The Chiral Condensate of One-Flavor QCD and the Dirac Spectrum at $\theta = 0$

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Relevant Papers

L. Ravagli and J.J.M. Verbaarschot, QCD in One Dimension at Nonzero Chemical Potential, Phys. Rev. D76 (2007) 05406 [arXiv:0704.1111]

J.C. Osborn K. Splittorff and J. J. M. Verbaarschot, Chiral Symmetry Breaking and the Dirac Spectrum at Nonzero Chemical Potential, Phys. Rev. Lett. 94 (2005) 202001 [arXiv[hep-th/0501210]].

J.J.M. Verbaarschot and T. Wettig, The Spectrum of the Dirac Operator for QCD with on flavor at fixed θ angle, (2014), in preparation.

P.H. Damgaard, Topology and the Dirac Operator Spectrum in Finite Volume Gauge Theory, Nucl. Phys. B556 327 (1999).

H. Leutwyler and A. Smilga, Spectrum of Dirac Operator and Role of Winding Number in QCD, Phys. Rev. D 46 (1992) 5607.

Contents

One Flavor QCD

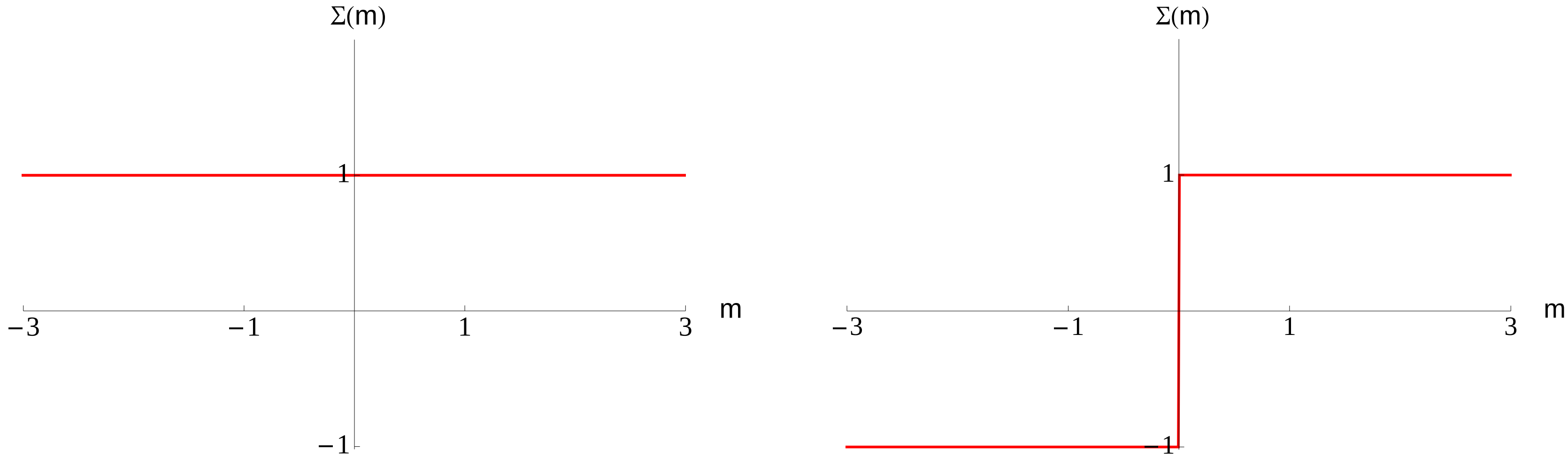
- Chiral symmetry is broken by the anomaly.
- There is no spontaneous symmetry breaking and no Goldstone bosons.
- The mass dependence of the one flavor QCD partition function is given by

$$Z = e^{mV\Sigma\cos\theta + O(m^2V)}.$$

- For $N_f = 2$ with spontaneous symmetry breaking, the mean field estimate of the partition function is given by (for $\theta = 0$)

$$Z = e^{|m|V\Sigma + O(m^2V)}.$$

Chiral Condensate



Behavior of the chiral condensate for $N_f = 1$ (left) and $N_f = 2$ (right).

$$\Sigma(m) = -\langle \bar{q}q \rangle \frac{d}{dm} \log Z(m)$$

The goal of this talk is to explain this behavior in terms of the Dirac spectrum.

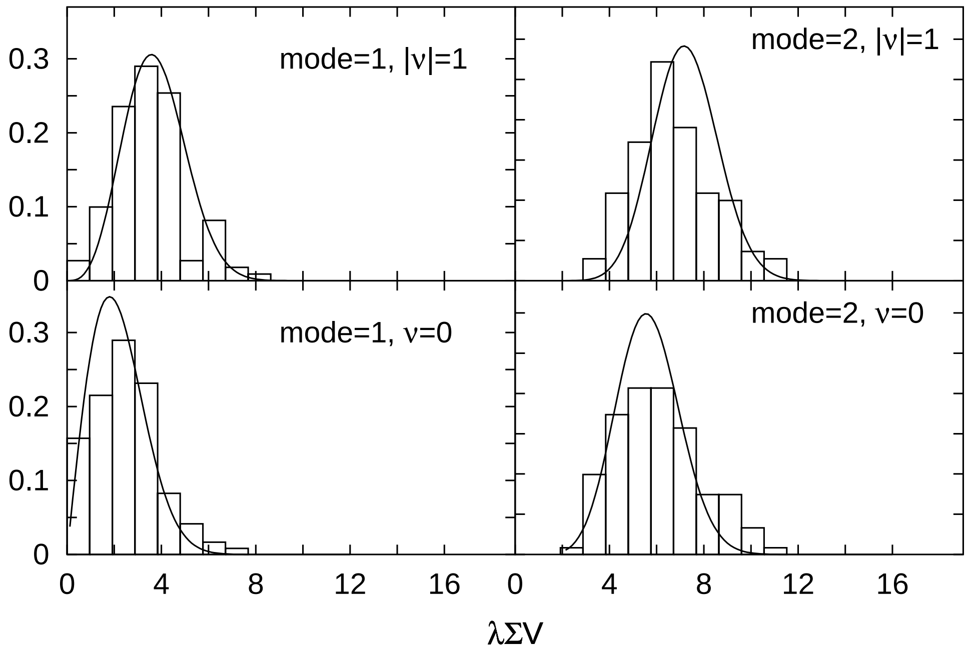
Banks-Casher

$$
\begin{aligned}
-\langle \bar{q}q \rangle &= \left\langle \frac{1}{V} \sum_k \frac{1}{i\lambda_k + m} \right\rangle \\
&= \left\langle \frac{1}{V} \sum_k \frac{m - i\lambda_k}{\lambda_k^2 + m^2} \right\rangle \\
&= \left\langle \frac{1}{V} \int d\lambda \rho(\lambda) \frac{m}{\lambda^2 + m^2} \right\rangle \\
&\underset{m \to 0}{=} \frac{\pi}{V} \rho(0) \mathrm{sign}(m)
\end{aligned}
$$

To obtain a continuous chiral condensate we need that $\rho(0_-) = -\rho(0_+)$.

Or could it be that $\rho(0) = 0$? Creutz-2007

Lattice results for one-flavor QCD



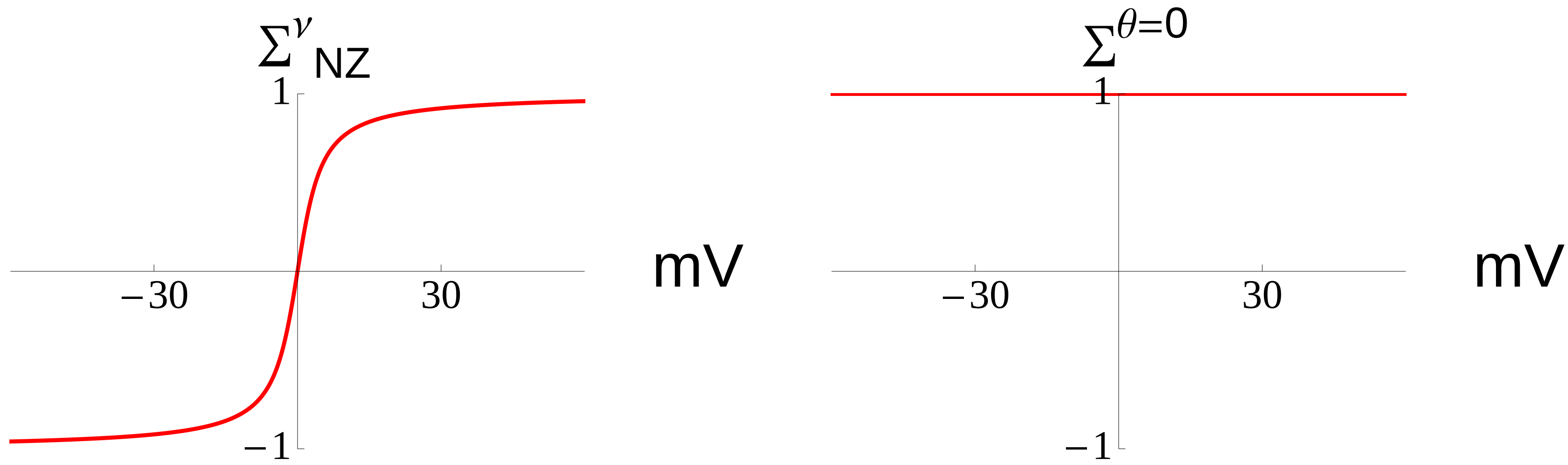
Distribution of the lowest two Dirac eigenvalues for QCD with one flavor compared to the result from chiral random matrix theory (solid curve).

DeGrand-Hoffmann-Schäfer-Liu-2006

What Can We Conclude from the Dirac Spectrum?

- The smallest Dirac eigenvalues of one-flavor QCD behave in exactly the same was as the Dirac spectrum of QCD and QCD-like theories with spontaneously broken chiral symmetry.
- The distribution of the smallest eigenvalues at fixed ν is given by random matrix theory or the ϵ-limit of the corresponding partially quenched chiral Lagrangian.
- Although the Dirac spectrum at fixed topology has all signatures of spontaneous chiral symmetry breaking, it should synthesize a chiral condensate that is due to explicit chiral symmetry breaking.

Chiral Condensate for $N_f = 1$



Mass dependence of the chiral condensate due to the nonzero modes for $\nu = 2$ (left) and the mass dependence of the chiral condensate at $\theta = 0$. Note that $\Sigma^{\nu}_{NZ}(m) = \Sigma^{\nu}(m) - \frac{|\nu|}{mV}$.

For $m < 0$, the negative part of $\Sigma^{\nu}(m)$ should average to a positive number.

Chiral condensate at $\theta = 0$

$$\Sigma(m, \theta = 0) = \frac{\sum_{\nu=-\infty}^{\infty} Z_\nu(m)\Sigma^\nu(m)}{\sum_{\nu=-\infty}^{\infty} Z_\nu(m)}.$$

This condensate follows from the spectral density at $\theta = 0$

$$\rho(m, \theta = 0) = \frac{\sum_{\nu=-\infty}^{\infty} Z_\nu(m)\rho^\nu(m)}{\sum_{\nu=-\infty}^{\infty} Z_\nu(m)}.$$

Can be evaluated numerically in the ϵ domain of QCD.

Damgaard-1999, Kanazawa-Wettig-2011

$$Z_\nu(-|m|) \sim (-1)^\nu |m|.$$

This makes it possible that the negative part of $\Sigma_\nu(m)$ averages to a positive number.

Silver Blaze Problem

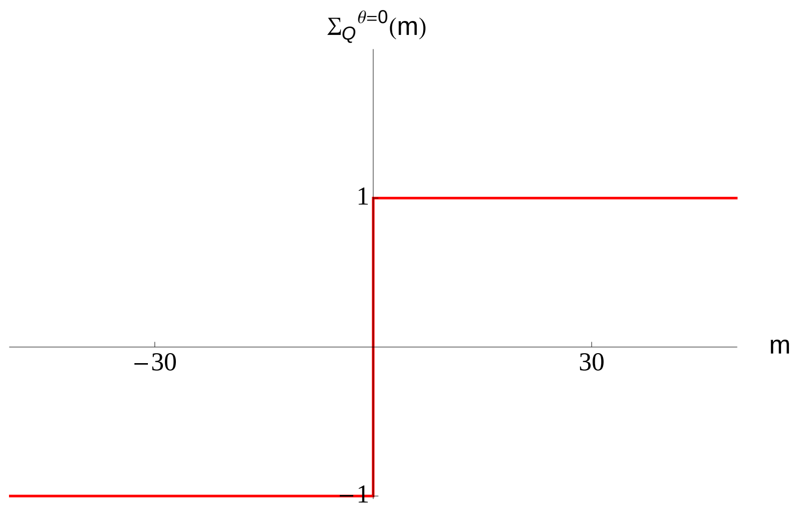
- In the thermodynamic limit, the chiral condensate at fixed ν has a discontinuity when the mass crosses the line of eigenvalues.
- The Silver Blaze Problem is that the chiral condensate at $\theta = 0$ does not have such discontinuity.
- A similar problem first arose in QCD at nonzero chemical potential, and the original motivatio for the present work was to improve our understanding of the relation between the chiral condensate and the Dirac spectrum for QCD at nonzero chemical potential.

OSV Mechanism at Nonzero Chemical Potential

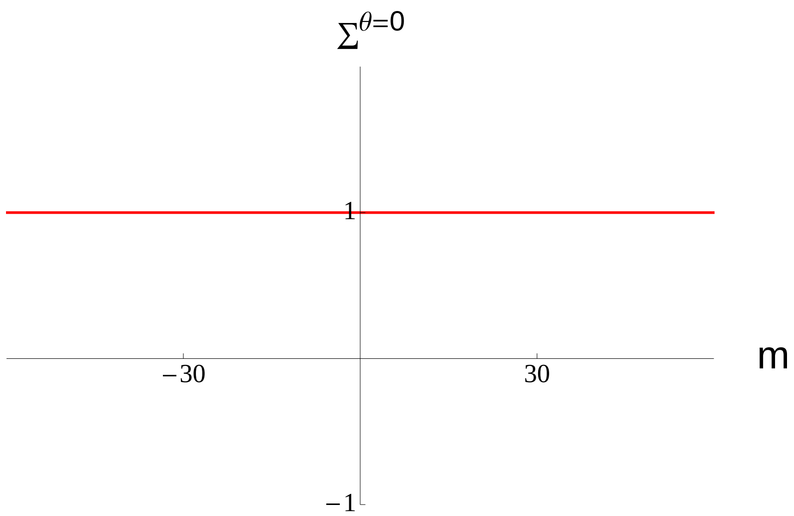
- The chiral condensate for QCD at nonzero chemical potential results from an oscillating spectral density with an amplitude that diverges exponentially with the volume and a period proportional to the inverse volume. Osborn-Splittorff-JV-2005
- When the eigenvalue density is not positive definite (due to the fermion determinant), the OSV mechanism replaces the Banks-Casher formula.

Let us see how this works for $N_f = 1$

How to get a Constant Chiral Condensate?



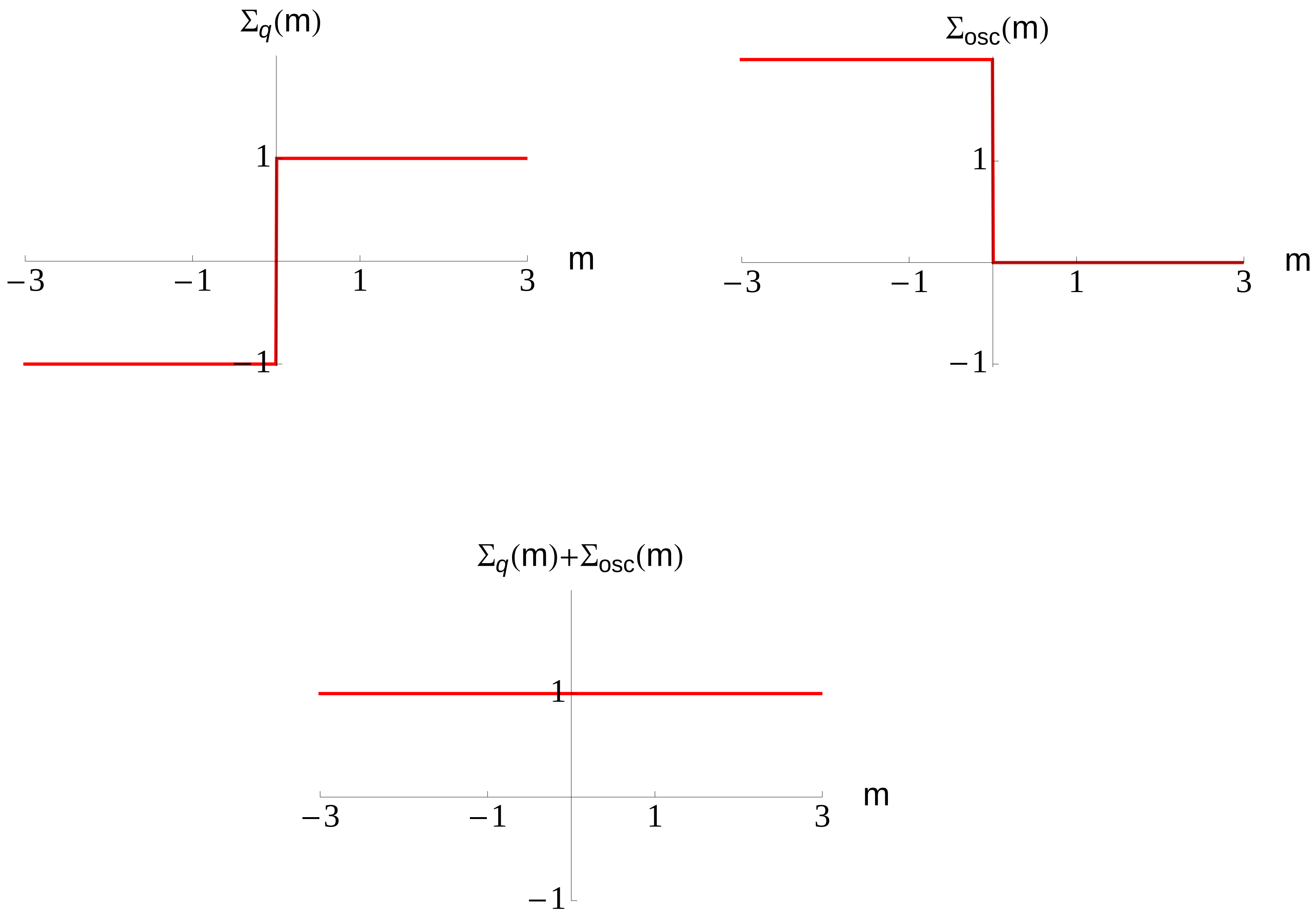
Behavior of the chiral condensate due to a line of eigenvalues for the quenched theory at $\theta = 0$.



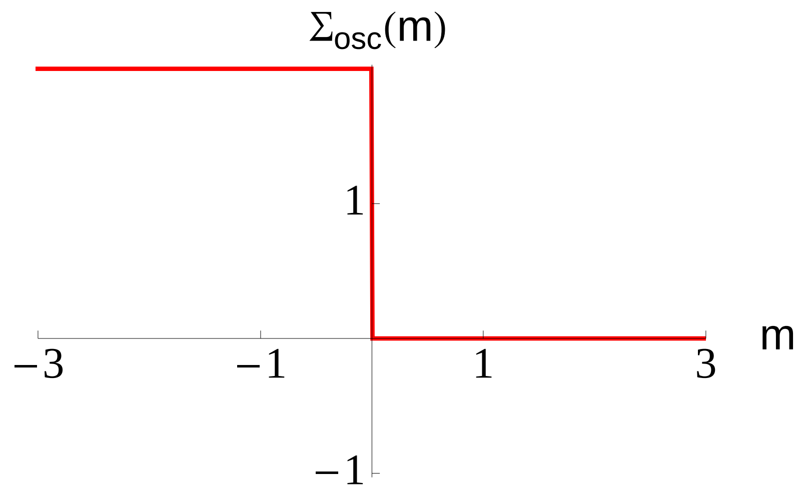
Behavior of the chiral condensate due to a line of eigenvalues for the one flavor theory at $\theta = 0$.

This implies that the not positive definite measure should give a correction to the spectral density that contributes to the chiral condensate as $\Sigma_{\rm osc}(m) = 2\theta(-m)$

OSV Mechanism in Pictures



How can this be Generated by a Spectral Density?



$$2\theta(-m) = \int d\lambda \frac{\rho_{osc}(\lambda, m)}{i\lambda - m}.$$

What is $\rho_{\rm osc}(\lambda, m)$?

Hint,

$$\theta(m) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\tau \frac{e^{im\tau + m\epsilon}}{\tau - i\epsilon}.$$

Solution

$$\rho_{\rm osc}(\lambda, m) = \frac{1}{\pi}(e^{iV\lambda - Vm} + e^{-iV\lambda - Vm})$$

$$\int_{-\infty}^{\infty} d\lambda \frac{1}{i\lambda - m}\frac{1}{\pi}(e^{V(i\lambda - m)} + e^{V(i\lambda - m)}) = 2\theta(-m) - 2\theta(m)e^{-2Vm}$$

- In the termodynamic limit we obtain a continuous chiral condensate.
- For $N_f = 1$ this should also happen when $mV \sim \mathcal{O}(1)$.

Let us calculate the spectral density for one flavor QCD.

The Microscopic Domain of QCD

In this domain, also know as the ϵ-domain, the quark mass and the Dirac eigenvalues scale as

$$m \sim \frac{1}{V}, \qquad \lambda \sim \frac{1}{V}.$$

Correction terms will enter when $m, \lambda \approx \sqrt{V}$.

In this domain it is possible to obtain exact analytical results for the spectral density at fixed ν and at fixed θ -angle.

We will use units where $\Sigma = 1$.

Microscopic Spectral Density at $\theta = 0$

One flavor spectral density at fixed ν (Note that $\Sigma = 1$)

$$\begin{aligned}\rho_\nu(x) &= \frac{\hat{x}}{2}(J_\nu^2(\hat{x}) - J_{\nu+1}(\hat{x})J_{\nu-1}(\hat{x})) + |\nu|\delta(\hat{x}) \\ &+ \frac{\hat{m}}{\hat{m}^2 + \hat{x}^2}\left[\hat{m}J_\nu(\hat{x})J_{\nu+1}(\hat{x}) - \hat{x}\frac{I_{\nu+1}(\hat{m})}{I_\nu(\hat{m})}J_\nu^2(\hat{x})\right]\Bigg|_{\hat{x}=xV,\hat{m}=mV}.\end{aligned}$$

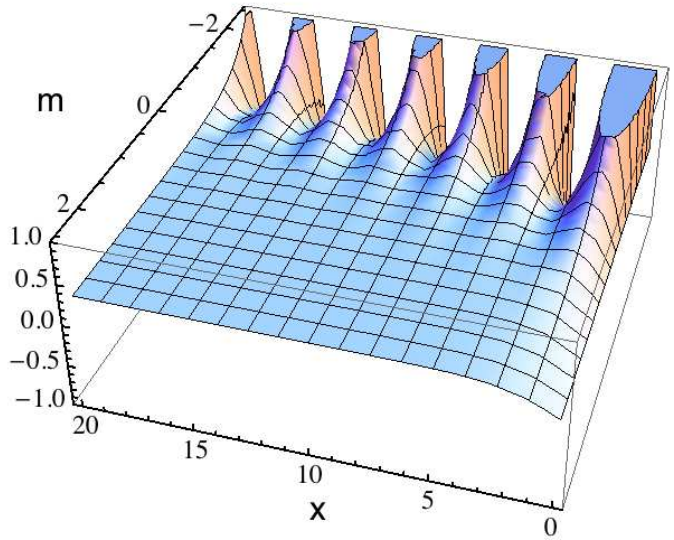
Damgaard-Osborn-Toublan-JV-1999

One flavor spectral density at θ = 0

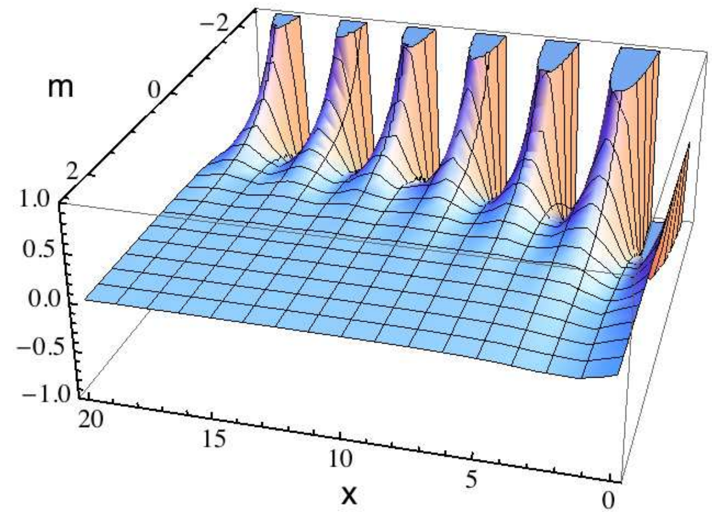
$$\begin{aligned}\rho(m, \theta = 0) &= \frac{\sum_{\nu=-\infty}^{\infty} Z_\nu(m)\rho^\nu(m)}{\sum_{\nu=-\infty}^{\infty} Z_\nu(m)} \\ &= \rho_q(x) + \rho_{ZM}(x) + \rho_d(x, m)\end{aligned}$$

with $Z_\nu(m) = I_\nu(mV\Sigma)$.

The Dirac Spectrum for $N_f = 1$ at $\theta = 0$



The quenched part of the spectral density at $\theta = 0$, $\rho_q(x, m)$.



The dynamical part of the spectral density at $\theta = 0$, $\rho_d(x, m)$.

$$\rho_q(x,m) \quad = \quad \frac{1}{\pi}\int_0^1 \frac{e^{-2mVt^2}dt}{t\sqrt{1-t^2}} J_1(2xVt).$$

$$\rho_d(x,m) \quad = \quad -\frac{2}{\pi}\frac{x}{x^2+m^2}\int_0^1 \frac{e^{-2mVt^2}dt}{\sqrt{1-t^2}}$$
$$\times\left[xtJ_1(2xVt) + m(1-2t^2)J_0(2xVt)\right].$$

JV-Wettig-2014

$$\rho_{ZM}(x,m) \quad = \quad e^{|m|V}\sum_\nu |\nu| I_\nu(mV)\delta(x) = e^{-mV}(I_0(mV) + I_1(mV))\delta(x)$$

Chiral Condensate

The chiral condensate can be obtained by integration over the spectral density

$$\Sigma(m) = \frac{1}{V}\int_{-\infty}^{\infty} \frac{2m\rho(\lambda, m)}{\lambda^2 + m^2}.$$

For $m < 0$ the contribution from the nonzero modes diverges in the thermodynamic limit as

$$\frac{e^{2|mV|}}{\sqrt{8\pi|mV|^3}}.$$

This contribution cancels against a similar contribution from the zero modes Kanazawa-Wettig-2012.

To extract the mass dependence of the chiral condensate we have to achieve this cancellation analytically and to all orders. JV-Wettig-2014

Contribution of Zero Modes

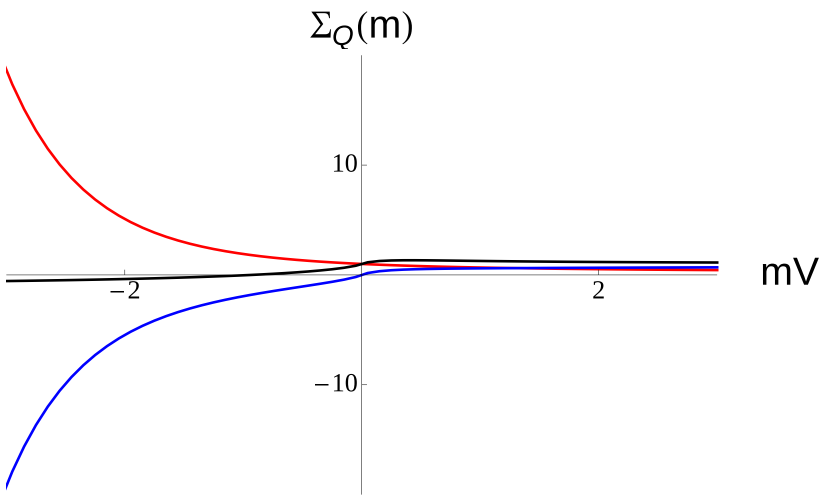
$$\begin{aligned}\Sigma_{\mathrm{ZM}}(m) &= e^{-mV}\left[I_0(mV)+I_1(mV)\right]\\ &= \frac{1}{\pi m V}\int_0^1 \frac{dt}{t^2\sqrt{1-t^2}}\left(1-e^{-2mVt^2}\right) \underset{\substack{V\to\infty\\ m<0}}{\sim} \frac{e^{2|mV|}}{\sqrt{8\pi|mV|^3}}.\end{aligned}$$

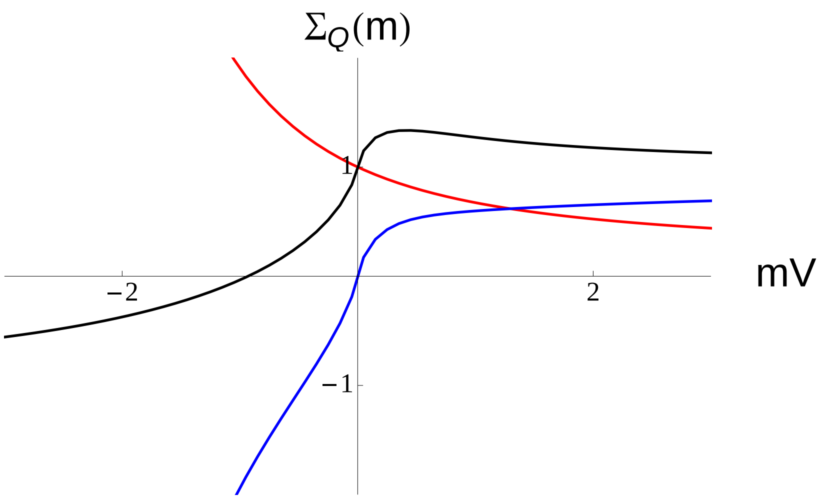
$$\Sigma_q(m) \quad = \tfrac{1}{\pi m V}\int_0^1 \tfrac{dt\, e^{-2mVt^2}}{t^2\sqrt{1-t^2}}\left[1-2t|mV|K_1(2t|mV|)\right] \underset{\substack{V\to\infty\\ m<0}}{\sim} -\tfrac{e^{2|mV|}}{\sqrt{8\pi|mV|^3}}.$$

The cancellation of the divergent part is true to all orders

$$\begin{aligned}\Sigma_Q \equiv \Sigma_q(m)+\Sigma_{\mathrm{ZM}}(m) &= \frac{1}{\pi m V}\int_0^1 \frac{dt}{t^2\sqrt{1-t^2}}\\ &\quad\times\left[1-e^{-2mVt^2}2t|mV|K_1(2t|mV|)\right].\end{aligned}$$

Exponential Cancellation



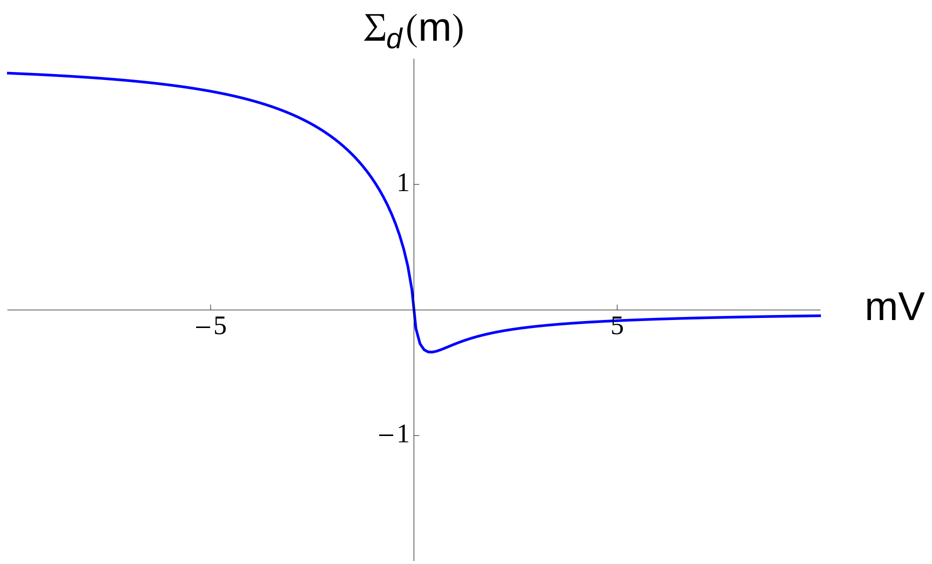


The exponentially large contribution of the zero modes (red) is canceled by the contribution from the the nonzero modes (blue). The sum of the two contributions is given by the black curve.

Contribution to the Chiral Condensate Due to Dynamical Quarks

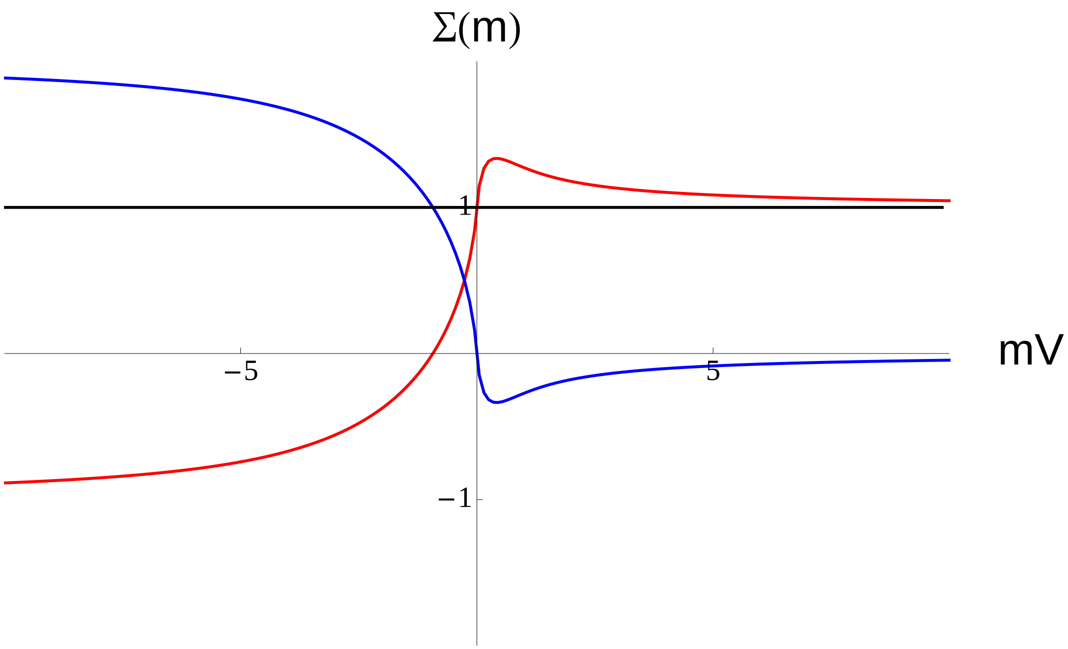
The integral over the dynamical part of the spectral density can also be evaluated analytically

$$\Sigma_d(m) \quad = -\frac{4}{\pi}\int_0^1 \frac{dt\, t\, e^{-2mVt^2}}{\sqrt{1-t^2}}\left[tmVK_0(2t|mV|) + (1-2t^2)|mV|K_1(2t|mV|)\right].$$



Contribution of the dynamical part of the Dirac spectrum to the chiral condensate.

Solution of the Silver Blaze Problem



Chiral condensate as a function of the quark mass m. The red curve shows the chiral condensate due to the quenched part of the spectral density, while the blue curve represents the condensate due to the oscillating part. The black curve is the sum of the two.

V. Conclusions

- The chiral condensate in the massless limit of one-flavor QCD is nonzero because of the zero modes but its value is determined by the nonzero modes.

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- One flavor QCD has a Silver Blaze problem when the chiral condensate remains constant while the quark mass crosses a line of eigenvalues.

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- The chiral condensate in the massless limit of one-flavor QCD is nonzero because of the zero modes but its value is determined by the nonzero modes.
- One flavor QCD has a Silver Blaze problem when the chiral condensate remains constant while the quark mass crosses a line of eigenvalues.
- From QCD at nonzero chemical potential we have learnt that the solution of the Silver Blaze problem requires an oscillating spectral density with period $\sim 1/V$ and an amplitude that grows exponentially with the volume.

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- The chiral condensate in the massless limit of one-flavor QCD is nonzero because of the zero modes but its value is determined by the nonzero modes.
- One flavor QCD has a Silver Blaze problem when the chiral condensate remains constant while the quark mass crosses a line of eigenvalues.
- From QCD at nonzero chemical potential we have learnt that the solution of the Silver Blaze problem requires an oscillating spectral density with period $\sim 1/V$ and an amplitude that grows exponentially with the volume.
- In the ϵ domain of QCD we have obtained simple exact analytical expressions for the eigenvalue density of the Dirac operator at $\theta = 0$ and $\theta = \pi$. Indeed, an oscillating contribution to the spectral density results in a constant chiral condensate.

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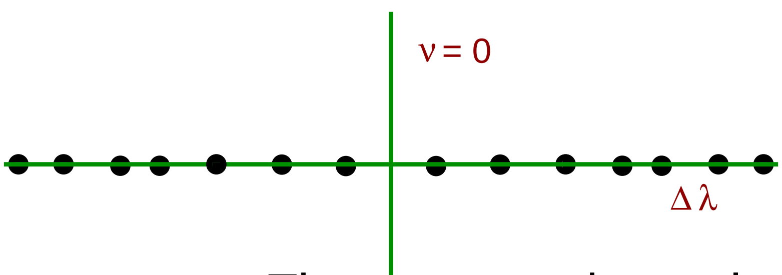
- The zero modes are essential for the continuity of the chiral condensate. Their exponentially increasing contribution is canceled against the contribution from the nonzero modes.
- Rooting fails at a fundamental level.

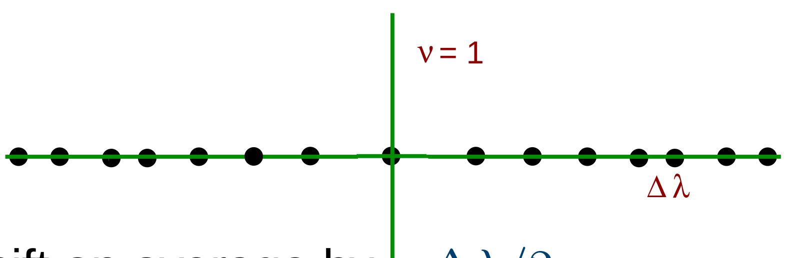
Could the Chiral Condensate be Due to the Zero Modes?

If we take the chiral limit before the thermodynamic limit then

$$-\langle \bar{q} q \rangle \quad = \quad \frac{1}{VZ(m)} \left\langle \sum_\nu \sum_k \frac{1}{i\lambda_k + m} m^{|\nu|} \prod_k (i\lambda_k + m) \right\rangle$$

$$\underset{m\to 0}{=} \quad \frac{\left\langle \prod_{\lambda_k \neq 0} i\lambda_k \Big|_{\nu=1} \right\rangle}{\left\langle \prod_{\lambda_k \neq 0} i\lambda_k \Big|_{\nu=0} \right\rangle} + \frac{\left\langle \prod_{\lambda_k \neq 0} i\lambda_k \Big|_{\nu=-1} \right\rangle}{\left\langle \prod_{\lambda_k \neq 0} i\lambda_k \Big|_{\nu=0} \right\rangle} \approx \frac{1}{\Delta\lambda}$$





The nonzero eigenvalues shift on average by $\nu\Delta\lambda/2$.

Even in the chiral limit, the value of the chiral condensate is due to the nonzero modes.

What Happens if we Reverse the Thermodynamical and Chiral Limits?

If the thermodynamic limit is taken before the chiral limit we have that for $mV\Sigma \gg 1$

$$
\begin{aligned}
-\langle \bar{q}q \rangle &= \frac{1}{V}\frac{\sum_\nu \frac{|\nu|}{m} e^{-\nu^2/2|m|V\Sigma}}{\int d\nu e^{-\nu^2/2|m|V\Sigma}} \\
&= \frac{1}{Vm}\frac{\sum_\nu |\nu| e^{-\nu^2/2|m|V\Sigma}}{\int d\nu e^{-\nu^2/2|m|V\Sigma}} \\
&= \mathrm{sign}(m)\frac{2\Sigma}{\sqrt{\pi 2|m|V\Sigma}}.
\end{aligned}
$$

To get a constant chiral condensate we need the contribution of the nonzero modes.

Because of the exponential cancellations, this argument is not correct for $m < 0$.

Technical Detail

To evaluate the microscopic spectral density at fixed θ-angle we need sums of the form

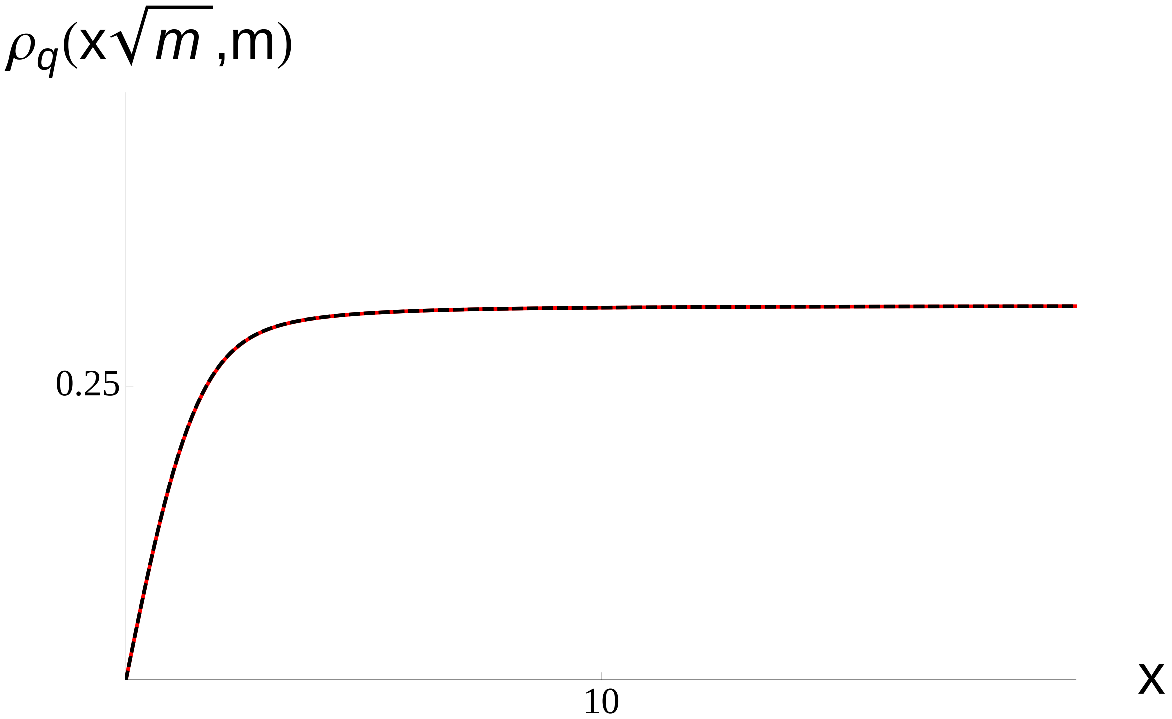
$$S_{a,b,c}(x,m,\theta) = \sum_{\nu=-\infty}^{\infty} e^{i\nu\theta} I_{\nu+a}(m) J_{\nu+b}(x) J_{\nu+c}(x)$$

They can be reduced to one-dimensional integrals. Examples are

$$\begin{aligned}
\sum_\nu I_\nu(m) J_\nu^2(x) &= \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^2}}\, e^{m-2mt^2} J_0(2xt)\,, \\
\sum_\nu I_\nu(m) J_{\nu+1}(x) J_{\nu-1}(x) &= -\frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^2}}\, e^{m-2mt^2} J_2(2xt)\,.
\end{aligned}$$

JV-Wettig-2014

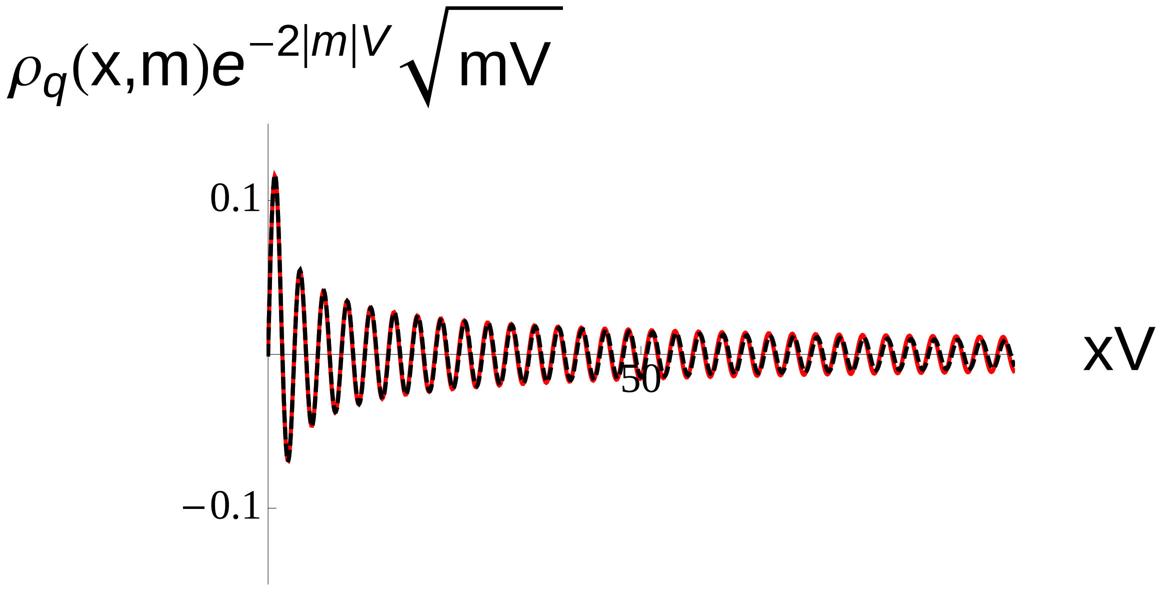
Asymptotic Scaling for $m > 0$



$$\rho_q(x\sqrt{m}, m > 0) \sim \frac{xV}{\sqrt{2\pi m V}} e^{-Vx^2/4m} [I_0(Vx^2/4m) + I_1(Vx^2/4m)]$$

Wettig-JV-2014

Asymptotic Scaling for $m < 0$



$$\rho_q(x, m < 0) \sim \frac{e^{2|m|V}}{\sqrt{8\pi|m|V}} J_1(2xV).$$

Wettig-JV-2014