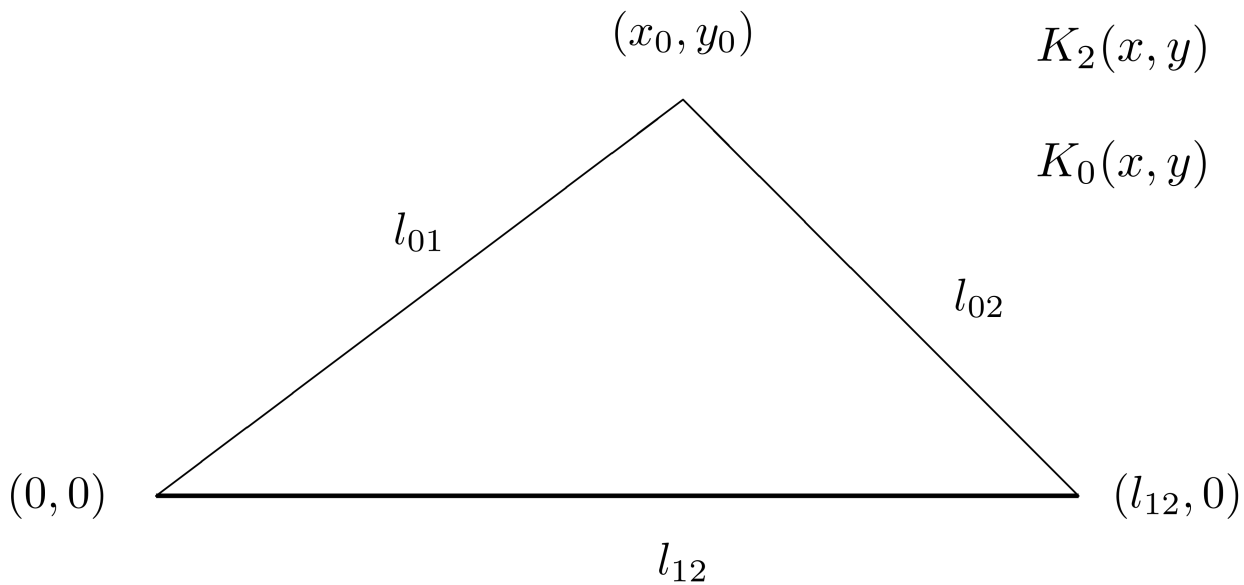
project spherical triangle onto local tangent plane

Kinetic term for Linear Element

$$K_1(x,y) = [l_{12} - x - \frac{(l_{12}-x_0)y}{y_0}]/l_{12}$$

$$K_2(x,y) = [x - \frac{x_0 y}{y_0}]/l_{12}$$

$$K_0(x,y) = \frac{y}{y_0}$$



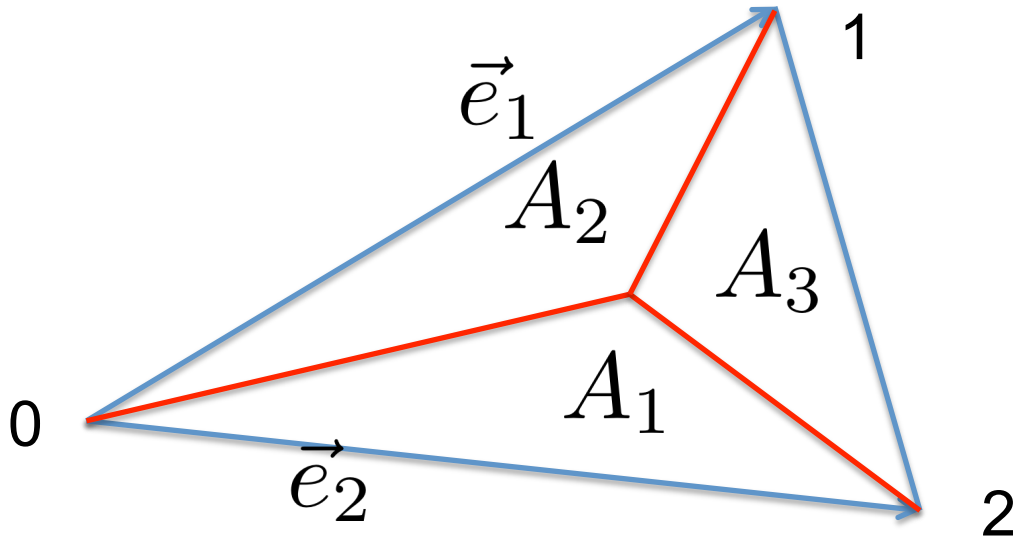
On each triangle expand: $\phi(x,y) = \sum_i K_i(x,y)\phi_i$ an integrate

$$\int_{A_{012}} dxdy\partial_\mu\phi(x,y)\partial_\mu\phi(x,y) = \frac{1}{2A_{012}}[(l_{01}^2 + l_{20}^2 - l_{12}^2)(\phi_1 - \phi_2)^2 + \text{cyclic}]$$

See also: Christ, Friedberg, Lee ion “Random Lattice” NP (1982)

Linear Finite Element Method for triangulate Manifold

$$g_{ij}(0) = \vec{e}_i \cdot \vec{e}_j$$



$$K_i^{\triangle}(r) = \frac{A_i}{A_{123}}$$

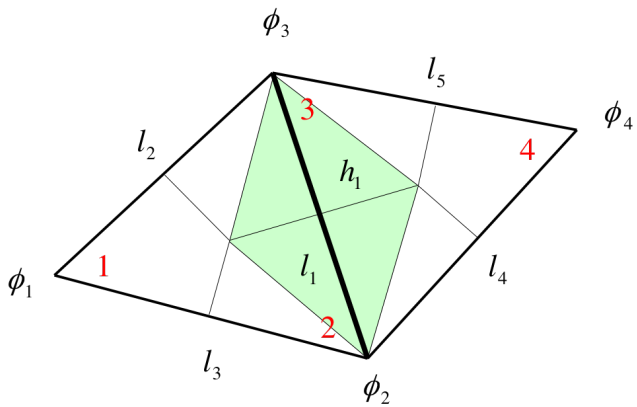
$$W_i(r) = \sum_{\triangle} K_i^{\triangle}(r)$$

$$\phi(r) = \sum_i W_i(r)\phi_i$$

Piecewise linear
subspace of Hilbert space

$$d\vec{x} = \vec{e}_i dx^i \qquad ds^2 = d\vec{x} \cdot d\vec{x} = g_{ij} dx^i dx^j$$

Regge Calculus formulation for smooth manifold.



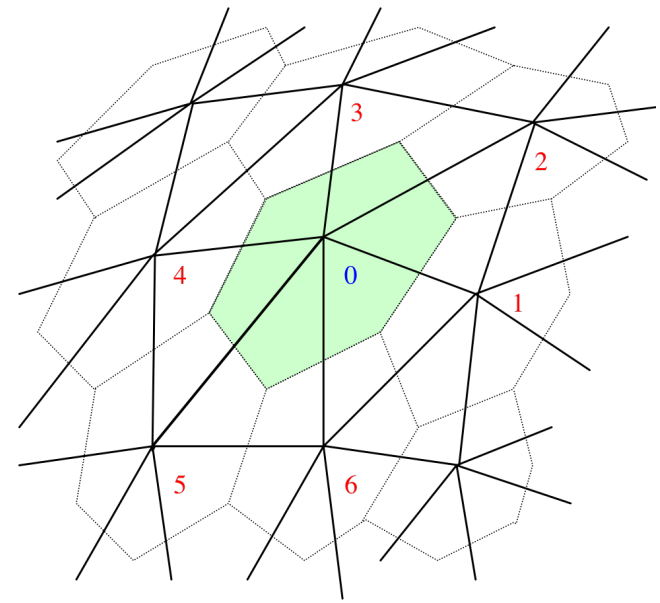
$$FEM : A_d \frac{(\phi_2 - \phi_3)^2}{l_1^2}$$

$$\text{Delaunay Link Area:} \quad A_d = h_1 l_1$$

$$\sum_{\triangle_{kij}} \sqrt{g(k)} g^{ij}(k) \frac{(\phi_k - \phi_i)(\phi_k - \phi_j)}{l_{ki} l_{kj}}$$

H. Hamber, S. Liu, *Feynman rules for simplicial gravity,* NP B475 (1996)

Einstein Regge Curvature



$$\delta_v = 2\pi - \sum_{i \in V} \theta_i$$

$$\sum_v \delta_v = 2\pi\chi = 2\pi(F - E + V)$$

Could Optimize adaptive Delaney triangles on unit sphere

$$\int d^2x\sqrt{g}[\lambda - kR^2 + aR^2] \implies \sum_v A_v[\lambda - 2kR_v + aR_v^2]$$

$$R_v = 2\delta_v/A_v$$

flat triangles: $\delta_v = 4\pi/A_v$

FEM have "spectral fidelity"

- Taylor expansion on hypercubic lattice:

$$a^{-1}\sum_{\pm\mu}(\phi(x)-\phi(x+a\mu)^2 \simeq (\nabla\phi)^2 + O(a^2)$$

- Taylor series for FEM does not work!

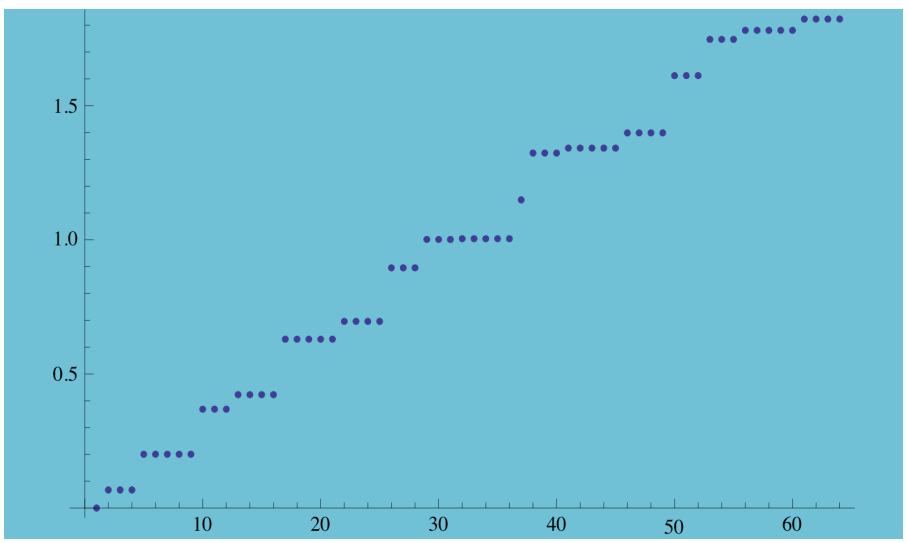
$$a^2\sum_{y}K(x,y)(\phi(x)-\phi(y))^2 \simeq c_{\mu\nu}\partial_\mu\phi\partial_\nu\phi + O(a^2)$$

- FEM "meta-theorem": spectrum < cut-off to O(a^2) if "triangles are regular enough"

FEM fixes the huge Spectral defects

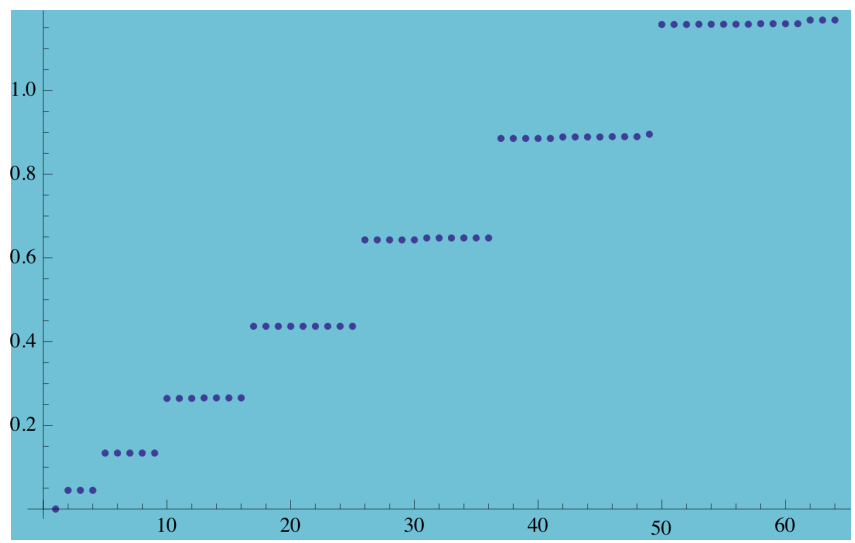
For s = 8 first (l+1)*(l +1) = 64 ev

BEFORE (K = 1)



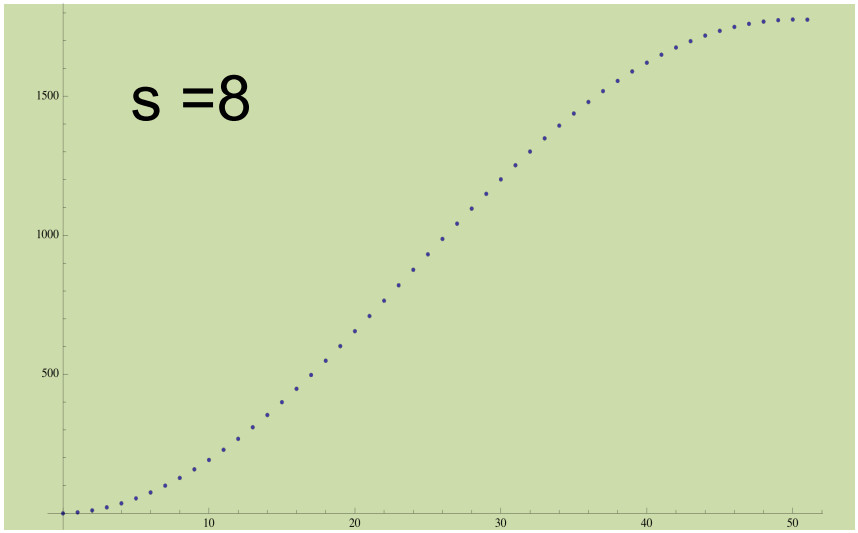
l, m

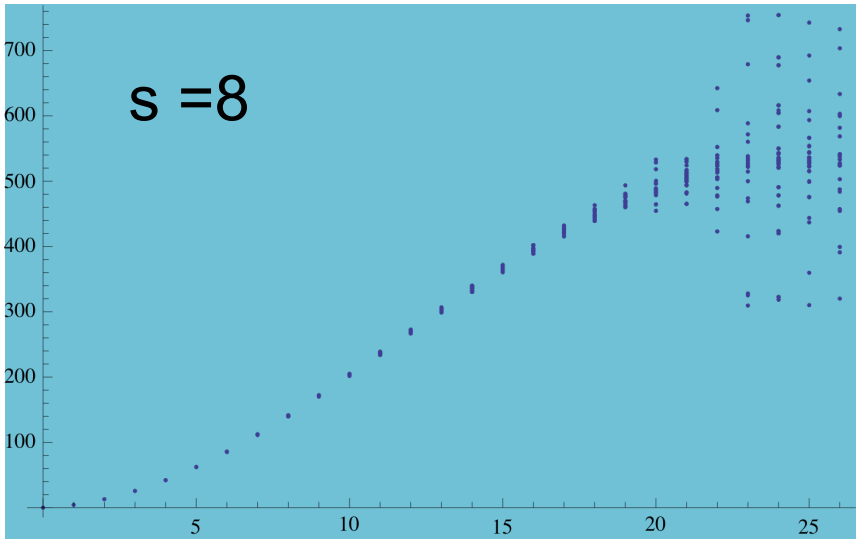
AFTER (FEM K's)

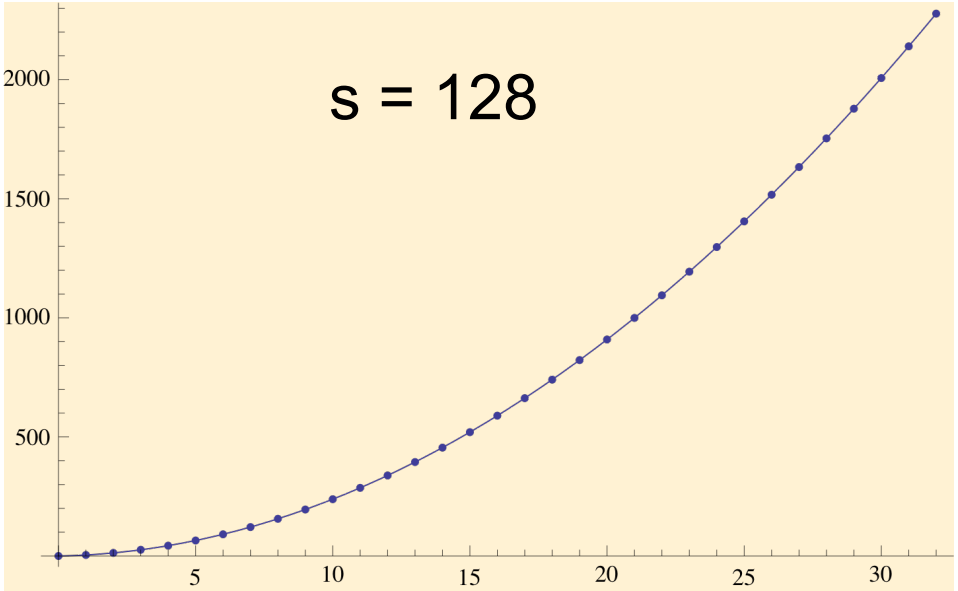


l, m

Spectrum of FE Laplacian on a sphere





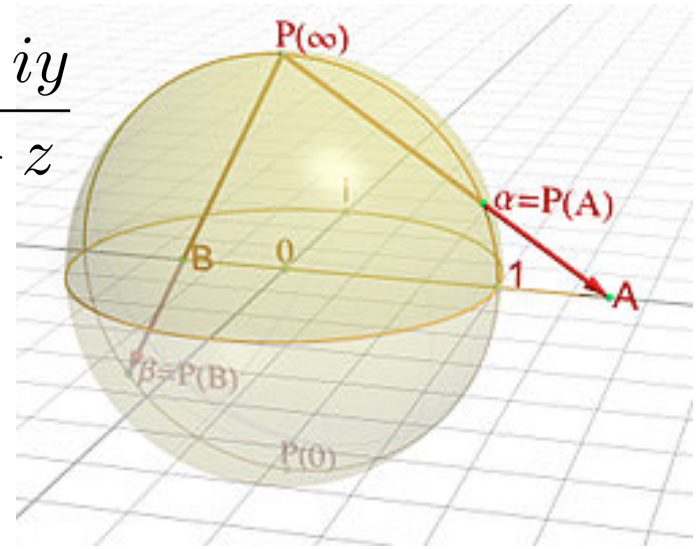


Fit

$$l + 1.00012\, l^2 - 13.428110^{-6}\, l^3 - 5.5724410^{-6}\, l^4$$

2D test: Conformal Projection to Riemann Sphere

projection $\xi = \tan(\theta/2)e^{-i\phi} = \dfrac{x+iy}{1+z}$



$$x^2 + y^2 + z^2 = 1$$

Exact Two point function

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \to \frac{1}{|1 - cos\theta_{12}|^{\Delta}}$$

$$\Delta = \eta/2 = 1/8$$

4 pt function

$$(x_1, x_2, x_3, x_4) = (0, \xi, 1, \infty)$$

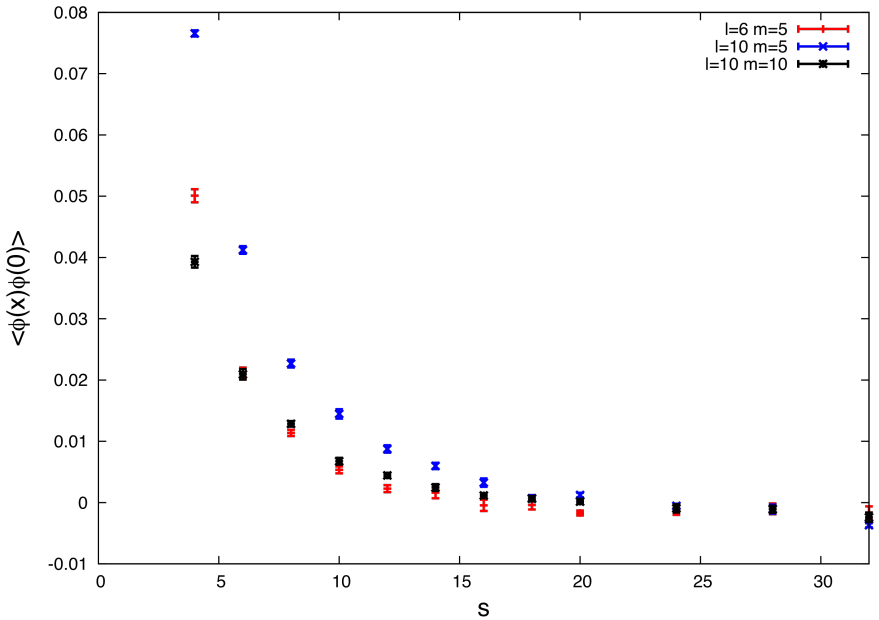
$$g(0,\xi,1,\infty) = \frac{1}{2|\xi|^{1/4}|1-\xi|^{1/4}}[1 + \sqrt{1-\xi}| + |1 - \sqrt{1-\xi}|]$$

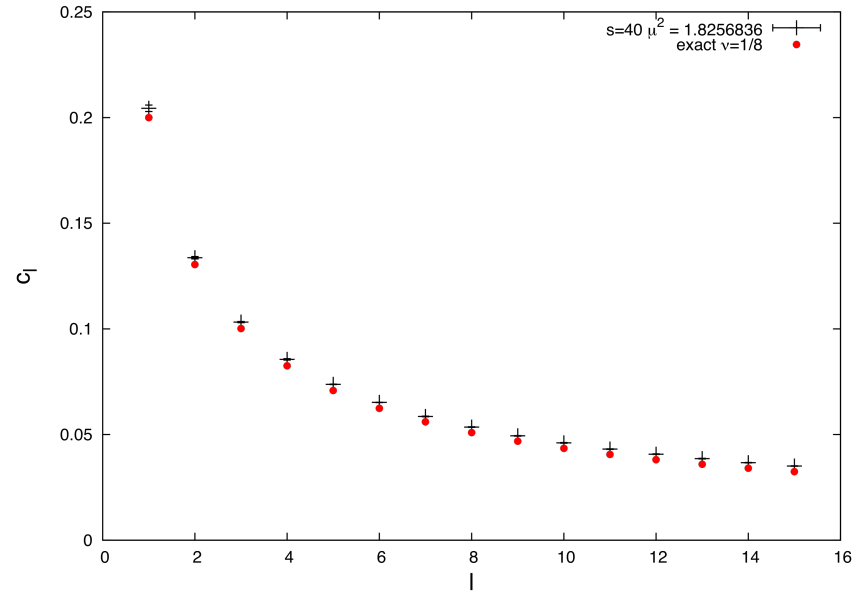
Critical Binder Commulant

$$U_B^* = 1 - \frac{\langle M^4\rangle}{3\langle M^2\rangle\langle M^2\rangle} = 0.567336$$

Test of rotational symmetry?

Ylm projection.





$$\Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

The simulation program is running

(1) Monte Carlo is a "standard" mixture of metropolis, over relax and Wolff methods from:

Ulli Wolff, "Collective Monte Carlo Updating for Spin Systems PRL 62: 361 (1989)

R.C.B. and P. Tamayo, "Embedded Dynamics for phi4 Theory", PRL 62:1087(1989)

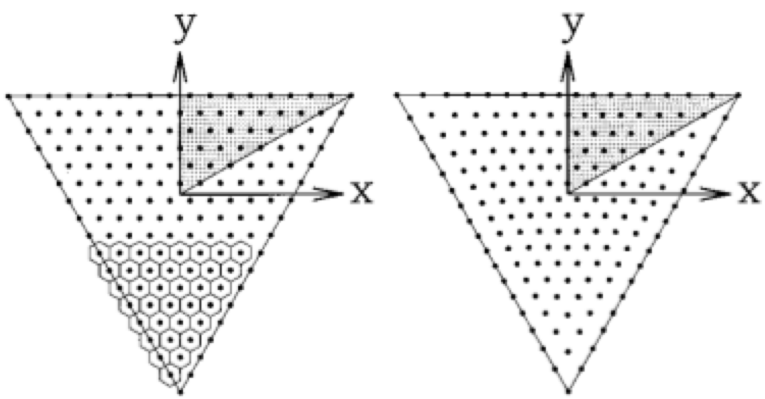
(2) Will compute higher primaries, even Z2 sector, Energy momentum tensor, Conformal Blocks partial waves

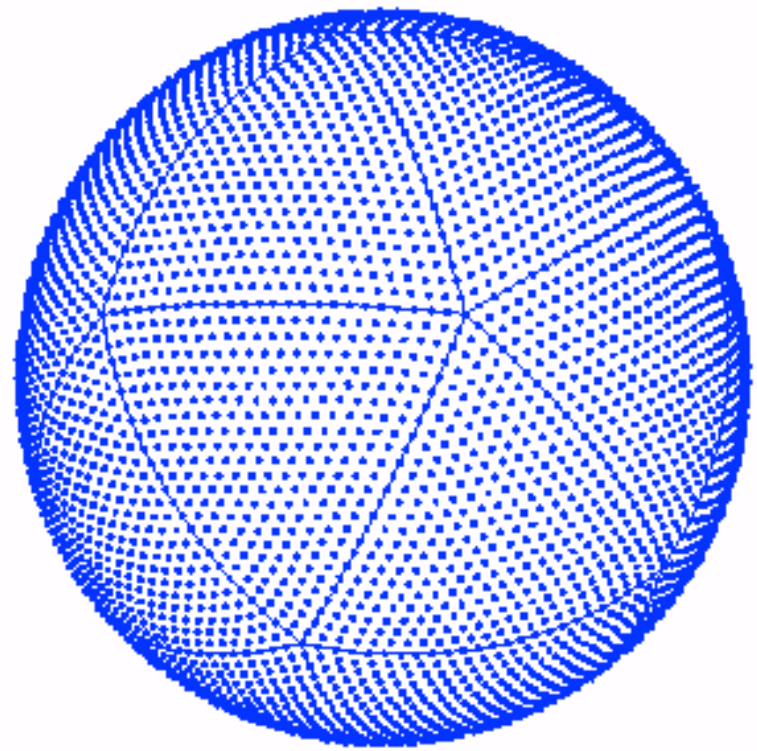
(3) The code can run any graph, so we will replace sphere by torus to reproduce phi 4th numbers from Hasenbusch,..

Hasenbusch,``A Monte Carlo study of leading order scaling corrections of phi**4 theory on a three-dimensional lattice" J.Phys. A 32 (1999) 4851 *

Accurate Results require
Better Triangulation, Counter Terms etc

Equal AREA simplex
for WMAP Ski!





An icosahedron-based method for pixelizing the celestial sphere
MAX TEGMARK THE ASTROPHYSICAL JOURNAL, 470:L81–L84, 1996 October 20

Future Challenges & Directions

- Many extensions are interesting
  - Easier problems:
    - Prove radial quantization for O(N) at large N
    - Strengthen bootstrap inequalities for spin systems?
  - Harder Problems:
    - Gauge fields (with discrete Chirstoffel connection)?
    - Fermions (with discrete spin connection) ?
    - Flow from UV to conformal IR fixed points for BSM? (Cross over from UV, to Hamiltonian to Dilation spectrum)

EXTRA SLIDES

Fancy stuff: "FEM" or discrete Exterior Calculus

Need a local reference tangent plane $\xi^a(x)$at x.
Introduce an ortho normal basis in the tangent space: $\vec{e}_a(x)$

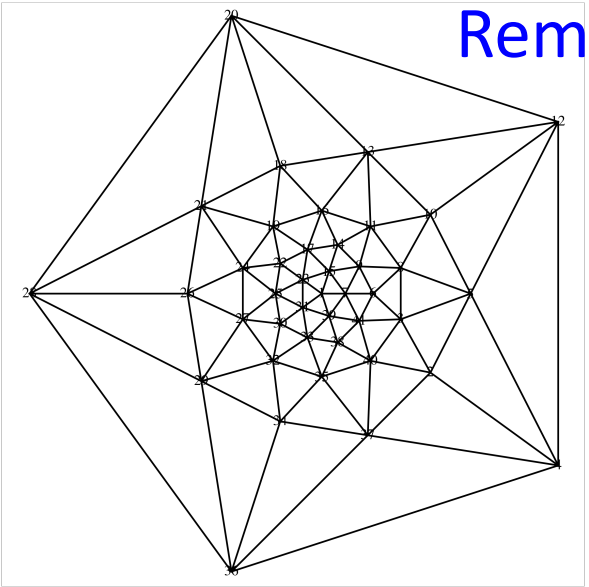
$$g_{\mu\nu}(x) = e^a_\mu(x) e^a_\nu(x) \qquad \bar{\psi} e^\mu_a \gamma^a D_\mu \psi$$

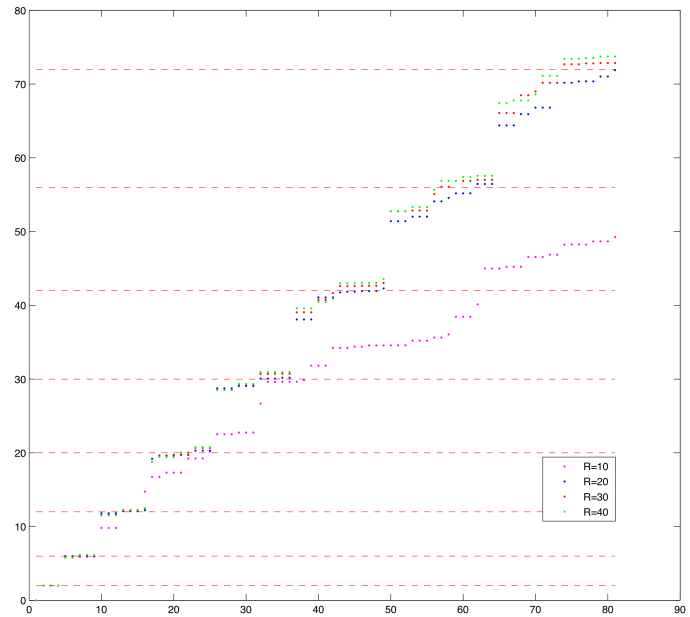
Lattice Fermions are on simplicial complex lattice manifolds with great care! Spin connection has be done carefully.

Compact gauge links can be represented also a la Christ, Friedberg, Lee! In weak field limit maybe equivalent to using Nedelic/Whitney "edge" elements etc.

Simplicial differential form, deRham complex a la Regge Calculus!

Remarks on Riemann Sphere





Deluanay triangulation commutes with projection!

Spectrum on Cubic Sphere!

- Testing
  - Exact 2D Ising correlators, exponents, central charge, etc.
  - Improved spherical and/, 2nd order elements, etc
  - Dynamical R^2 Curvature Constrained Triangulation.