Update on the critical endpoint of the finite temperature phase transition for three flavor QCD with clover type fermions

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in collaboration with

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\( m_\pi \) at the endpoint at \( \mu = 0 \) (bottom-left corner of Columbia plot)

<table>
<thead>
<tr>
<th>( N_t )</th>
<th>action</th>
<th>( m_\pi^E [\text{MeV}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>unimproved staggered</td>
<td>260</td>
</tr>
<tr>
<td>6</td>
<td>unimproved staggered</td>
<td>150</td>
</tr>
<tr>
<td>4</td>
<td>p4-improved staggered</td>
<td>70</td>
</tr>
<tr>
<td>6</td>
<td>stout-improved staggered</td>
<td>( \leq 50 )</td>
</tr>
<tr>
<td>6</td>
<td>HISQ</td>
<td>( \leq 45 )</td>
</tr>
<tr>
<td>4</td>
<td>unimproved Wilson</td>
<td>( \sim 1100 )</td>
</tr>
</tbody>
</table>

  - \( m_\pi^E \) decreases with decreasing lattice spacing
  - the crossover may persist down to \( \sim 0.1 m^{\text{phy}} \)

- Wilson type: [Iwasaki, et. al. ’96]
  - 1st order at rather heavy \( m_q \)
Motivation

- Critical endpoint (CEP) obtained with staggered and Wilson type fermions is inconsistent
- Results in the continuum limit is necessary

We determine CEP on $m_l = m_s$ line with clover fermions

$N_f = 3$ study is a stepping stone
- to the physical point
- curvature of critical surface

→ talk by S. Takeda [15:15 Tue]
### Distinguishing between 1st, 2nd and crossover criterion

<table>
<thead>
<tr>
<th>Distribution</th>
<th>First order</th>
<th>Second order</th>
<th>Crossover</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_{\text{peak}}$</td>
<td>$\propto N_l^d$</td>
<td>$\propto N_l^{\gamma/\nu}$</td>
<td>-</td>
</tr>
<tr>
<td>$\beta(\chi_{\text{peak}}) - \beta_c$</td>
<td>$\propto N_l^{-d}$</td>
<td>$\propto N_l^{-1/\nu}$</td>
<td>-</td>
</tr>
<tr>
<td>Kurtosis at $N_l \to \infty$</td>
<td>$K = -2$</td>
<td>$-2 &lt; K &lt; 0$</td>
<td>-</td>
</tr>
</tbody>
</table>

- Scaling might work with wrong exponents near CEP
- Peaks in histogram might emerge only at large $N_l$ on weak 1st order
- $K$ does not depend on volume at 2nd order phase transition point

\[
M = N_l^{-\beta/\nu} f_M(tN_l^{1/\nu})
\]

\[
K + 3 = B_4(M) = \frac{N_l^{-4\beta/\nu} f_M^4(tN_l^{1/\nu})}{[N_l^{-2\beta/\nu} f_M^2(tN_l^{1/\nu})]^2} = f_B(tN_l^{1/\nu})
\]
Method to determine CEP (kurtosis intersection)

- determine the transition point (peak position of susceptibility)
- determine kurtosis at transition point at each spatial lattice size
- find intersection point of kurtosis by fit, $K_E + aN_l^{1/\nu}(\beta - \beta_E)$
  → other method (gap of masses), talk by X.-Y. Jin [14:55 Tue]

- interpolate/extrapolate $(m_{PS}/m_V)_t$ measured at transition point to $\beta_E$
- extrapolate $(m_{PS}/m_V)_E$ to the continuum limit
Simulations

- action: Iwasaki gluon + $N_f = 3$ clover (non perturbative $c_{SW}$, degenerate)
- temporal lattice size $N_t = 4, 6, 8$ for continuum extrapolation
- statistics: $O(200,000)$ traj.
- observables: gauge action density, plaquette, Polyakov loop, chiral condensate and their higher moments
plaquette at $\beta = 1.60$, $N_t = 4$

The diagram shows the susceptibility, skewness, and kurtosis as functions of $\kappa$ for different values of $N_f$ (6, 8, 10, 12, 16). The data points are labeled with error bars indicating uncertainties. The plot demonstrates the phase transition behavior at finite temperatures.
plaquette at $\beta = 1.65, N_t = 4$
Kurtosis intersection at \( N_t = 4 \)
Kurtosis intersection at $N_t = 4$
\[ \chi_{\text{max}} \text{ fit: } aN_l^{b} \]
\( \gamma / \nu \) v.s. \( \beta \)

<table>
<thead>
<tr>
<th>( N_t )</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_E(E_m) )</td>
<td>1.6130(55)</td>
<td>1.7269(42)</td>
<td>1.7505(19)</td>
</tr>
<tr>
<td>( \beta_E(P, s_g, L) )</td>
<td>1.6238(21)</td>
<td>1.7361(16)</td>
<td>1.75491(92)</td>
</tr>
</tbody>
</table>
continuum extrapolation for \((m_{PS}/m_V)_E\)

\[
\begin{align*}
\text{\textbullet: } & \quad m_{PS}^{\text{phy}; \text{sym}} / m_V^{\text{phy}; \text{sym}} = \sqrt{(m_{\pi}^2 + 2m_K^2)/3}/[(m_\rho + 2m_{K^*})/3] \sim 0.4817 \\
\text{\textDelta: } & \quad m_{\eta_{ss}} / m_\phi \sim 0.6719
\end{align*}
\]
We have investigated the critical endpoint of QCD with clover fermions and determined the critical endpoint by using the intersection points of the Binder cumulants at $N_t = 4, 6, 8$ and extrapolated to the continuum limit.

- Kurtosis intersection analysis is consistent with $\chi_{\text{max}}$ analysis.
- $(m_{PS}/m_V)_E$ at $N_t = 4$ is out of scaling region.
- $(m_{PS}/m_V)_E$ in the continuum limit is smaller than the SU(3) symmetric point, not so small as staggered type fermions at $N_t = 6$ and it will be controlled by values at larger $N_t$. 
Backup slides
inconsistent results: Wilson and staggered type fermion

\[ m_s \]

\[ N_F = 2 \]

pure gauge

1st order

2nd order
crossover

\[ m_s^* \]

\[ m_u \]

\[ m_d \]

\[ m_{ud} \]

\[ m_{u,d}/m_{phys} \]

physical point
crossover region

\[ \approx 140 \]

\[ \approx 150 \]

\[ \approx 250 \text{ MeV} \]

\[ \approx 1.2 \text{ GeV} \]

\[ \approx 50 \]

Wilson

staggered
Higher moments

$i$-th derivative of $\ln Z$ with respect to control parameter $c$:

$$E = \frac{\partial \ln Z}{\partial c}$$

- Variance

$$V = \frac{\partial^2 \ln Z}{\partial c^2} = \sigma^2$$

- Skewness (e.g. right-skewed $\rightarrow S > 0$, left-skewed $\rightarrow S < 0$)

$$S = \frac{1}{\sigma^3} \frac{\partial^3 \ln Z}{\partial c^3}$$

- Kurtosis (e.g. Gaussian $\rightarrow K = 0$, $2\delta$ func. $\rightarrow K = -2$)

$$K = \frac{1}{\sigma^4} \frac{\partial^4 \ln Z}{\partial c^4} = B_4 - 3$$
Plaquette v.s. $\kappa$ at lowest $\beta$ (= 1.60)

- no bulk phase transition
$\beta = 1.60$ and $\kappa = 0.14345$ on $10^3 \times 4$, clear two states, $K \sim -1.5$

$\beta = 1.70$ and $\kappa = 0.13860$ on $10^3 \times 4$, one state, $K \sim -0.5$
CEP of the finite temperature phase transition

Yoshifumi Nakamura (RIKEN AICS)
Critical endpoint at $N_t = 6, 8$

$N_t = 6$

$N_t = 8$
continuum extrapolation for \((T/m_V)_E\)
$K_E$ and critical exponent $\nu$

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>$K_E$</th>
<th>$\nu$</th>
<th>class</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-1.363(88)</td>
<td>0.64(11)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-1.323(76)</td>
<td>0.60(14)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>-1.199(72)</td>
<td>0.48(14)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1.396</td>
<td>0.63</td>
<td>3D Z2</td>
</tr>
<tr>
<td></td>
<td>-1.758</td>
<td>0.67</td>
<td>3D O(2)</td>
</tr>
<tr>
<td></td>
<td>-1.908</td>
<td>0.75</td>
<td>3D O(4)</td>
</tr>
</tbody>
</table>

$K_E$ and $\nu$ are consistent with values of 3D Z2.