The Yang-Mills gradient flow and renormalization

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NIC, DESY

Outline

- Yang-Mills flow.
- Renormalized couplings, step-scaling.
- Cutoff effects, improvement.
- Small flow time expansion.
- Flow for fermion fields.

Gradient flow

Add "extra" (flow) time coordinate t ($[t] = -2$). Define gauge field $B_\mu(x,t)$

$$\begin{aligned} G_{\nu\mu}(x,t) &= \partial_\nu B_\mu(x,t) - \partial_\nu B_\mu(x,t) + [B_\nu(x,t), B_\mu(x,t)] \\ \frac{dB_\mu(x,t)}{dt} &= D_\nu G_{\nu\mu}(x,t) \quad \left(\sim -\frac{\delta S_{\mathrm{YM}}[B]}{\delta B_\mu}\right) \end{aligned}$$

with initial condition $B_\mu(x, t=0) = A_\mu(x)$.

- Geometry of 4-manifolds **[Atiyah, Bott, Donaldson,...]**.
- Continuous smearing **[R. Narayanan, H. Neuberger. '06]**.

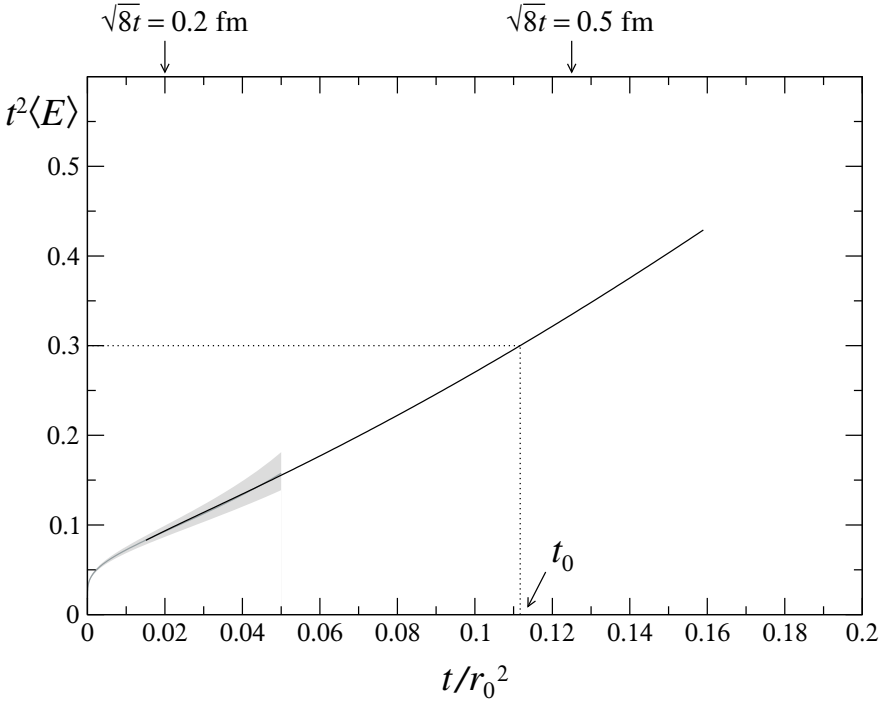
Renormalization and continuum limit.

- Composite gauge invariant operators are renormalized observables defined at a scale $\mu = 1/\sqrt{8t}$ **[M. Lüscher '10; M. Lüscher, P. Weisz '11]**.
- Continuum limit to be taken at fixed t.
- Example

$$\langle E(t)\rangle = -\frac{1}{2}\mathrm{Tr}\langle G_{\mu\nu}(x,t)G_{\mu\nu}(x,t)\rangle$$

finite quantity for $t > 0$.

Scale setting and renormalized couplings



- $t^2\langle E(t)\rangle$ is dimensionless.
- Depends on scale $\mu = 1/\sqrt{8t}$
- Ideal candidate for scale setting: t_0 **[M. Lüscher JHEP 1008 '10]**.
- Similar quantities: $t_1, w_0, \ldots$ **[Borsanyi et. al. '12; R. Sommer Latt. '14]**.
- Renormalized couplings at scale $\mu = \frac{1}{\sqrt{8t}}$

$$t^2\langle E(t)\rangle = \frac{3}{16\pi^2} g^2_{\overline{MS}}(\mu)\left[1 + c_1 g^2_{\overline{MS}}(\mu) + \mathcal{O}(g^4_{\overline{MS}})\right]$$

Non-perturbative coupling definition

$$g^2_{\mathrm{GF}}(\mu) = \frac{16\pi^2}{3} t^2\langle E(t)\rangle\Big|_{\mu=1/\sqrt{8t}}$$

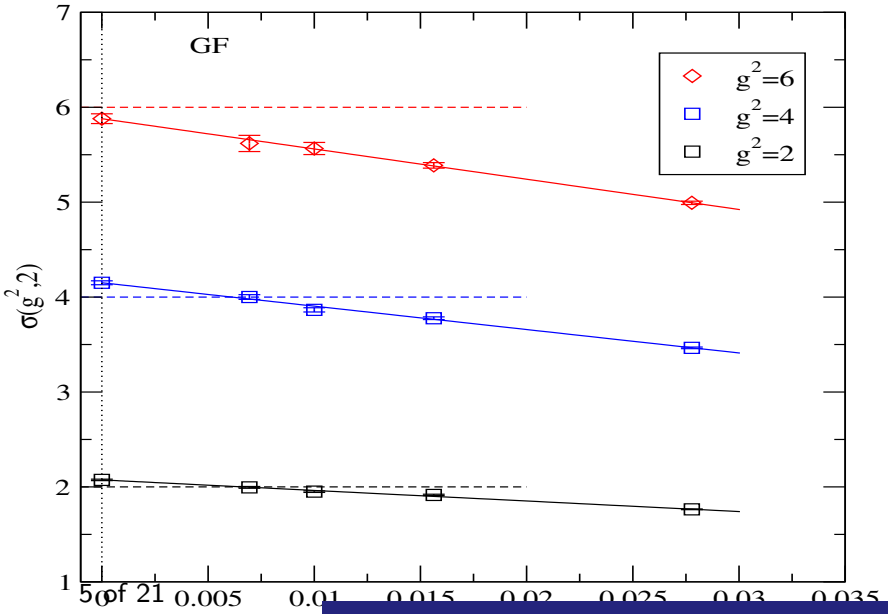
- On the lattice, infinite volume $a \ll \sqrt{8t} \ll L$.

Finite volume renormalization schemes

$$g^2_{\rm GF}(\mu) = \mathcal{N}^{-1} t^2 \langle E(t) \rangle \Big|_{\mu = 1/\sqrt{8t}}$$

- To avoid the need of a window $a \ll \sqrt{8t} \ll L$, use $\mu = 1/cL$.
- Boundary conditions become relevant, and $\frac{16\pi^2}{3} \to \mathcal{N}^{-1}$:
  - Periodic **[Z. Fodor et al. '12]**, SF **[P. Fritzsch, A. Ramos '13]**, Twisted (a là t'Hooft) **[A. Ramos '13]**, SF-open **[M. Lüscher '14]**.

[J. Rantaharju '13: SU(2) with 2 adjoint fermions]



- Main ingredient: step scaling function

$$\sigma(u,s) = g^2_{\rm GF}(\mu/s)\Big|_{g^2_{\rm GF}(\mu)=u}$$

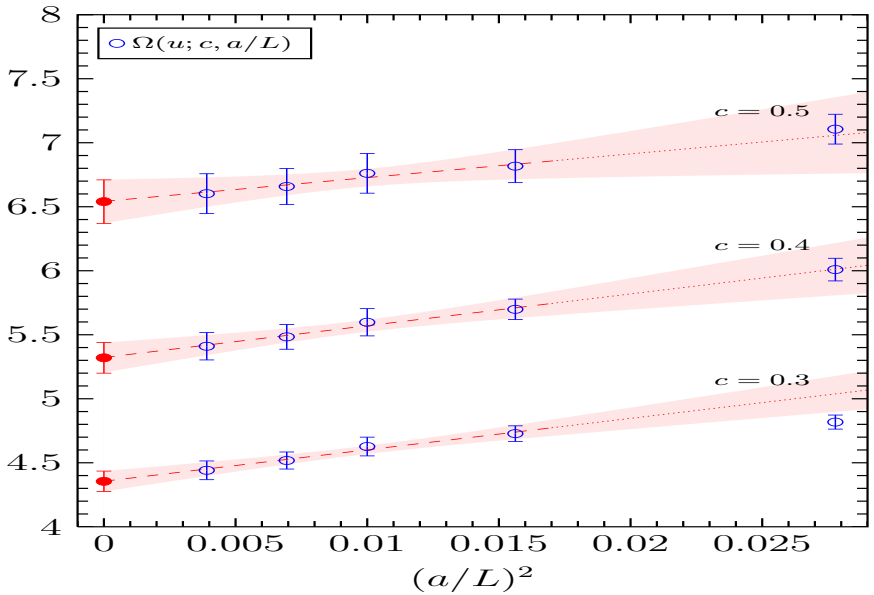
- Simple on the lattice ($L/a \to sL/a$)

$$\sigma(u,s) = \lim_{a/L \to 0} \Sigma(u,s,a/L)$$

- QCD: follow $g^2(\mu)$ from $\mu \sim 0.5\mathrm{GeV}$ to $\mu \sim 100\mathrm{GeV}$ **[P. Fritzsch Thu@14:15]**
- BSM: conformal window without FV effects

Main advantage: precision [P. Fritzsch, A. Ramos. '13]

L/a	6	8	10	12	16
β	5.2638	5.4689	5.6190	5.7580	5.9631
$\kappa_{\rm sea}$	0.135985	0.136700	0.136785	0.136623	0.136422
$N_{\rm meas}$	12160	8320	8192	8280	8460
$\overline{g}^2_{\rm SF}(L_1)$	4.423(75)	4.473(83)	4.49(10)	4.501(91)	4.40(10)
$\overline{g}^2_{\rm GF}(\mu)\,(c=0.3)$	4.8178(46)	4.7278(46)	4.6269(47)	4.5176(47)	4.4410(53)
$\overline{g}^2_{\rm GF}(\mu)\,(c=0.4)$	6.0090(86)	5.6985(86)	5.5976(97)	5.4837(97)	5.410(12)
$\overline{g}^2_{\rm GF}(\mu)\,(c=0.5)$	7.106(14)	6.817(15)	6.761(19)	6.658(19)	6.602(24)



Dependence on c: If c grows:

- Smaller cutoff effects.
- Larger statistical errors.

Main advantage: precision [P. Fritzsch, A. Ramos. '13]

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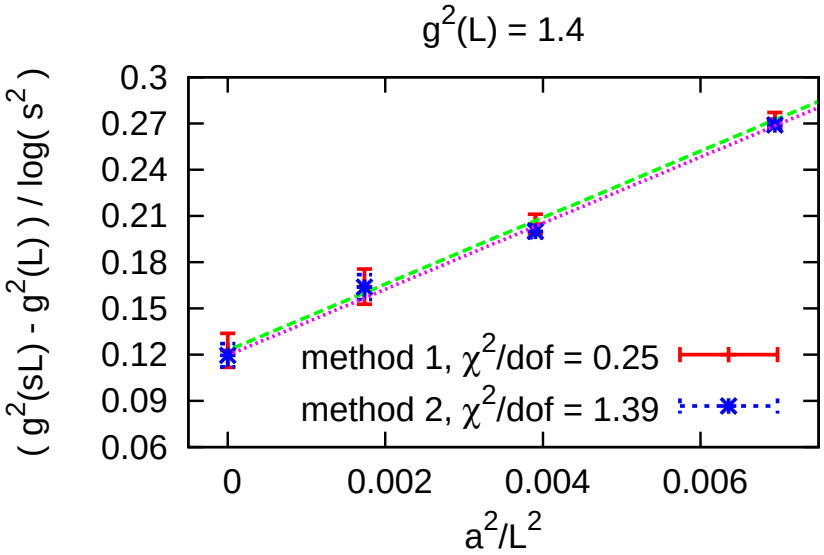
$g_{\rm SF}$ is not dead!

- $g^2 \to 0$: $\delta(g^2_{\rm SF}) \sim g^4_{\rm SF}$
- $a \to 0$: $\delta(g^2_{\rm SF}) \sim 1/a$
- $g^2 \to 0$: $\delta(g^2_{\rm GF}) \sim g^2_{\rm GF}$
- $a \to 0$: $\delta(g^2_{\rm GF}) \sim \text{constant}$

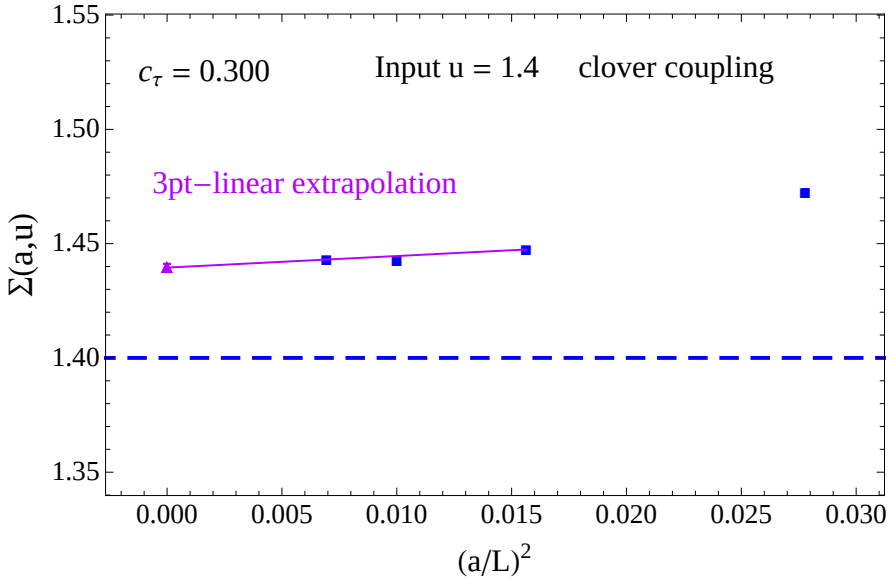
- $g^2_{\rm SF}$ better to match with perturbation theory ($g^2 \sim 1-2$)
- $g^2_{\rm GF}$ better to match with hadronic scale ($g^2 \gtrsim 2$)
- Optimal strategy to determine α_s [P. Fritzsch Thu@14:15].

Cutoff effects of flow observables

Periodic **[Z. Fodor et al. 2012. JHEP 1211 (2012) 007]**



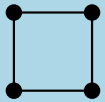
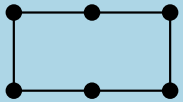
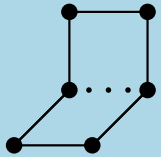
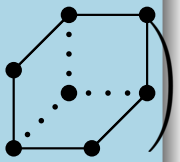
Twisted **[D. Lin Wed@9:40]**



Do boundary conditions play a role in cutoff effects?

Cutoff effects of flow observables

Contribution to $\mathcal{O}(a^2)$ cutoff effects

action : $S(c_i^{(a)}) = \frac{1}{g_0^2} \sum_x \mathrm{Tr} \Big(1 - c_0^{(a)}$  $- c_1^{(a)}$  $- c_2^{(a)}$  $- c_3^{(a)}$  $\Big)$

flow : $\frac{\mathrm{d}}{\mathrm{d}t} V_\mu(x,t) = -g_0^2 \frac{\delta S(c_i^{(f)})}{\delta V_\mu(x,t)} V_\mu(x,t)$

obs : $E(t) = -\frac{1}{2} \mathrm{Tr} G_{\mu\nu}(x,t) G_{\mu\nu}(x,t) = S(c^{(o)})$

- i.e. Wilson action ($c_0^{(a)} = 1, c_1^{(a)} = c_2^{(a)} = c_3^{(a)} = 0$).
- i.e. Symanzik flow ($c_0^{(f)} = 5/3, c_1^{(f)} = -1/12, c_2^{(f)} = c_3^{(f)} = 0$).
- Clover observable. Symanzik observable (use $c_0^{(o)} = 5/3, c_1^{(o)} = -1/12$).

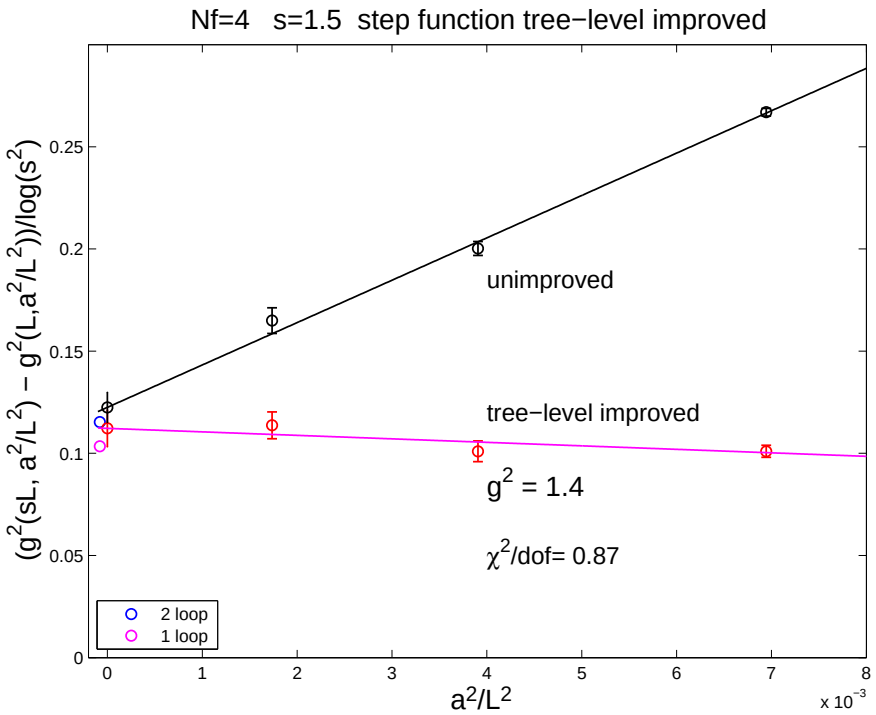
Tree level improvement common in many works

Define coupling with the lattice tree-level computation $\hat{\mathcal{N}} \Longrightarrow$ No tree level cutoff effects.

$$g_{\mathrm{GF}}^2(\mu) = \hat{\mathcal{N}}^{-1} t^2 \langle E(t) \rangle \Big|_{\mu = 1/\sqrt{8t}}$$

Cutoff effects of flow observables

Periodic **[Z. Fodor et al. '14; D. Nogradi Thu@15:40]**

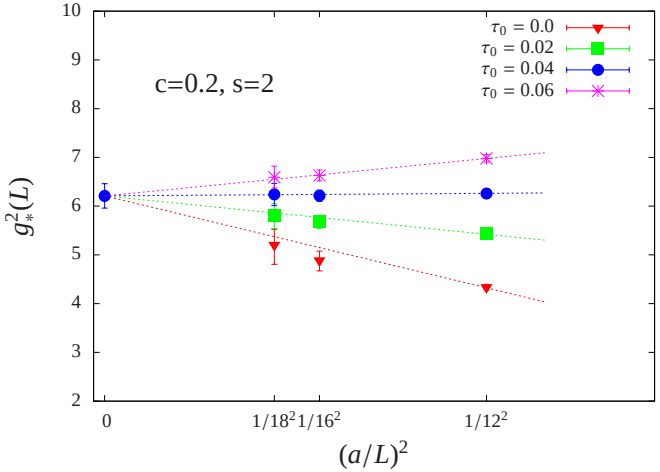


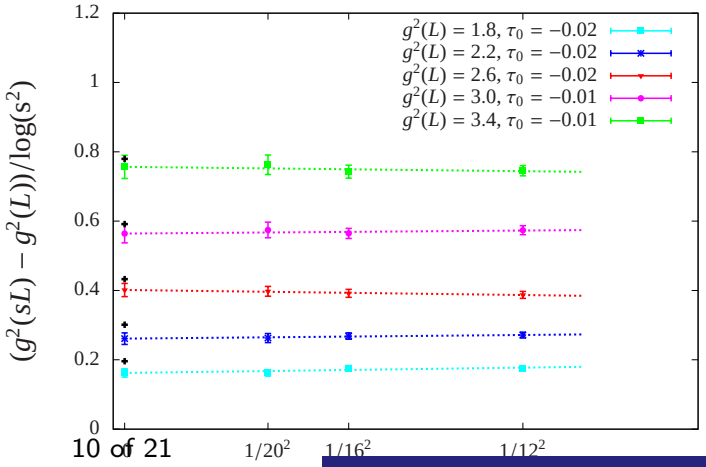
- After subtracting tree-level correction cutoff effects reduced.
- Applications beyond running coupling:
- Removing $\mathcal{O}(a^{2,4,6})$ cutoff effects of $t^2\langle E(t)\rangle$ in infinite volume by choosing $c_1^{(a)}, c_1^{(f)}, c_1^{(o)}$.
- Application to t_0 and w_0.
- Value of the coefficients vary with observable and volume.

- Boundary conditions seem irrelevant for size of cutoff effects.
- Tree-level improvement has a big effect in step-scaling analysis.

More improvement: $t-$shift [Cheng et al. JHEP 1405 (2014) 137; A. Hasenfratz Wed@9:00]

$$g^2(L) = \mathcal{N}^{-1} t^2 \langle E(t + a^2\tau_0)\rangle$$





- Determine $\tau_{\rm opt}$ from large volume simulations (i.e. improving t_0).
- In step scaling analysis τ_0 depends on $g^2_{\rm GF}$.
- The value of τ_0 depends also on the observable.
- Also applied in [J. Rantaharju Wed@9:20].
- Warnings:
  - Careful making $\tau_0(g_0)$.
  - Do not "take τ_0 blindly".
- It comes for free! So try it!

Symanzik improvement and the Zeuthen flow [S. Sint Thu@15:55]

- Symanzik effective action describes cutoff effects of all (improved) observables

$$S^{\mathrm{latt}} = S^{\mathrm{cont}} + a^2 S^{(2)} + \mathcal{O}(a^4)$$
$$\langle O \rangle_{\mathrm{latt}} = \langle O \rangle_{\mathrm{cont}} + a^2 \langle O S^{(2)} \rangle_{\mathrm{cont}} + \mathcal{O}(a^4)$$

- Aim: Choose S^{latt} so that $S^{(2)} = 0$.
- 5D local field theory. Lagrange multiplier imposes flow equation on the bulk.

$$S^{\mathrm{cont}} = -\frac{1}{2g_0^2} \int \mathrm{d}^4x \, \mathrm{Tr}\left\{F_{\mu\nu}F_{\mu\nu}\right\} - 2\int_0^\infty \mathrm{d}t \int \mathrm{d}^4x \, \mathrm{Tr}\left\{L_\mu(x,t)[\partial_t B_\mu(x,t) - D_\nu G_{\nu\mu}]\right\} .$$

- Ansatz for improved action: boundary ($c_i^{(a)}$ and c_4) and bulk ($c_i^{(f)}$) parameters.

$$S^{\mathrm{latt}} = S^{\mathrm{g}}(c_i^{(a)}) + c_4 a^4 \sum_x \mathrm{Tr}\left\{L_\mu(0,x)\left[g^2 \partial^a_{x,\mu} S^{\mathrm{w}}\right]\right\}$$
$$+ a^4 \sum_x \int_0^\infty \mathrm{d}t \, \mathrm{Tr}\left\{L_\mu(x,t)\left[\partial_t V_\mu(x,t) V_\mu^{-1}(x,t) + g^2 \partial_{x,\mu} S^{\mathrm{g}}(c_i^{(f)})\right]\right\} .$$

- Bulk improvement coefficients can not depend on g^2: non-perturbative improvement.

Symanzik improvement and the Zeuthen flow [S. Sint Thu@15:55]

Zeuthen flow: a^2-improved flow equation to all orders in g^2!

$$\partial_t V_\mu(x,t) V_\mu^{-1}(x,t) = -g^2 \partial_{x,\mu} S(c_i^{(f)}) + c_4^{(f)} a^2 \hat{D}_\mu \hat{D}_\mu \partial_{x,\mu} S(V)$$

$c_0^{(f)} = 5/3, c_1^{(f)} = -1/12, c_2^{(f)} = c_3^{(f)} = 0, c_4^{(f)} = 1/12.$

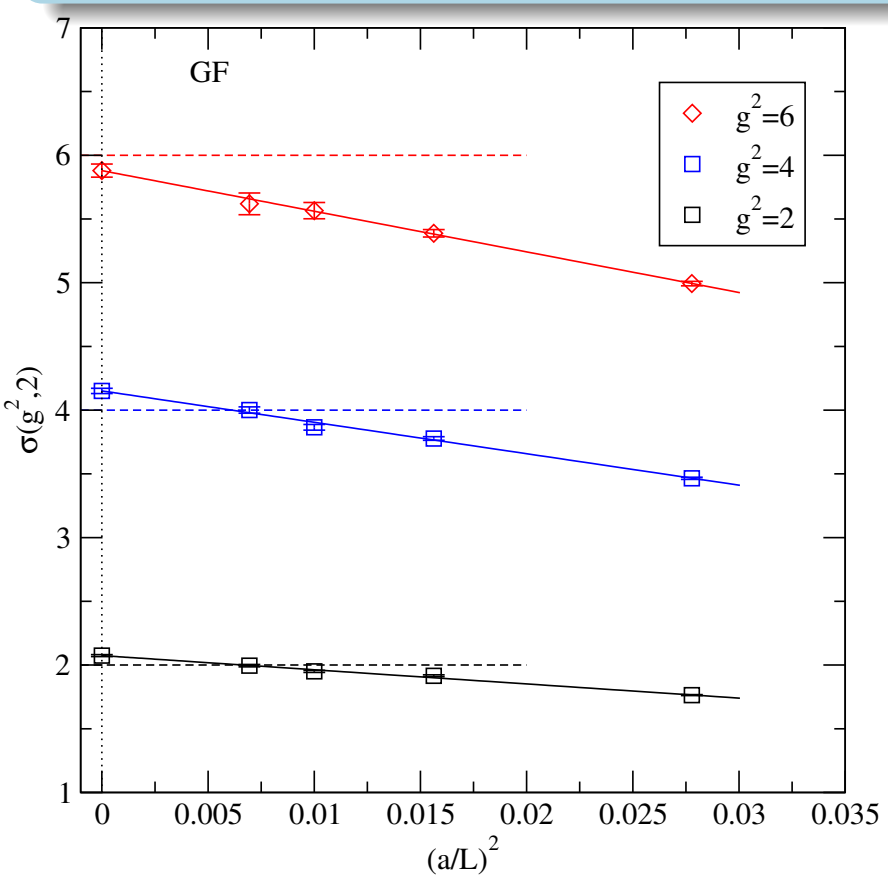
- Observables improved (also all orders in g^2) by demanding classical improvement

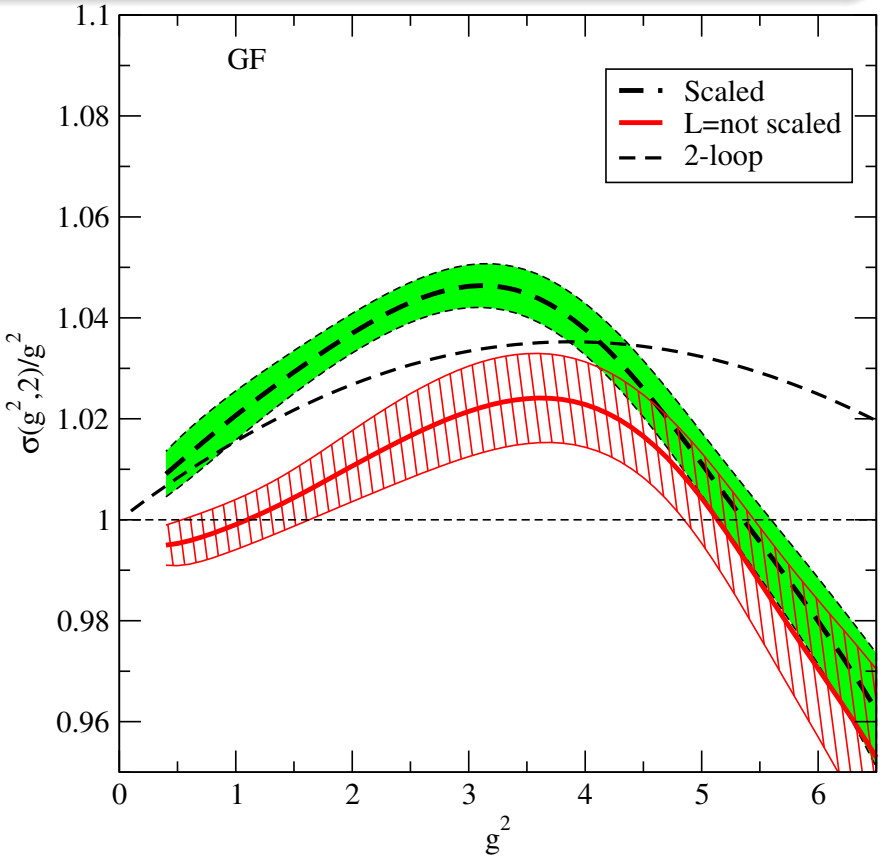
$$c_0^{(f)} = 5/3, c_1^{(f)} = -1/12 \Longrightarrow E^{(improved)}(t)$$

- No $\mathcal{O}(a^2)$ cutoff effects from the flow or from the observable: Only (simulated, 4d) action cutoff effects, and boundary counterterms!
- Tests to leading order in PT.
  - $t^2 \langle E(t) \rangle$ in infinite volume.
  - $t^2 \langle E(t) \rangle$ in finite volume with Twisted bc.
  - Correlators $t^2 s^2 \langle E(t) E(s) \rangle$.
  - ...

Flash of some results: Step scaling and the gradient flow

$SU(2)$ with two adjoint fermions **[J. Rantaharju '13]**

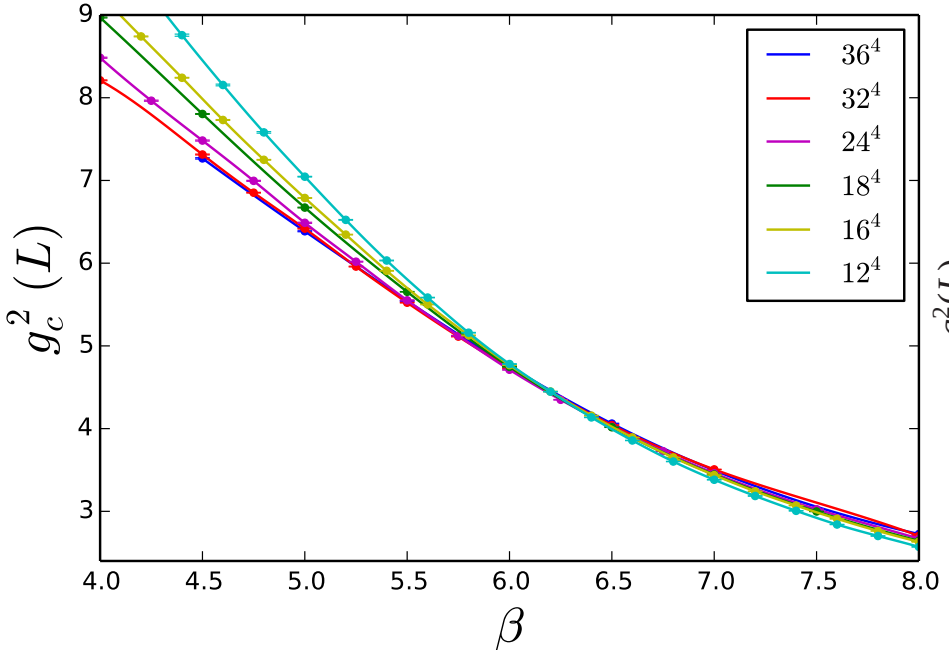


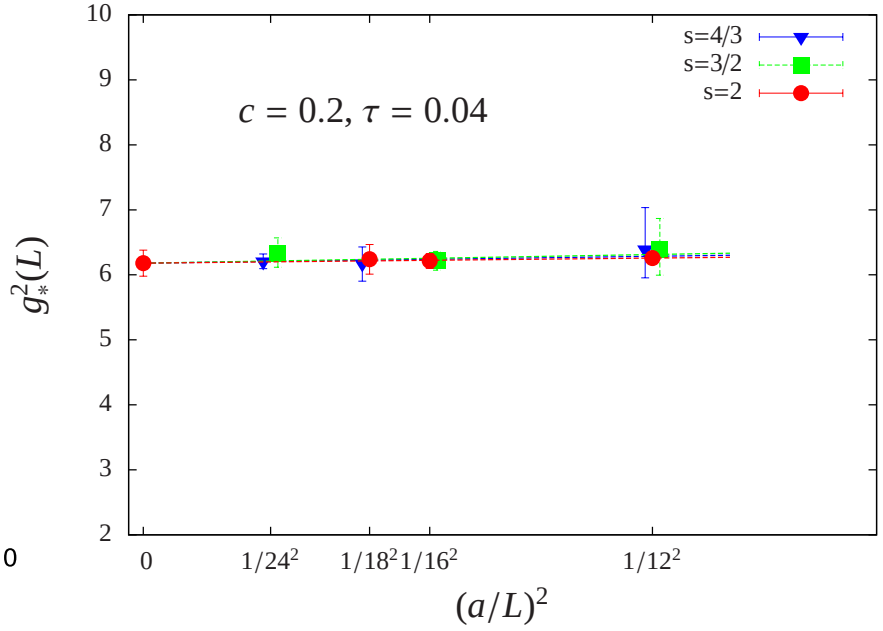


non-perturbative IRFP.

Flash of some results: Step scaling and the gradient flow

$SU(3)$ with 12 fundamental fermions **[Cheng et al. JHEP 1405 (2014) 137]**





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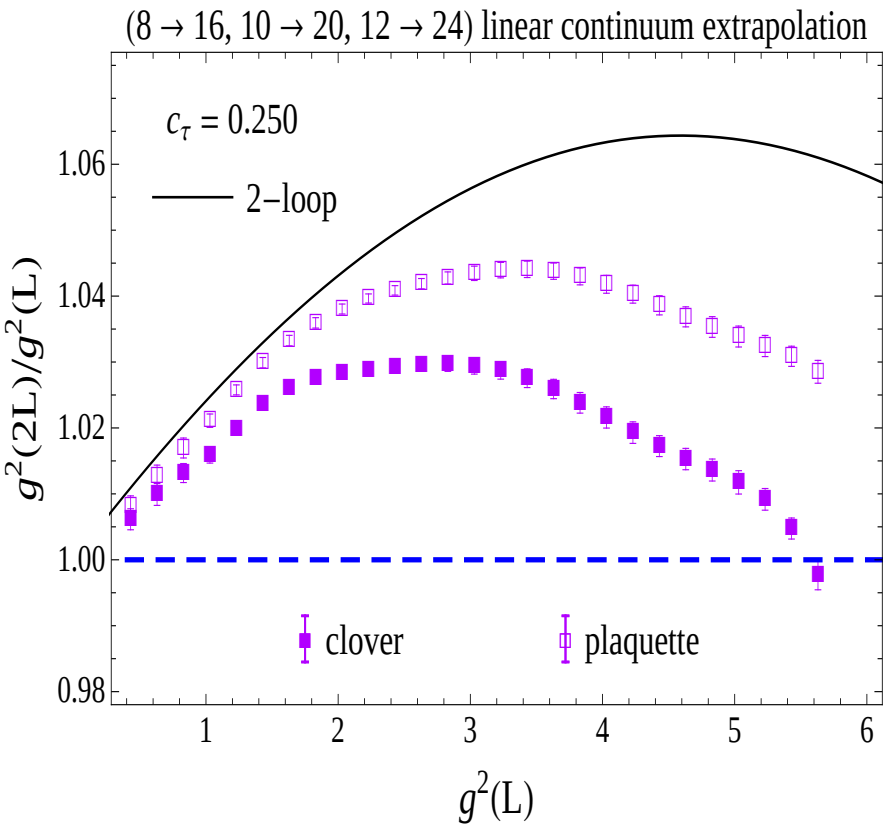
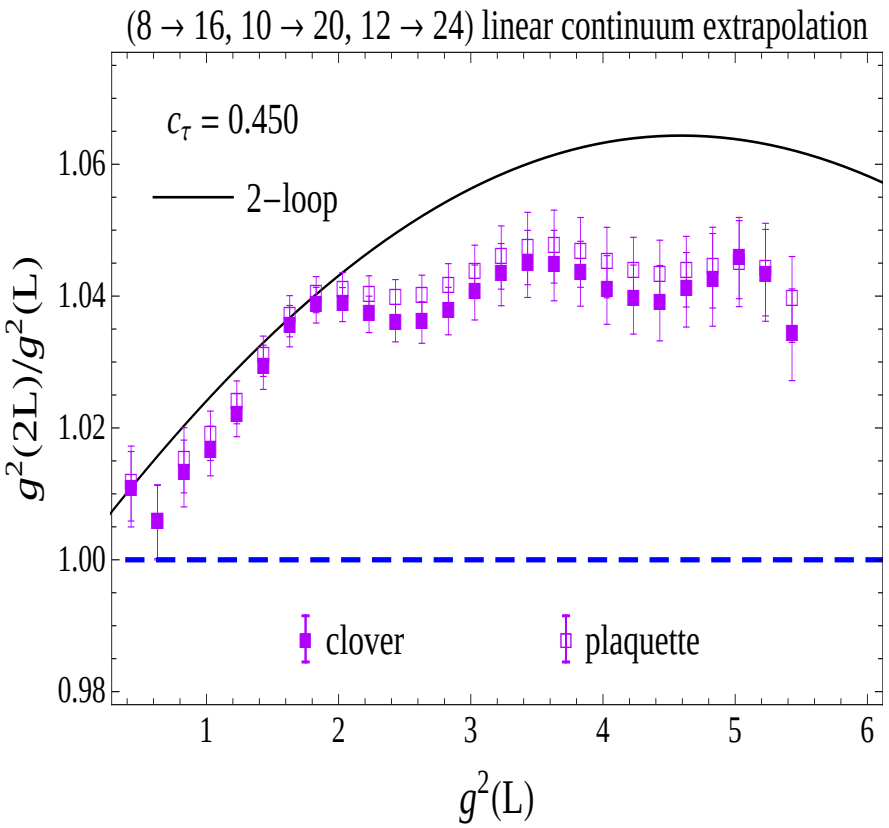
Flash of some results: Step scaling and the gradient flow

$SU(3)$ with 12 fundamental fermions **[D. Lin Wed@9:40]**

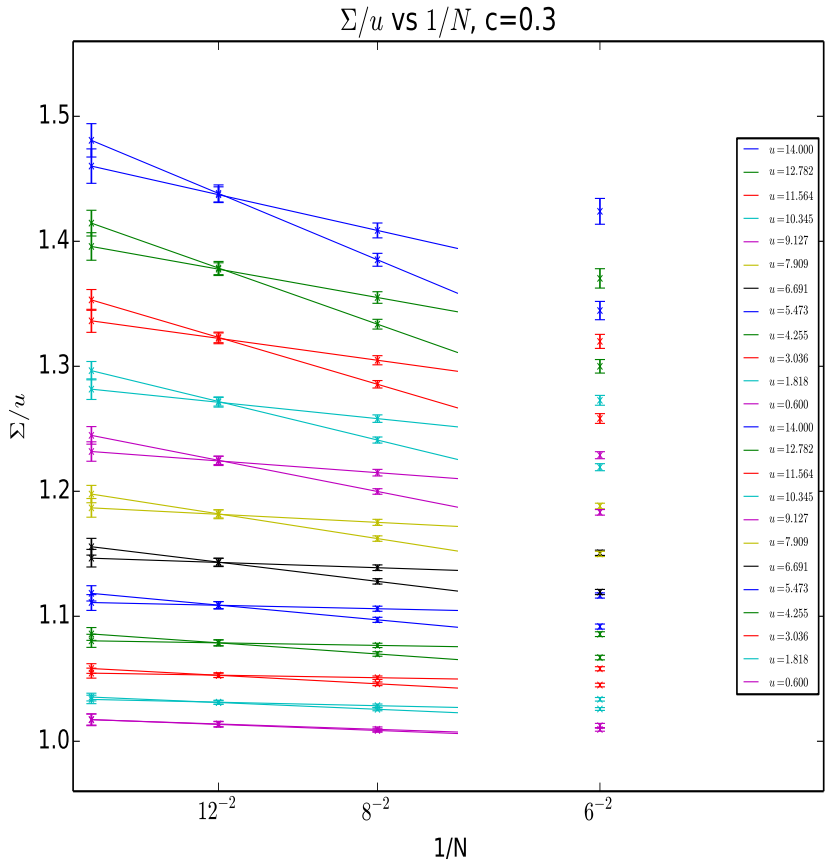


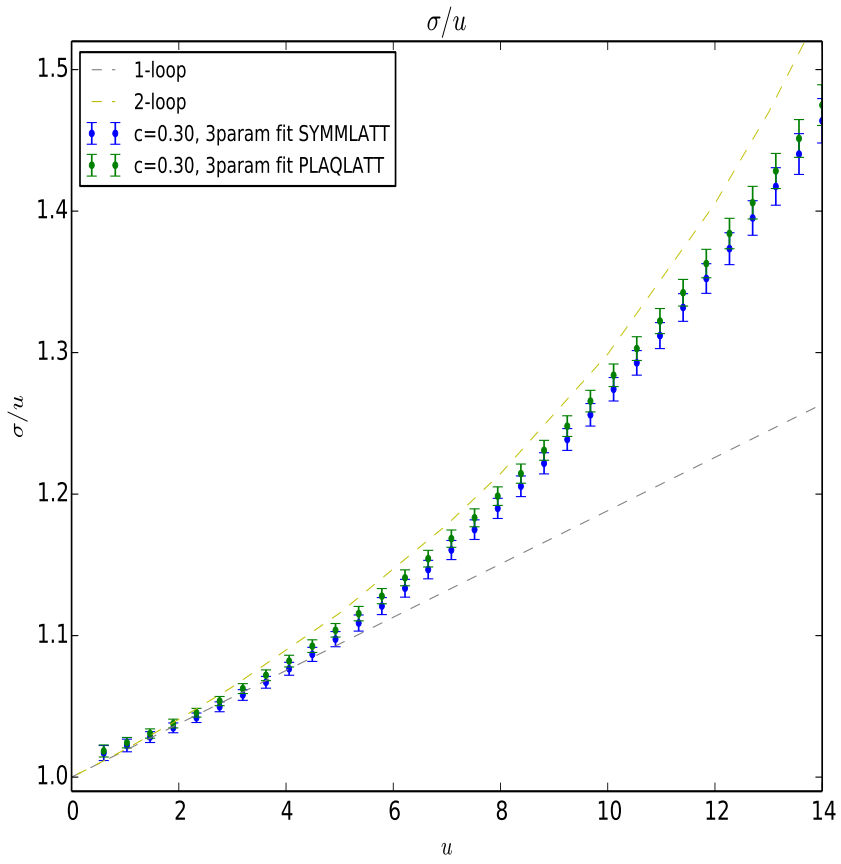


No non-perturbative IRFP. Larger couplings, smaller lattices spacings to come.

Flash of some results: Step scaling and the gradient flow

$SU(N)$ for large N and Reduction **[L. Keegan Tue@14:35]**.





One point lattice and change of scale by $N \to N'$

Small flow time expansion

Any operator at positive flow time has an expansion in terms of renormalized fields

$$O(x,t) = \sum_\alpha c_\alpha(t)\{O^\alpha\}_R(x) + \mathcal{O}(t)$$

Mixing pattern determined by continuum symmetries!

Example: $E(t,x)$

$$E(t,x) = c_1(t)\mathbf{1} + c_2(t)\{F_{\mu\nu}F_{\mu\nu}\}_R(x) + \mathcal{O}(t)$$

can be used to determine spin-0 component of the EM tensor

$$T_{\mu\mu}(x) = \{F_{\mu\nu}F_{\mu\nu}\}_R(x) = \lim_{t\to 0} c_2^{-1}(t)\left[E(t,x) - \langle E(t,x)\rangle\right]$$

We need

- Determination of $c_2(t)$
  - Perturbation theory **[Kitazawa Wed@9:40]**.
  - Non-perturbative determination (analysis correlation function) **[A. Patella Thu@14:55]**.
  - Ward-Identities **[M. Luscher '13, A. Shindler '13, Del Debbio et al '13]**
- Take the continuum limit $a \to 0$ at fixed t.
- Take the limit $t \to 0$.
- Scaling window $a \ll \sqrt{8t} \ll 1/\Lambda$

Fermion flow

Flow for fermion fields **[M. Lüscher, '13]**

$$\partial_t \chi(x,t) = D_\mu D_\mu \chi(x,t); \quad D_\mu = \partial_\mu + B_\mu$$

with initial condition $\chi(x,t)|_{t=0} = \psi(x)$.

- Composite operators O made of $\chi(x,t), \overline{\chi}(x,t)$ renormalize multiplicatively ($t > 0$)

$$\langle O_{\mathrm{R}} \rangle = (Z_\chi)^{(n+n')/2} \langle O \rangle; \quad n \text{ and } n' \text{ number of } \chi \text{ and } \overline{\chi} \text{ fileds.}$$

- Chiral condensate does not mix for $t > 0$ **[M. Lüscher, '13]**

$$\Sigma(t) = \langle \overline{u}(t,x) u(t,x) \rangle$$

- Compute proton strange content **[A. Shindler Tue@14:15]**.

$$m_s \langle N | \overline{s} s(t) | N \rangle_{\mathrm{c}} = c_3(t) m_s \langle N | \overline{s} s(0) | N \rangle_{\mathrm{c}} + \mathcal{O}(t)$$

but chiral symmetry relates $c_3(t)$ with the $G_\pi(t) = |\langle 0 | \pi(t) \rangle|^2$

$$c_3(t) = \frac{G_\pi(t)}{G_\pi(0)}$$

Conclusions

- Many applications of the gradient flow still to come

Concepts

- Step scaling and the gradient flow **[M. Lüscher, '14]**.
- Locally smeared operator product expansions **[Monahan, C. Thu@16:15]**.
- Stochastic perturbation theory **[M. Dalla Brida, D. Hesse '13]**
- Automatic $\mathcal{O}(a)$-improvement and the gradient flow **[A. Shindler '13]**

Applications

- Testing the WittenVeneziano mechanism with the YM gradient flow **[CÈ Marco. Wed@11:10]**.
- The gradient flow running coupling in $SU(2)$ with 8 flavors **[Rantaharju Wed@9:20]**.
- Thermodynamics using Gradient Flow **[Kitazawa Sat@9:30]**.
- String tension from smearing and Wilson flow methods **[M. Okawa Thu@15:15]**.
- Topology density correlator on dynamical domain-wall ensembles with nearly frozen topological charge **[H. Fukaya Wed@12:50]**.
- Beyond the Standard Model Matrix Elements with the gradient flow **[Shindler, A. Tue@14:15]**.
- Shear Viscosity from Lattice QCD **[S. Mages Fri@15:35]**.

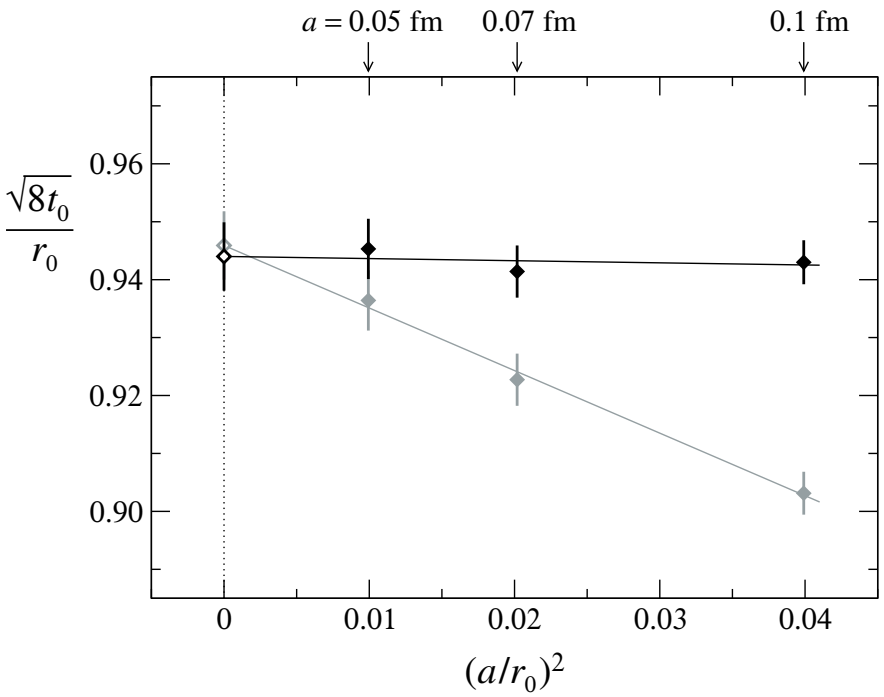
Special thanks

All of you for your attention.

P. Fritzsch, M. Garcia Perez, A. Gonzalez-Arroyo, A. Hasenfratz, P. Korcyl, D. Lin, A. Patella, D. Nogradi, S. Schafer, H. Simma, J. Rantaharju, C. Monahan, S. Sint, R. Sommer, A. Shindler, U. Wolff.

Backup: An urban legend

The symmetric (clover) definition of $E(t)$ produce smaller cutoff effects.



- In **[M. Lüscher '10]** never stated that "clover is better".
- This plot only shows that the Wilson action (pure gauge), with Wilson flow and clover observable produce smaller cutoff effects in $\sqrt{8t_0}/r_0$
- But different sources of cutoff effects can be responsible of this behavior.
- In fact we think that this is an accidental cancellation.
- Not to be expected in general.

Backup: An urban legend

The symmetric (clover) definition of $E(t)$ produce smaller cutoff effects.

"Anatomy" of $\mathcal{O}(a^2)$ cutoff effects: Leading Perturbation theory.

Observable	Action	Flow	Total
Clover	Wilson	Wilson	
15	-3	-9	3
Clover	Lüscher-Weisz	Symanzik	
15	1	3	19

- Something is flat does not mean that there is improvement.

Backup: boundary conditions (I).

Periodic

- Dificult perturbation theory and non-universal $\beta-$function [A. Gonzalez-Arroyo et al. '83].

$$SU(N) \text{ and } N > 2: \; g^2 = g^2_{\rm MS}(1 + \mathcal{O}(g_{\rm MS}))$$
$$SU(2): \; g^2 = g^2_{\rm MS}(1 + \mathcal{O}(1/\log g^2_{\rm MS}))$$

- But no problem non-perturbatively!
- Automatic $\mathcal{O}(a)$-improvement with "massless" Wilson quarks.

Twisted (à la t'Hooft) boundary conditions.

- $SU(N)$ gauge fields and N_f fermions in the fundamental representation requires $N_f/N \in \mathbb{Z}$.
- Ok for adjoint (multi-index) representations.
- Automatic $\mathcal{O}(a)$-improvement with "massless" Wilson quarks.
- Universal β-function.

Backup: boundary conditions (II).

Schrödinger-Functional (Dirichlet) ($L^3 \times T$)

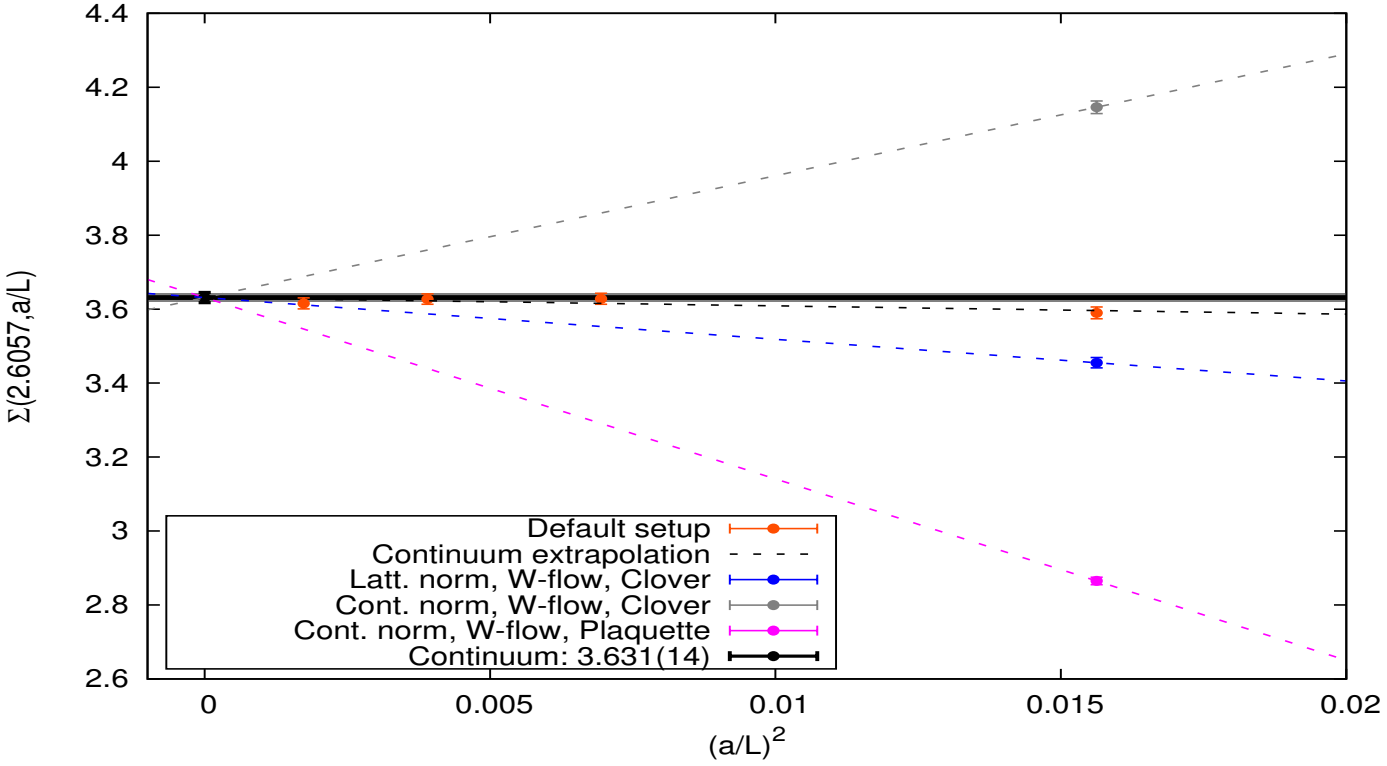
- Work required to keep $\mathcal{O}(a)$ boundary cutoff effects negligible.
- Need c_{SW} for $\mathcal{O}(a)$-improvement (but see the χSF variation **[S. Sint '11]**).

Schrödinger-Functional-Open (Mixed) ($L^3 \times T$)

- Work required to keep $\mathcal{O}(a)$ boundary cutoff effects negligible.
- Need c_{SW} for $\mathcal{O}(a)$-improvement (but χSF idea could be implemented).
- Alleviate critical slowing down: Good ergodicity.
- Topology freezing (<u>might</u>) be difficult to spot in "periodic" schemes.

Backup: Zeuthen flow v1.0

Imposing tree-level improvement to $\langle E(t) \rangle$ in finite volume: $c_0 = 1, c_1 = -1/12, c_2 = 1/24$.



Currently we do not know if this condition is enough to remove all $\mathcal{O}(a^2)$ cutoff effects from the flow.