Multi-point Reweighting Method and beta-functions for the calculation of QCD equation of state

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Introduction

Reweighting method for QCD at high T and μ.

– Reweighting method is one of popular method.

This method have two problems.

– Sign problem
– **Overlap problem**

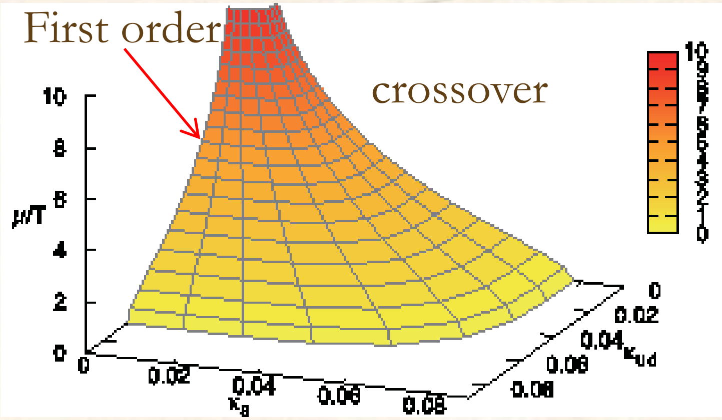
In this talk,

- **Overlap problem** in the reweighting method.
  – Focusing on multi-parameter and multi-simulation-point
  – Multi-simulation-point reweighting is useful for gauge action.
  – We deal with parameters in the fermion action.
- Application:
  – We compute beta-functions for the Equation of State.

Multi-parameter reweighting method

- Reweighting method in the **heavy** quark region
  (WHOT-QCD, Phys. Rev. D89, 034507, (2014))

  They calculate critical surface @ $\mu \neq 0$
  - Hopping parameter expansion
  - Reweighting factor is given by Plaquette term and Polyakov loop term.
  - The system is controlled by two combinations of parameters,



$$h = \sum_{f=1}^{N_f} \kappa_{\mathrm{cp}\,f}^{N_t} \cosh(\mu_f / T), \qquad \beta^* \equiv \beta + \prod_{f=1}^{N_f} 48\kappa_f^4,$$

- Expect such combinations of parameters in the light quark region also.
  ⇨ We consider mixture of the parameters.
  ⇨ Multi-parameter and Multi-simulation-point reweighting for fermion action in the light quark region.

Reweighting method (Histogram method)

- We compute $W(X;\beta,\kappa)$ and $\langle X \rangle_{(\beta,\kappa)}$ by configurations at (β_0,κ_0)
- Expectation values in Monte Carlo simulations

$$\langle X \rangle_{(\beta,\kappa)} = \frac{1}{Z}\int DU\, X\, e^{-S} \approx \frac{1}{N_{\text{conf.}}}\sum_{\{\text{conf.}\}} X \qquad S = N_f \ln\det M(\kappa, c_{SW}) - S_G$$

- Histogram fixing $\hat{X}_i$s$(i = 1, 2, \cdots, N)$

$$W(\vec{X};\beta,\kappa) \equiv \int \mathcal{D}U \prod_i \delta(X_i - \hat{X}_i) e^{-S}$$

Constraint: $\vec{X} = (X_1, X_2, X_3, \cdots, X_n)$

- Using $W(\vec{X};\beta,\kappa)$, we can rewrite $\langle X \rangle_{(\beta,\kappa)}$.

$$\langle X_1 \rangle_{(\beta,\kappa)} = \frac{1}{Z(\beta,\kappa)} \int X_1 W(\vec{X};\beta,\kappa) \prod_i dX_i \quad Z(\beta,\kappa) = \int W(X_i;\beta,\kappa) \prod_i dX_i$$

- When we choose $\vec{X} = (X, S, S_0)$, $\mathrm{S} \equiv \mathrm{S}(\beta,\kappa)$, $\mathrm{S}_0 \equiv \mathrm{S}(\beta_0,\kappa_0)$

$$W(X, S, S_0;\beta,\kappa) = e^{-S+S_0} W(X, S, S_0;\beta_0,\kappa_0)$$

- Using this equation, $\langle X \rangle_{(\beta,\kappa)}$ calculable by simulation at (β_0,κ_0)

Multi-point-Reweighting method

- We compute $W(X;\beta,\kappa)$ and $\langle X\rangle_{(\beta,\kappa)}$ by simulations at many (β_i,κ_i)
- Histograms fixing $\vec{X}=(X,\vec{S})$: $\vec{S}=(S(\beta,\kappa),S(\beta_1,\kappa_1),S(\beta_2,\kappa_2),\cdots)\equiv(S,S_1,S_2,\cdots)$

$$W(X,\vec{S};\beta_i,\kappa_i)=e^{-S(\beta_i,\kappa_i)+S(\beta,\kappa)}W(X,\vec{S};\beta,\kappa)$$

- Naïve sum of the histograms

N_i : Number of configurations at (β_i,κ_i)

$$\sum_{i=1}^{N_{SP}} N_i Z^{-1}(\beta_i,\kappa_i)W(X,\vec{S};\beta_i,\kappa_i)=\sum_{i=1}^{N_{SP}} N_i Z^{-1}(\beta_i,\kappa_i)e^{-S_i+S}\underline{W(X,\vec{S};\beta,\kappa)}$$

$$\therefore\ \underline{W(X,\vec{S};\beta,\kappa)}=\boxed{G(X,\vec{S};\beta,\kappa)}\sum_{i=1}^{N_{SP}} N_i Z^{-1}(\beta_i,\kappa_i)W(X,\vec{S};\beta_i,\kappa_i)$$

$$\boxed{G(X,\vec{S};\beta,\kappa)=\frac{e^{-S}}{\sum_i N_i Z^{-1}_{(\beta_i,\kappa_i)}e^{-S_i}}}$$

- Expectation value by simulations at (β_i,κ_i)

$$\langle X\rangle_{(\beta,\kappa)}=\frac{1}{Z(\beta,\kappa)}\int XG\sum_{i=1}^{N_{SP}} N_i Z^{-1}_{(\beta_i,\kappa_i)}W(X,\vec{S};\beta_i,\kappa_i)\,dX\prod_i dS_i=\frac{\sum_{i=1}^{N_{SP}} N_i\langle XG\rangle_{(\beta_i,\kappa_i)}}{\sum_{i=1}^{N_{SP}} N_i\langle G\rangle_{(\beta_i,\kappa_i)}}$$

Naïve average

Fermion determinant for the reweighting

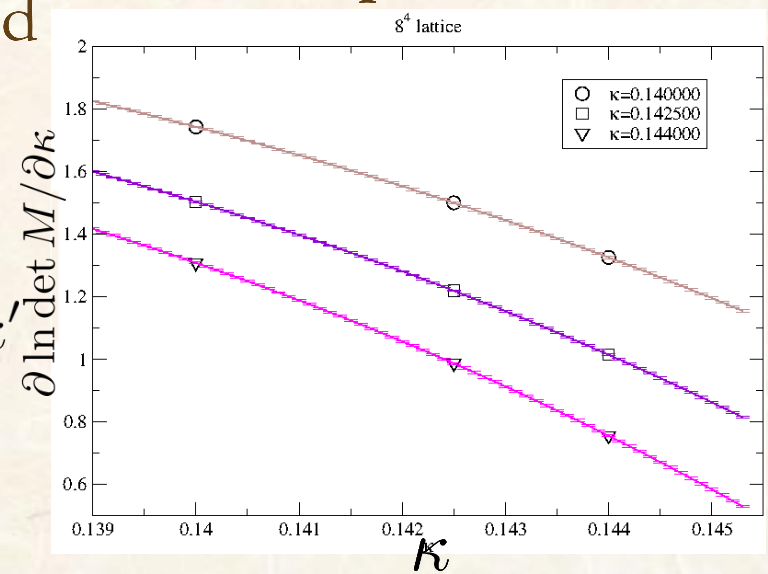
- The reweighting factor is given by the fermion determinant.
- We calculate $\frac{\partial \ln \det M}{\partial \kappa}, \frac{\partial^2 \ln \det M}{\partial \kappa^2}$ by the random noise method,

$$\frac{\partial \ln \det M}{\partial \kappa} = \mathrm{tr}\left(\frac{\partial M}{\partial \kappa} M^{-1}\right) \simeq \frac{1}{N_{noise}} \sum_i \eta_i^\dagger \frac{\partial M}{\partial \kappa} M^{-1} \eta_i$$

- We interpolate $\frac{\partial \ln \det M}{\partial \kappa}$ assuming cubic function in terms of κ on each configuration.
- The reweighting factor is computed by this integral,

$$\ln\left(\frac{\det M(\kappa)}{\det M(\kappa_0)}\right) = \int_{\kappa_0}^{\kappa} \frac{\partial \ln \det M(\kappa')}{\partial \kappa} d\kappa'$$

➢ **interpolation**



Simulation set up

We choose Iwasaki gauge and clover Wilson fermion actions.

$$Z(\beta, m) = \int \mathcal{D}U e^{-S} \quad S = N_f \ln \det M(\kappa, c_{SW}) - S_G$$

$$S_g = S_g^{1\times1} + S_g^{1\times2} = -\beta \sum_{x,\mu>\nu} \left[c_0 W_{\mu\nu}^{1\times1}(x) + 2c_1 W_{\mu\nu}^{1\times2}(x) \right]$$

$$M_{x,y} = \delta_{x,y} - \kappa \sum_i \left[(1-\gamma_i)\, \mathrm{U}_i \delta_{x+\hat{i},y} + (1+\gamma_i)\, \mathrm{U}_i^\dagger (x-\hat{i}) \delta_{x-\hat{i},y} \right]$$

$$- \delta_{x,y} c_{SW} \kappa \sum_{\mu>\nu} \sigma_{\mu\nu} F_{\mu\nu}$$

$$c_1 = -0.331 \quad c_0 = 1 - 8c_1 \qquad c_{SW} = \left(1 - 0.8412\beta^{-1}\right)^{-3/4}$$

T=0 simulations with N_f=2

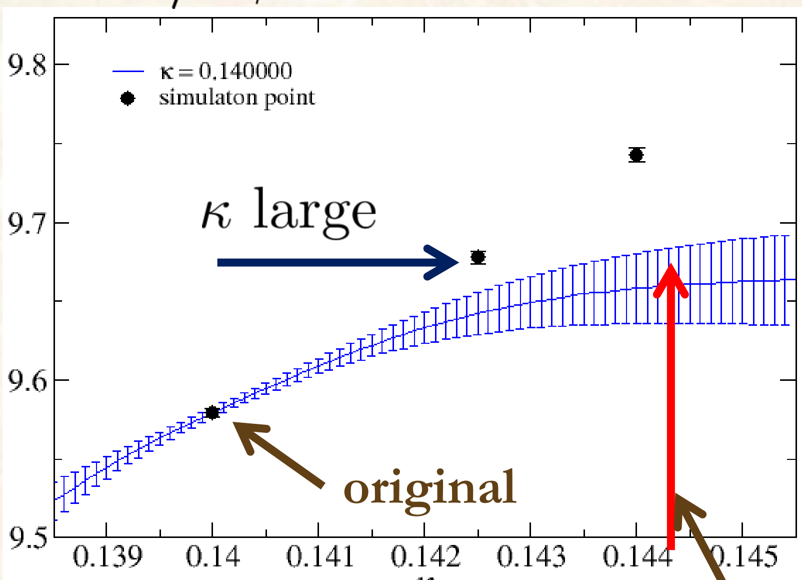
on an 8^4 lattice at 9 simulation points (8^4) (test)

and on a 16^4 lattice at 35 simulation points (16^4)

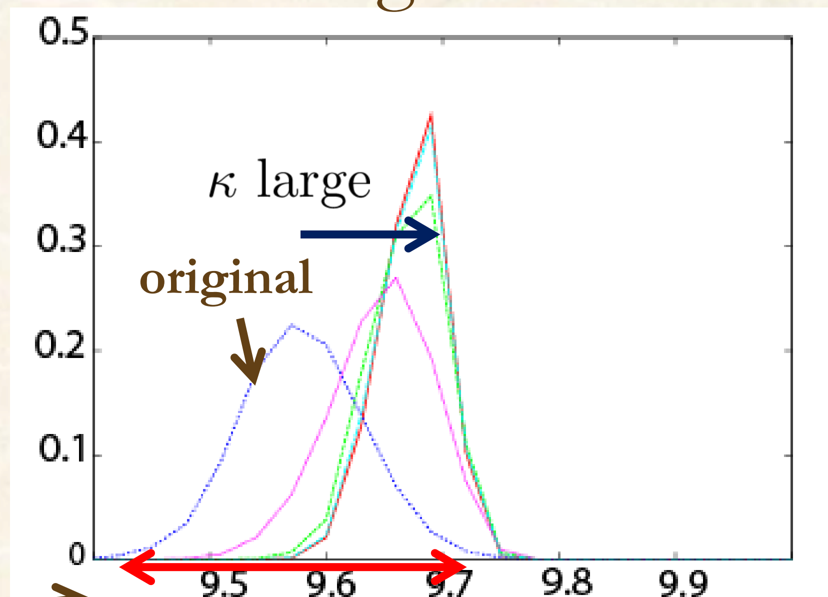
$$0.7 < m_\pi / m_\rho < 0.8$$

Reweighting method and **Overlap problem**

➢ $\partial S/\partial\beta$



➢ histogram



Range of original distribution

$$P \equiv \frac{\partial S}{\partial \beta} = \left[\sum_{x,\mu>\nu} c_0 W^{1\times1}_{\mu\nu}(x) + \sum_{x,\mu\neq\nu} c_1 W^{1\times2}_{\mu\nu}(x) \right] + (clover\ \text{term})$$

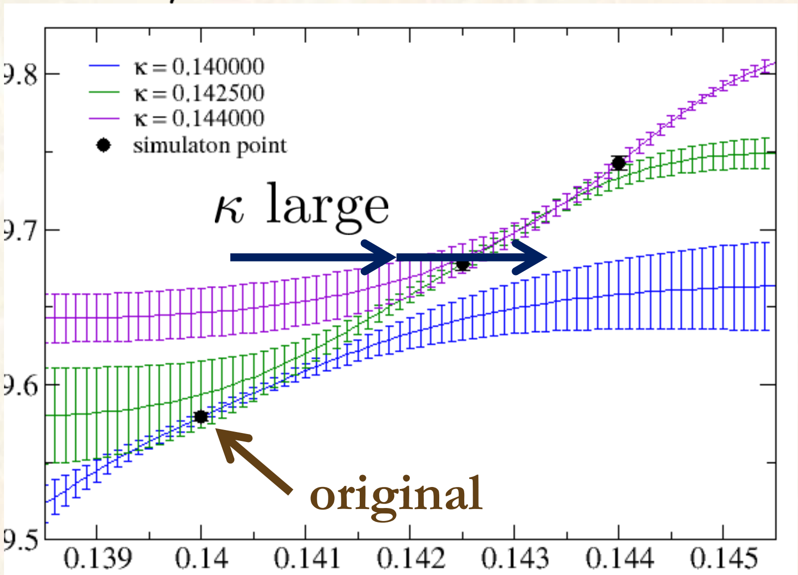
- Black symbols: Expectation values at simulation point.
- blue curve: Results by the reweighting method at κ=0.140000
- Just multiply the original histogram by the reweighting factor :

$$W(P;\beta,\kappa) = e^{-S(\beta,\kappa)+S(\beta_0,\kappa_0)} W(P;\beta_0,\kappa_0)$$

- The histogram cannot go out of the range of the original distribution.

Multi-simulation-point reweighting

- $\partial S/\partial\beta$
- histogram



κ

- We performed simulations at 3 points (blue, green, purple).
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-
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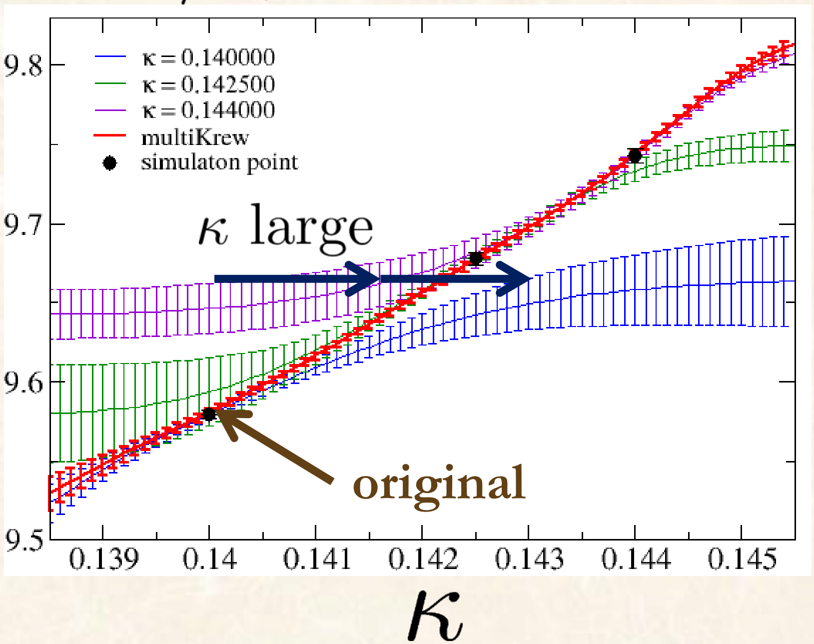
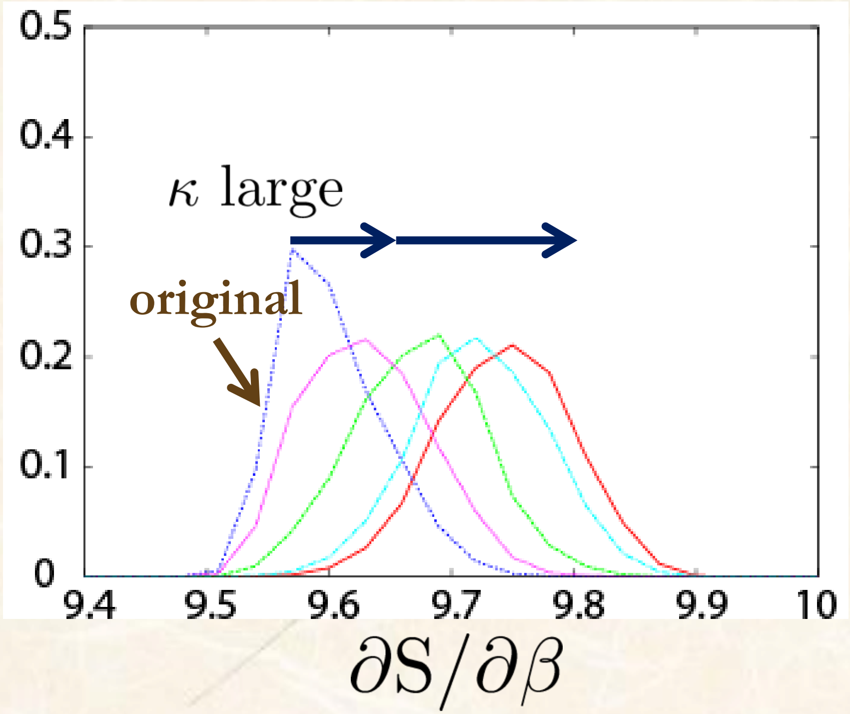
Multi-simulation-point reweighting

- $\partial S/\partial \beta$



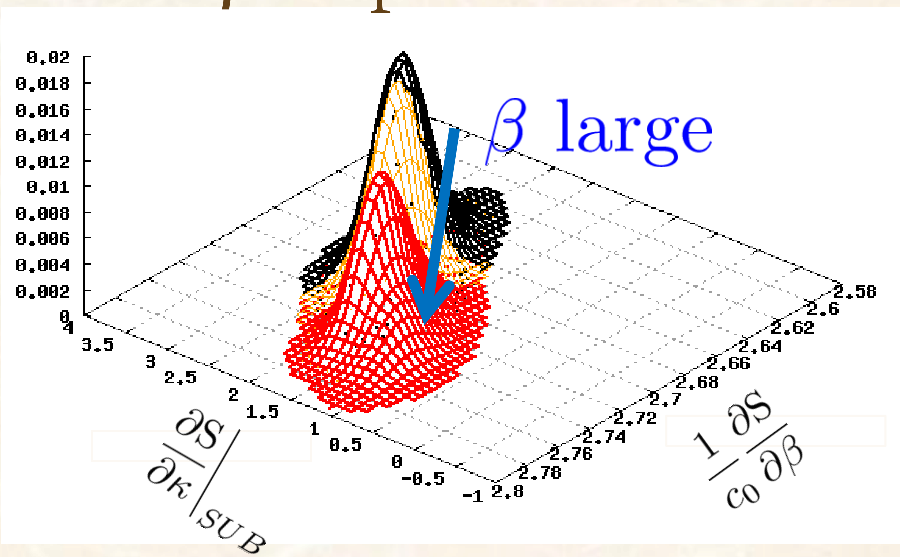
- histogram

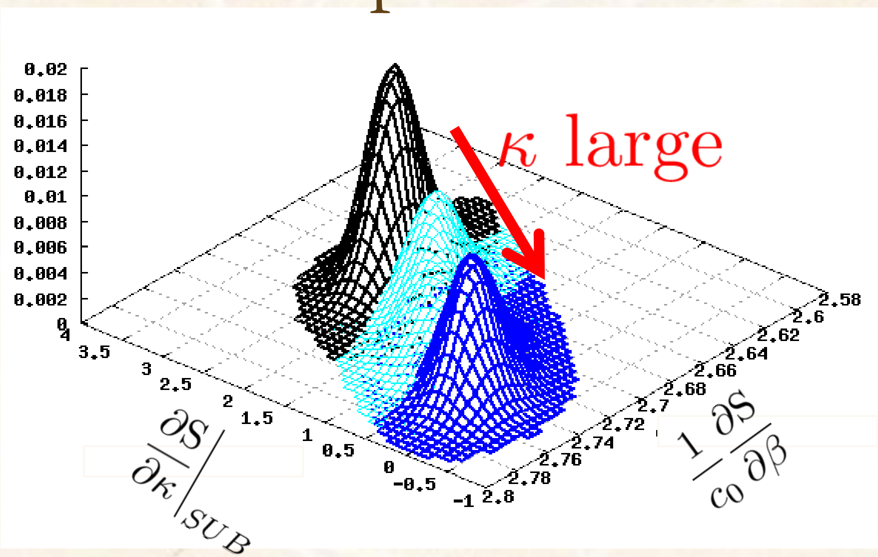


- We performed simulations at 3 points (blue, green, purple).
- These are combined by the multi-point-reweighting. (red)
- The result is natural in a wide range.
- The histogram moves naturally as κ changes.

Distribution functions in the $\left(\frac{\partial S}{\partial \beta}, \frac{\partial S}{\partial \kappa}\right)$ plane

- Multi-point reweighting is powerful calculation of multi dim. histogram
  - ➢ β-dependence
  - ➢ κ-dependence





$$\left.\frac{\partial S}{\partial \kappa}\right|_{SUB} \equiv \frac{\partial S}{\partial \kappa} - \frac{288 N_f \kappa^4}{c_0}\frac{\partial S}{\partial \beta} = N_f \frac{\partial \ln \det M}{\partial \kappa} - (\text{Hopping parameter exp. leading term})$$

- The histogram moves as changing β and κ.
- The peak position corresponds to the expectation value of $\left(\frac{\partial S}{\partial \beta}, \frac{\partial S}{\partial \kappa}\right)$.

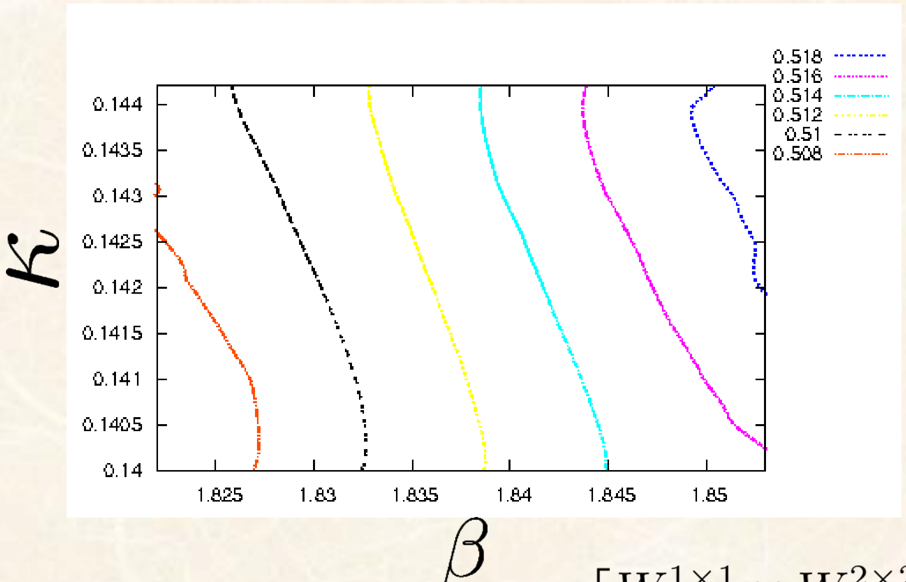
These value are important to calculate EOS.

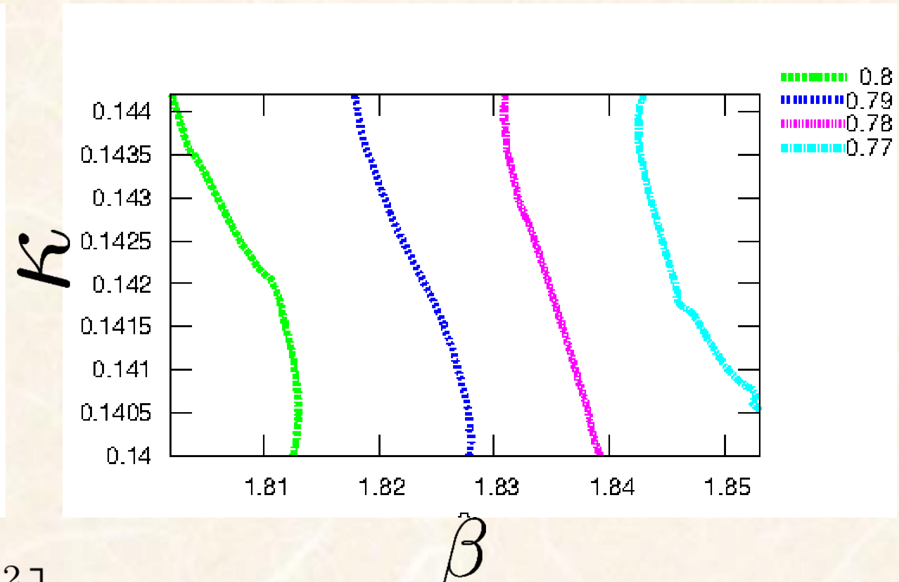
Lines of constant physical quantities in the (β,κ) plane

- Multi-point reweighting is useful calculation of

  lines of constant of physical quantities.

  - $W^{1\times 1} = Const.$
  - $\sqrt{\chi(1,1)} = Const.$





$$\chi(1,1) = -\ln\left[\frac{W^{1\times 1}\times W^{2\times 2}}{W^{1\times 2}\times W^{2\times 1}}\right] \to \sigma a^2 \text{ (strong coupl. limit)}$$

- (left) The plaquette is constant along these lines.
- (right) A Creutz ratio is constant along these lines.

Beta-function for the calculation of EOS

Derivatives of β and κ are needed for the calculation of EOS.

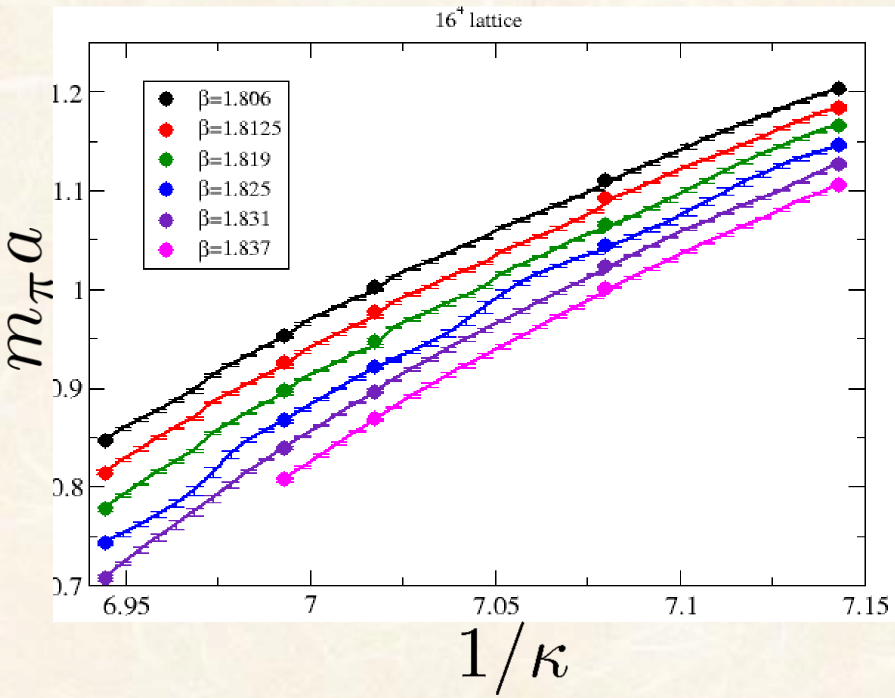
$$T\frac{d}{dT}\left(\frac{P}{T^4}\right) = \frac{\epsilon - 3P}{T^4} = \frac{N_t^3}{N_s^3}\left(a\frac{d\beta}{da}\left\langle\frac{\partial S}{\partial\beta}\right\rangle + a\frac{d\kappa_{ud}}{da}\left\langle\frac{\partial S}{\partial\kappa_{ud}}\right\rangle\right)_{\mathrm{LCP}}$$

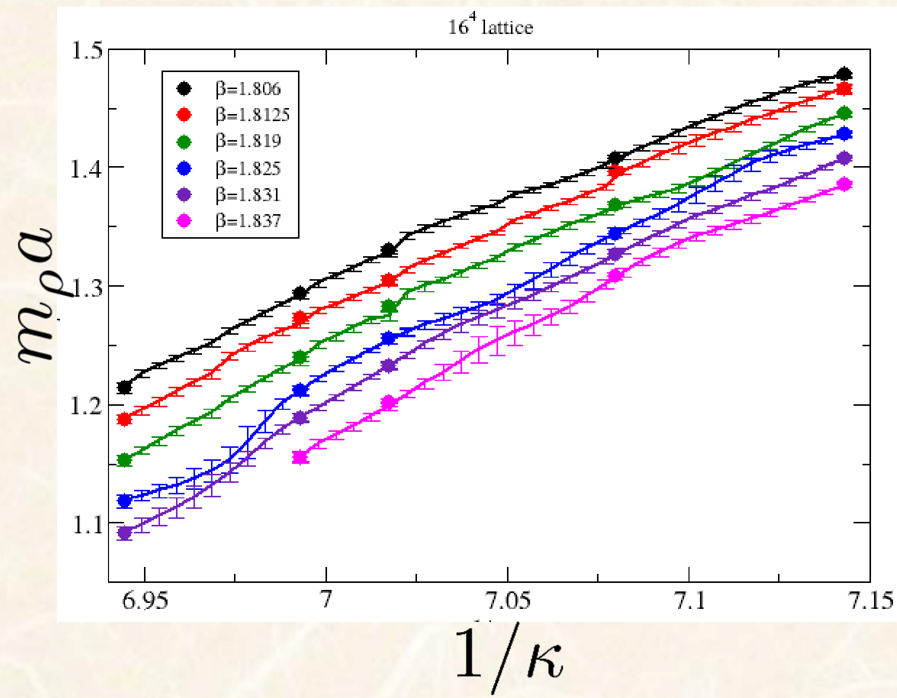
a : lattice spacing

- The derivatives must be measured along the lines of constant physics (quark mass) (LCP).
- The multi-point reweighting is useful for the calculation of the derivatives and lines of constant physical quantities.
- First, we determine LCP. We define LCP as $m_\pi/m_\rho = Const.$
- Then, we measure β and κ dependence of the lattice spacing a.

Hadron masses

- We measure hadron correlators using multi-point reweighting and obtain the hadron masses as continuous function of (β, κ)
  - $m_\pi a$ (Pseudo scalar mass)
  - $m_\rho a$ (Vector mass)





- (black,red,green,blue,violet,pink) symbols:

  Results by the reweighting method

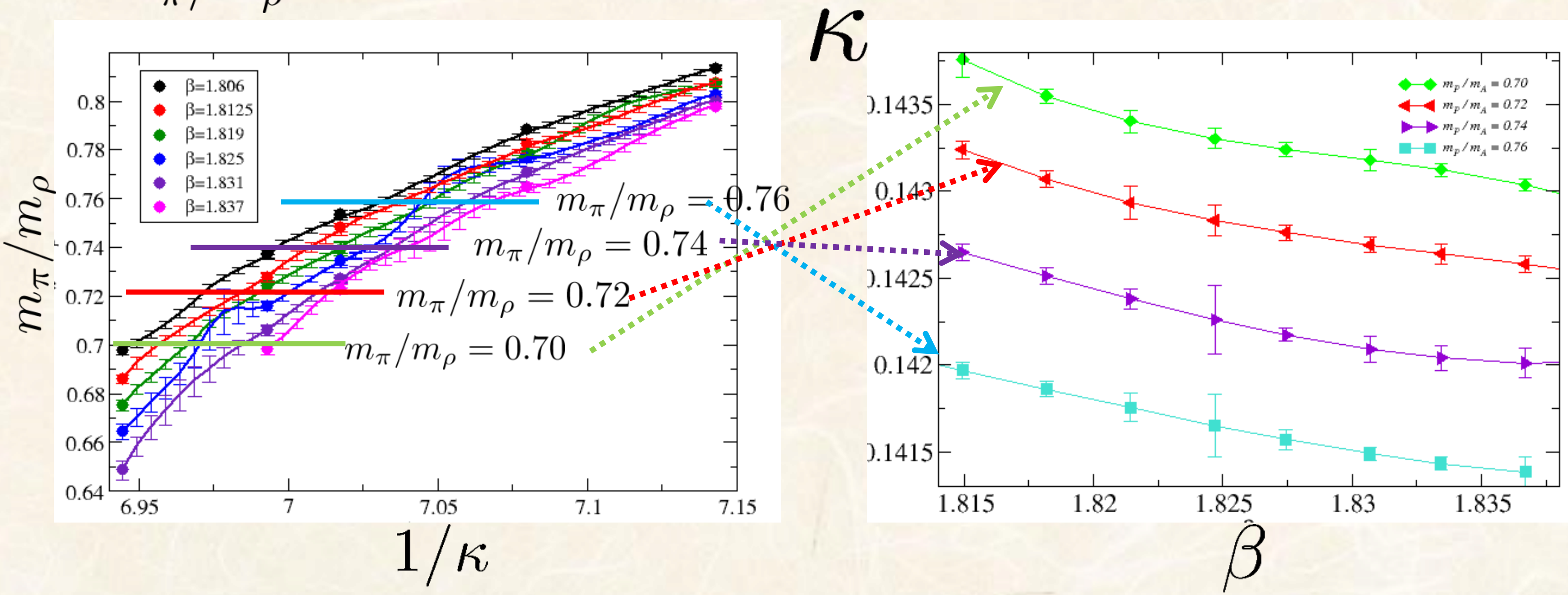
  @ $\beta = (1.806, 1.8125, 1.819, 1.825, 1.831, 1.837)$

Lines of Constant Physics

- We define the lines of constant physics as $m_\pi / m_\rho = Const.$
- From data of m_π / m_ρ , we determine the LCP.

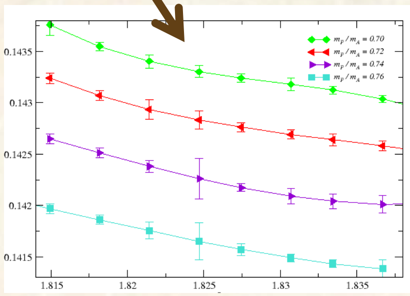
➢ m_π / m_ρ

➢ LCP



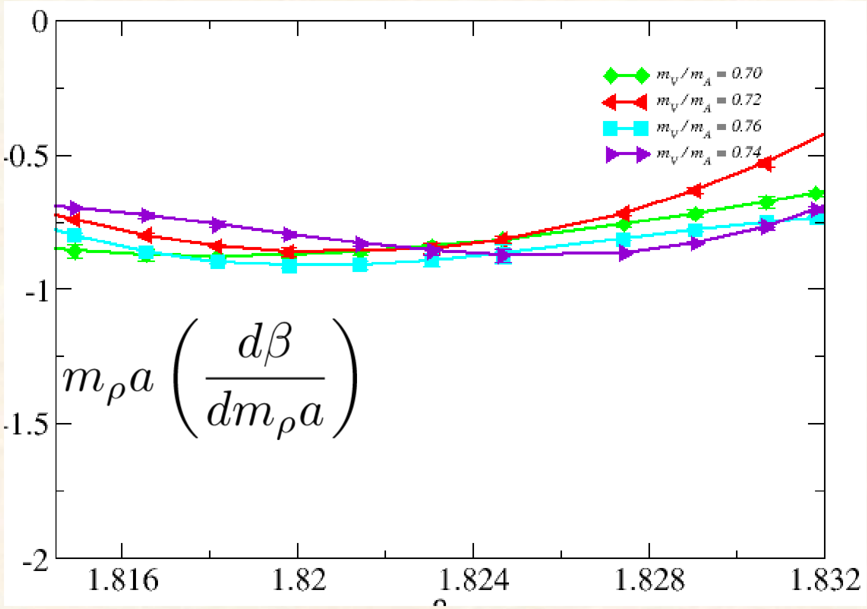
Beta-functions

LCP



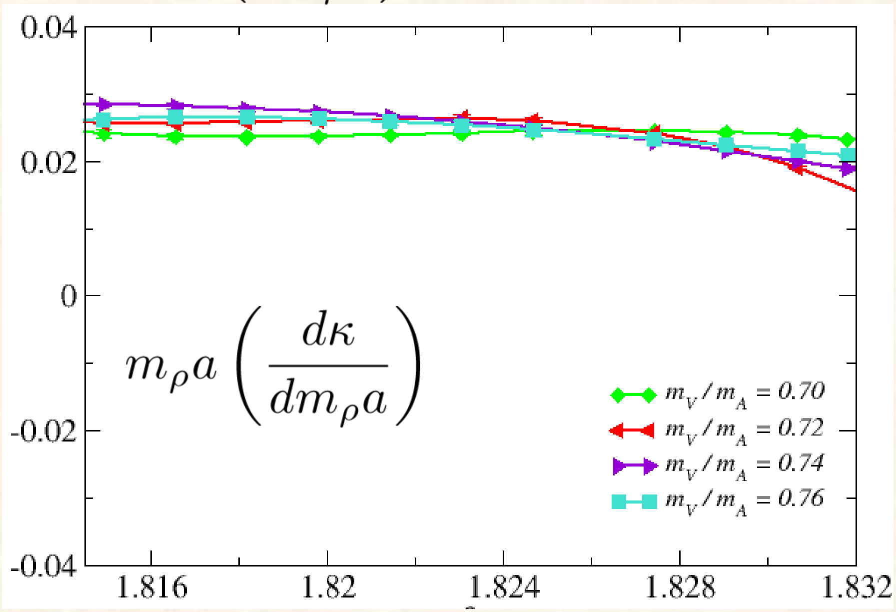
Along LCP, we measure $m_\rho a$ and calculate the derivatives with respect to β and κ.

- $m_\rho a \left(\dfrac{d\beta}{dm_\rho a} \right)$



β

- $m_\rho a \left(\dfrac{d\kappa}{dm_\rho a} \right)$



β

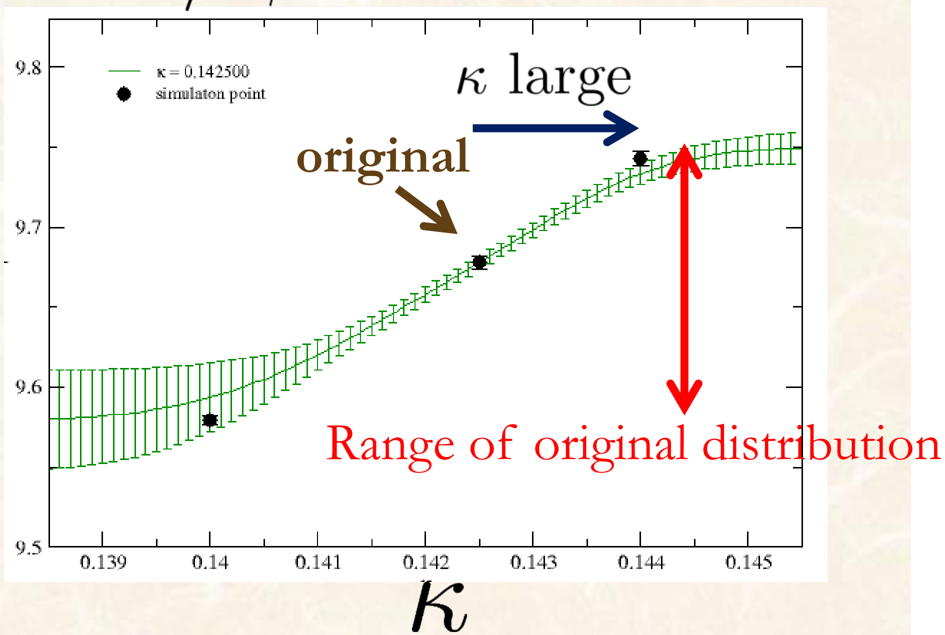
It is useful for calculation of EOS.

SUMMARY

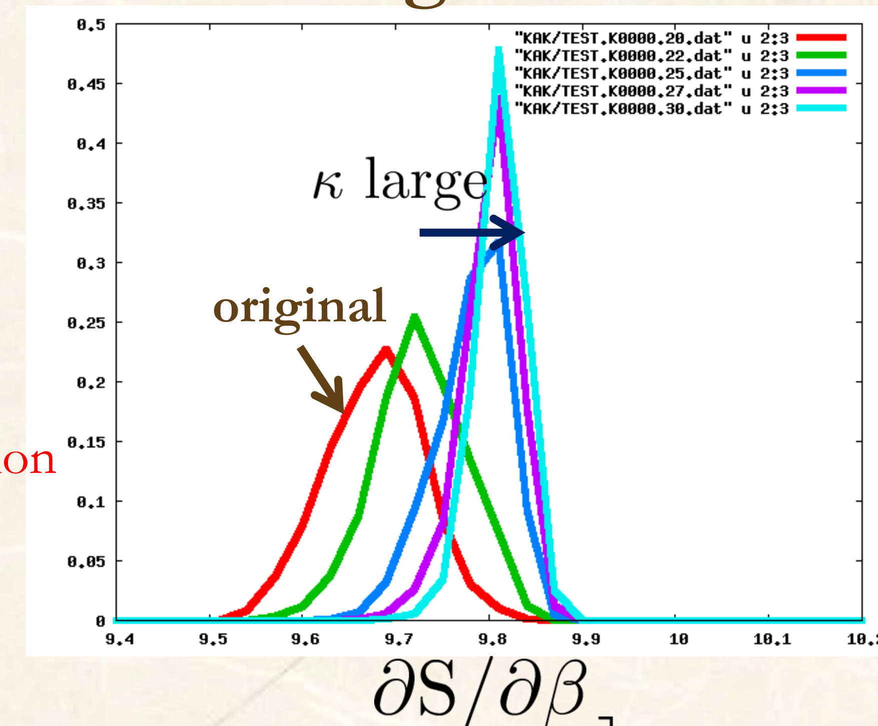
- We discussed multi-parameter and multi-simulation-point reweighting method to avoid the **overlap problem**.
- Using the method,
  - we can calculate physical quantities as functions of (β,κ).
  - histograms of physical quantities changes continuously as functions of (β,κ).
- Lines of physical quantity constant can be measured in the (β, κ) plane.
- We compute the beta-functions for the EOS.

Reweighting method and Overlap problem

- $\partial S/\partial\beta$



- **histogram**



$$S = N_f \ln \det M(\kappa, c_{SW}) - S_G \quad \frac{\partial S}{\partial \beta} = \left[\sum_{x,\mu>\nu} c_0 W^{1\times1}_{\mu\nu}(x) + \sum_{x,\mu\neq\nu} c_1 W^{1\times2}_{\mu\nu}(x) \right] + (clover\ \mathrm{term})$$

- Black symbols: Expectation values at simulation point.
- Green curve: Results by the reweighting method
- The expectation value does not go out of the original distribution.
- Also, the histogram cannot go out of the original distribution.