Pion-pion scattering phase shifts with the stochastic LapH method

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Scattering using Lattice QCD

- energies will be shifted from their non-interacting values.
- Accurate measurements of single and two-hadron energies below inelastic thresholds can give information of the phase shifts.
- Method:
  - Compute energies of $\pi\pi$ and ρ like operators in many channels with different momentum
  - Get the energy E_{cm} in the center of mass frame
  - Then p_{cm} is computed from $E_{cm} = 2\sqrt{{p_{cm}}^2 + m_\pi^2}$ which can be used to find the elastic pion-pion phase shift.

Computing the phase shift

- Lüscher's method for phase shifts:

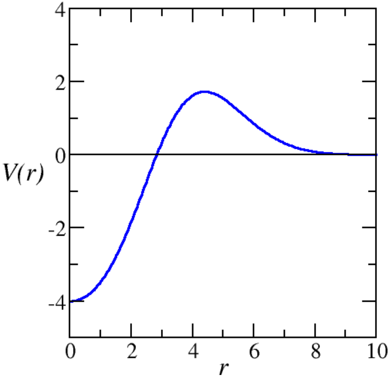
$$\det\left[e^{2i\delta(p_{cm})} - U^{(P,\Lambda)}\left(\frac{p_{cm}L}{2\pi}\right)\right] = 0$$

- The first term is diagonal and independent upon the lattice volume
- The matrix U has rows corresponding to different angular momenta ℓ which subduce onto the irrep Λ of the little group.
- Assume that $\delta_1 \gg \delta_{l>1}$, neglect all but the lowest partial wave making this is a single equation with one unknown.
- The U can be expressed $U = (M+i)/(M-i)$ where M is dependent upon the irrep and p_{cm}
- This M is expressed in terms of generalized zeta functions which have an integral representation and can be numerical estimated.

Resonances in a box: an example

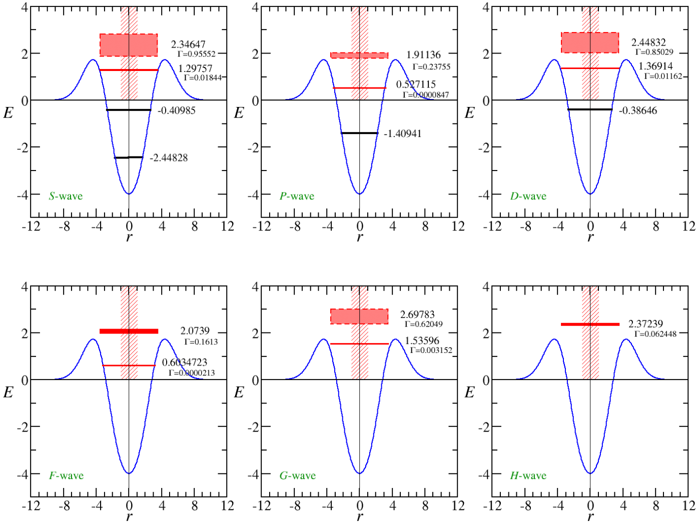
- Consider a simple quantum mechanical example.
- Hamiltonian

$$H = \frac{1}{2}\boldsymbol{p}^2 + V(r), \qquad V(r) = (-4 + \tfrac{1}{16}r^4)\, e^{-r^2/8}$$



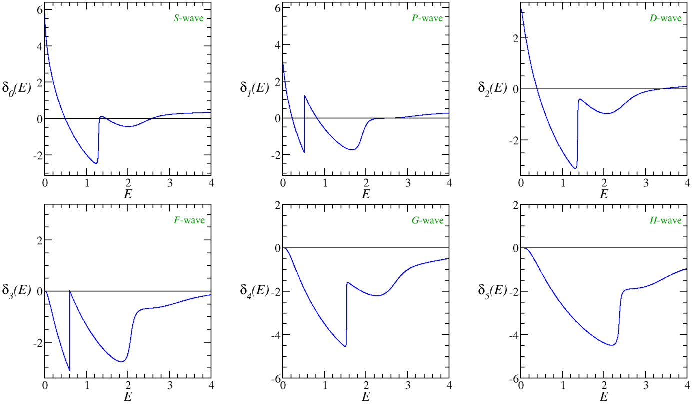
Spectrum of example Hamiltonian

- spectrum for $E < 4$ and $l = 0, 1, 2, 3, 4, 5$ of example system



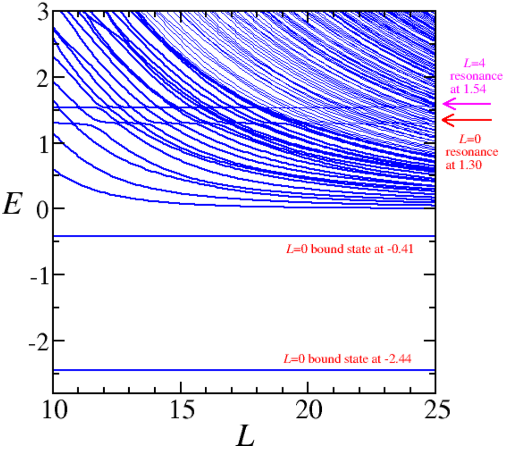
Scattering phase shifts

- scattering phase shifts for various partial waves



Spectrum in box: A_{1g} channel

- spectrum discrete in box, periodic b.c., momenta quantized
- stationary-state energies in A_{1g} channel shown below
- narrow resonance is avoided level crossing



The challenge

- To compute the phase shift requires very precise measurements of the energy eigenstates in finite volume
- Must construct operators which strongly overlap with the states of interest
- Smeared operators will overlap strong with the low-laying elastic two pion states
  - Stout link smearing
  - LapH quark smearing
- Two hadron (pion-pion) operators can be efficiently computed using the stochastic-LapH method, including disconnected diagrams
- Using a matrix of correlators and diagonalizing using a GEVP to extract the low energy eigenstates

Stochastic estimation of quark propagators

- do not need exact inverse of Dirac matrix $K[U]$
- use noise vectors η satisfying $E(\eta_i) = 0$ and $E(\eta_i \eta_j^*) = \delta_{ij}$
- Z_4 noise is used $\{1, i, -1, -i\}$
- solve $K[U]X^{(r)} = \eta^{(r)}$ for each of N_R noise vectors $\eta^{(r)}$, then obtain a Monte Carlo estimate of all elements of K^{-1}

$$K_{ij}^{-1} \approx \frac{1}{N_R} \sum_{r=1}^{N_R} X_i^{(r)} \eta_j^{(r)*}$$

- variance reduction using noise dilution
- dilution introduces projectors

$$P^{(a)} P^{(b)} = \delta^{ab} P^{(a)}, \qquad \sum_a P^{(a)} = 1, \qquad P^{(a)\dagger} = P^{(a)}$$

- define

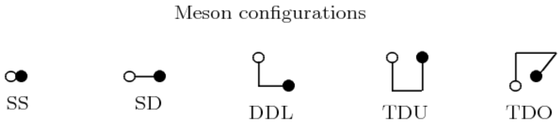
$$\eta^{[a]} = P^{(a)} \eta, \qquad X^{[a]} = K^{-1} \eta^{[a]}$$

to obtain Monte Carlo estimate with drastically reduced variance

$$K_{ij}^{-1} \approx \frac{1}{N_R} \sum_{r=1}^{N_R} \sum_a X_i^{(r)[a]} \eta_j^{(r)[a]*}$$

Extended operators for single hadrons

- quark displacements build up orbital, radial structure



$$\overline{\Phi}^{AB}_{\alpha\beta}(\boldsymbol{p},t) \quad = \quad \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot(\mathbf{x}+\frac{1}{2}(\boldsymbol{d}_\alpha+\boldsymbol{d}_\beta))}\delta_{ab}\ \overline{q}^{B}_{b\beta}(\boldsymbol{x},t)\ q^{A}_{a\alpha}(\boldsymbol{x},t)$$

- group-theory projections onto irreps of lattice symmetry group

$$\overline{M}_l(t) = c^{(l)*}_{\alpha\beta}\ \overline{\Phi}^{AB}_{\alpha\beta}(t)$$

- definite momentum $\boldsymbol{p}$, irreps of little group of $\boldsymbol{p}$

Two-hadron operators

- our approach: superposition of products of single-hadron operators of definite momenta

$$c^{I_{3a} I_{3b}}_{\boldsymbol{p}_a \lambda_a;\ \boldsymbol{p}_b \lambda_b} \ M^{I_a I_{3a} S_a}_{\boldsymbol{p}_a \Lambda_a \lambda_a i_a} \ M^{I_b I_{3b} S_b}_{\boldsymbol{p}_b \Lambda_b \lambda_b i_b}$$

- fixed total momentum $\boldsymbol{p} = \boldsymbol{p}_a + \boldsymbol{p}_b$, fixed $\Lambda_a, i_a, \Lambda_b, i_b$
- group-theory projections onto little group of $\boldsymbol{p}$ and isospin irreps
- Included momentum (n_x, n_y, n_z):
  - $n^2 = 1 \quad (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$
  - $n^2 = 2 \quad (\pm 1, \pm 1, 0), (0, \pm 1, \pm 1), (\pm 1, 0, \pm 1)$
  - $n^2 = 3 \quad (\pm 1, \pm 1, \pm 1)$
  - $n^2 = 4 \quad (\pm 2, 0, 0) \ldots$
  - $n^2 = 5 \quad (\pm 2, \pm 1, 0) \ldots$
  - $n^2 = 6 \quad (\pm 2, \pm 1, \pm 1) \ldots$
  - $n^2 = 8 \quad (\pm 2, \pm 2, 0) \ldots$
- efficient creating large numbers of two-hadron operators
- generalizes to three, four, ... hadron operators

Excited states from correlation matrices

- in finite volume, energies are discrete (neglect wrap-around)

$$C_{ij}(t) = \sum_n Z_i^{(n)} Z_j^{(n)*} \, e^{-E_n t}, \qquad Z_j^{(n)} = \langle 0 | \, O_j \, | n \rangle$$

- not practical to do fits using above form to the whole matrix
- define new correlation matrix $\widetilde{C}(t)$ using a single rotation

$$\widetilde{C}(t) = U^\dagger \, C(\tau_0)^{-1/2} \, C(t) \, C(\tau_0)^{-1/2} \, U$$

- columns of U are eigenvectors of $C(\tau_0)^{-1/2} \, C(\tau_D) \, C(\tau_0)^{-1/2}$
- choose τ_0 and τ_D large enough so $\widetilde{C}(t)$ diagonal for $t > \tau_D$
- fits to the diagonal correlators give the N lowest-lying stationary state energies

Fitting

- Energies from correlated- χ^2 fits over some range (t_{min}, t_{max})
- Fit all energy levels below inelastic threshold
- Fits functions account for around the world effects
- Fixed $t_{max} = 38a_t$, t_{min} adjusted to find region without excited state contamination
- Fitting was done using MINUIT minimization routine on many bootstrap samples to determine statistical errors.

Ensembles and run parameters

- plan to use three Monte Carlo ensembles
  - $(32^3|240)$: 412 configs $32^3 \times 256$, $m_\pi \approx 240$ MeV, $m_\pi L \sim 4.4$
  - $(24^3|390)$: 551 configs $24^3 \times 128$, $m_\pi \approx 390$ MeV, $m_\pi L \sim 5.7$
- anisotropic improved gluon action, clover quarks (stout links)
- QCD coupling $\beta = 1.5$ such that $a_s \sim 0.12$ fm, $a_t \sim 0.035$ fm
- strange quark mass $m_s = -0.0743$ nearly physical (using kaon)
- work in $m_u = m_d$ limit so $SU(2)$ isospin exact
- generated using RHMC, configs separated by 20 trajectories

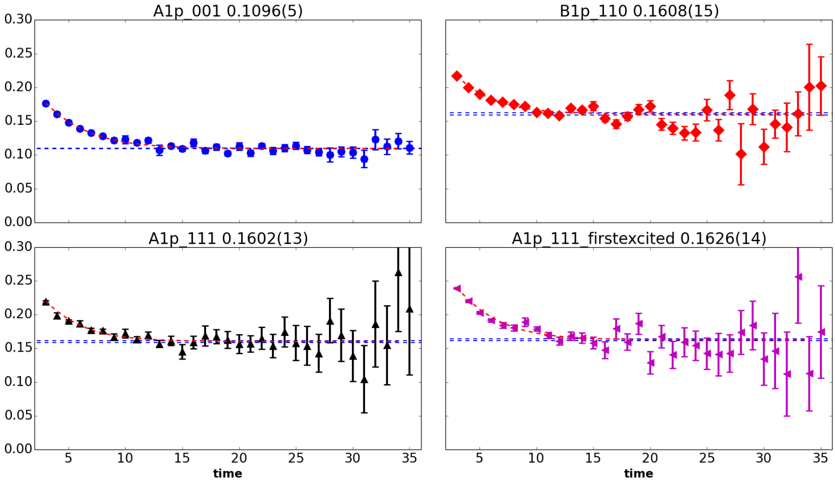
- stout-link smearing in operators $\xi = 0.10$ and $n_\xi = 10$
- LapH smearing cutoff $\sigma_s^2 = 0.33$ such that
  - $N_v = 112$ for 24^3 lattices
  - $N_v = 264$ for 32^3 lattices
- source times:
  - 4 widely-separated t_0 values on 24^3
  - 8 t_0 values used on 32^3 lattice

Irreps and momentum

Irrep	P_{tot}	# of directions	# of levels	# of operators
A1p	$n^2 = 1$	3	3	10
Ep	$n^2 = 1$	3	1	7
A1p	$n^2 = 2$	6	4	10
B1p	$n^2 = 2$	6	2	9
B2p	$n^2 = 2$	6	1	7
A1p	$n^2 = 3$	4	4	12
Ep	$n^2 = 3$	4	2	8
A1p	$n^2 = 4$	3	1	7
Ep	$n^2 = 4$	3	2	9

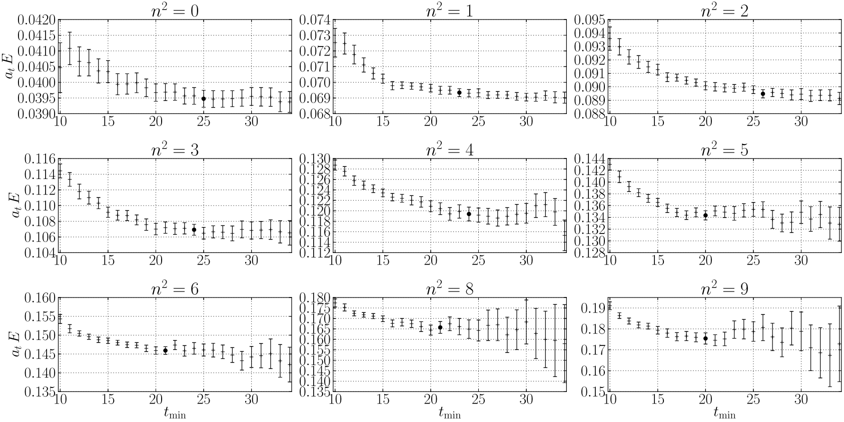
32^2 $I = 1$ Fits PRELIMINARY

Effective masses (with fits), various irreps and momenta



Renormalized anisotropy

anisotropy from energy of πs with different momenta

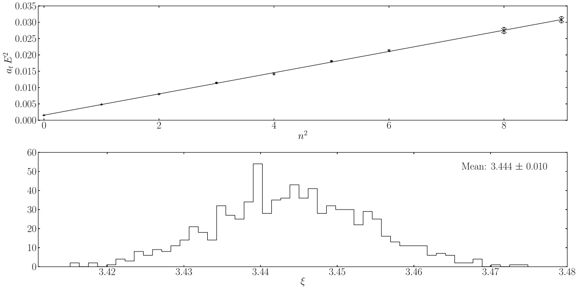


Dispersion Relation

- dispersion relation to determine the anisotropy

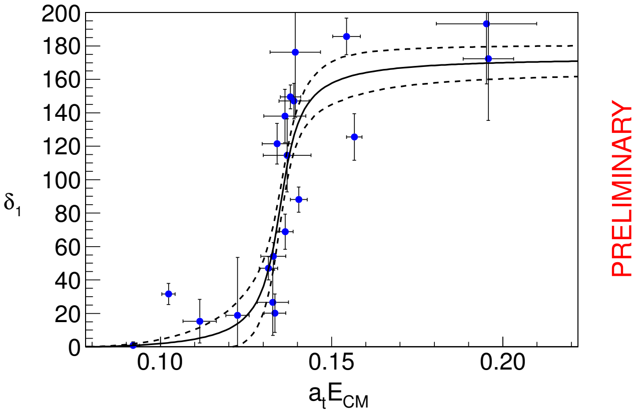
$$(a_t E_{n^2})^2 = (a_t m)^2 + \frac{1}{\xi^2}\left(\frac{2\pi}{4a_s}\right)n^2$$

- Computed on each bootstrap
- Top: E^2 vs total momentum squared n^2
- Bottom: Distribution of anisotropy on each bootstrap



δ_1 32^3 results

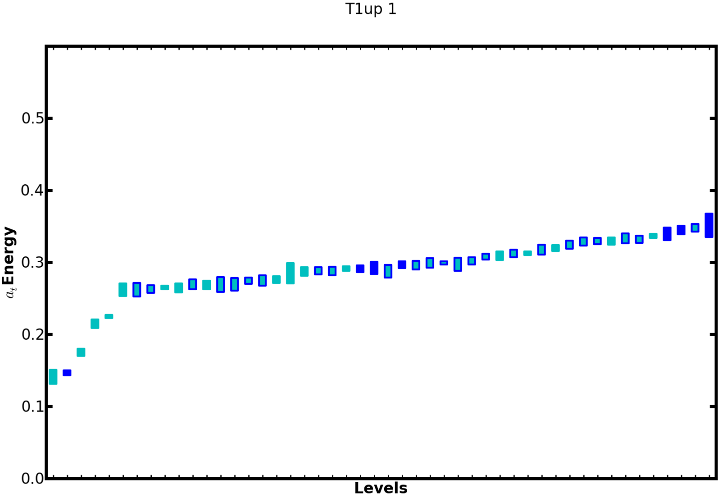
The phase shift for $l = 1$, $I = 1$ $\pi - \pi$ on 32^3 lattice, $m_\pi \approx 240$ MeV



- Breit-Wigner fit $a_t m_R = 0.1355 \pm 0.0019$, $g = 4.3 \pm 1.6$

Wednesday

Preview of talk on Wednesday. C. Morningstar



Conclusions

- The stochastic LapH method is suitable for phase shift calculations
- Able to extract the $\rho - \pi\pi$ phase shift from fits on a single lattice
- This method allows us to compute on reasonably large lattices with minimal inversions compared to other methods
- Can extract energies of excited states using correlator matrices involving many multi-hadron operators even in channels which have disconnected diagrams
- Other channels and phase shifts are possible with this method in the future