Induced QCD with N_c auxiliary bosonic fields

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Contents

1. Motivation

Limitations of LQCD – Why changing the gauge action?

Main problem for studies of the QCD phase diagram:

- **Simulating QCD at (real) non-zero chemical potential.** (sign problem)

Possible solutions:

- Use complex Langevin for simulations. [Parisi, PLB 131 (1983); Aarts, Stamatescu, JHEP 0809 (2008); Sexty, arXiv:1307.7748]
- Simulate on a Lefschetz thimble? [Christoforetti *et al*, PRD 86 (2012); PRD 88 (2013)]
- Dual variables and worm algorithms [e.g. Delgado Mercado *et al*, PRL 111 (2013), Gattringer, Lattice 2013]
- Fermion bags [e.g. Chandrasekharan, EPJA 49 (2013)]

Often it is the gauge action which makes it difficult to find solutions.
(see e.g. strong coupling solution to sign problem [Karsch, Mütter, NPB 313 (1989)])

Idea: Find an alternative discretisation of pure gauge theory which allows the use of strong coupling methods!

$\Rightarrow$ A gauge action which is linear in the gauge fields might do this job!

Induced QCD

This idea is not new!

Ansatz: Induce pure gauge dynamics using auxiliary fields.

- Using fermionic fields:
  - with standard (Wilson) fermions. [Hamber, PLB 126 (1983)]
  - Standard fermions + 4-fermion current-current interaction. [Hasenfratz, Hasenfratz, PLB 297 (1992)]

  Need the limit $N_f \to \infty$, $\kappa \to 0$.
- Using scalar fields:
  - Spin model. [Bander, PLB 126 (1983)]
    Needs the limit $N_s \to \infty$ and $g_s \to \infty$.
  - Adjoint scalar fields. [Kazakov, Migdal, NPB 397 (1992)]
    No "exact" pure gauge limit.
    It is interesting since it allows a solution in terms of large N_C.

⇒ This is where our induced model offers improvement!

Lattice regularised path integrals – fixing notations

Expectation value of operator O:

$$\langle O \rangle = \frac{1}{Z} \int [dU][d\psi][d\bar{\psi}] \; O \; \omega_G[U] \; \omega_F[\psi, \bar{\psi}, U]$$

- $\omega_G[U]$: Pure gauge weight factor.
- $\omega_F[\psi, \bar{\psi}, U]$: Quark weight factor.

Typically: $\omega_G[U] \, \omega_F[\psi, \bar{\psi}, U] = \exp\left[-S[\psi, \bar{\psi}, U]\right]$.

Basic demands:

- The discretised action has to reproduce the continuum Yang-Mills action.
- All weight factors should be gauge invariant.

2. The new weight factor

Zirnbauer's weight factor

Consider the weight factor: [Budczies, Zirnbauer, math-ph/0305058]

$$\omega_{\mathrm{BZ}}[U] \sim \prod_p \left|\det\left(m_{\mathrm{BZ}}^4 - U_p\right)\right|^{-2N_b}$$

Here:

- p is an index running over unoriented plaquettes U_p.
- m_{BZ} is a real parameter with $m_{\mathrm{BZ}} \geq 1$
  (or more generally $m_{\mathrm{BZ}} \in \mathbb{C}$ with $\mathrm{Re}(m_{\mathrm{BZ}}) \geq 1$)
- N_b is an integer number
- we consider a hypercubic lattice

Does this weight factor have anything to do with continuum Yang-Mills theory?

Why is this weight factor interesting?

Non-trivial pure gauge limit

There is a trivial pure gauge limit for $\alpha_{\mathrm{BZ}} \, (= m_{\mathrm{BZ}}^{-4}) \to 0 \quad N_b \to \infty$.
(I will not discuss this here)

Zirnbauers conjecture: [Budczies, Zirnbauer, math-ph/0305058]

At fixed $N_b \geq N_c$ and $d \geq 2$ the theory has a continuum limit for $\alpha_{\mathrm{BZ}} \to 1$ which reproduces continuum Yang-Mills theory.

(excluding the case $d = 2$ and $N_b = N_c$)

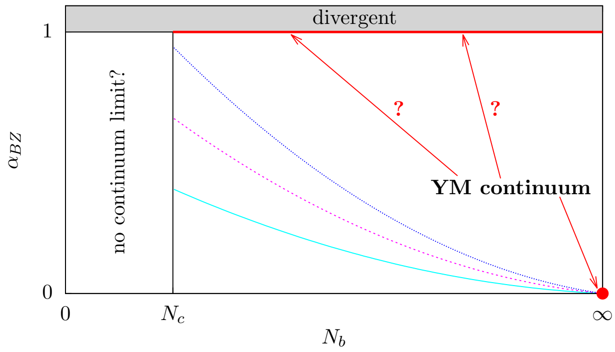
- This can be shown rigorously for $d = 2$ and $N_b > N_c$.

  The proof for $\mathrm{U}(N_c)$ is given in [math-ph/0305058] .

  It is straightforwardly extended to $\mathrm{SU}(N_c)$.
  (we will not go through the details here)
  (probably $N_b > N_c - 1$ is sufficient for $\mathrm{SU}(N_c)$)

- For $d > 2$ the equivalence with Yang-Mills theory is only a conjecture and relies on the increase of the collective behaviour when going to $d > 2$.

Phases in the $(N_b, \alpha_{\mathrm{BZ}})$ parameter space



$\Rightarrow$ We will now test this limit numerically!

3. Numerical tests

Basic idea and setup

Consider the cheap case: SU(2) at $d = 3$!

Suitable observables for a first test:

- $T = 0$ observables:
  Quantities connected with the $q\bar{q}$ potential.
- $T \neq 0$ observables:
  Transition temperature and order of the transition.

Simulation setup:

- Wilson theory: Standard mixture of heatbath and overrelaxation updates.
- Induced theory: Local metropolis with random link proposal.
- Computation of $q\bar{q}$ potential: Lüscher-Weisz algorithm

  [Lüscher, Weisz, JHEP 0109 (2010)]
- Scale setting: Sommer parameter r_0

  [Sommer, NPB 411 (1994)]

Scale setting and matching

First step: Matching between α ($\sim m^{-4}$) and β.

- Start with some information from $\langle U_p \rangle$.
- Compute r_0 in the interesting region:

$\Rightarrow$ Matching ($N_b = 2$): $$\beta(\alpha) = \frac{2.47(1)}{1-\alpha} - 2.70(3)$$

Second step:

Simulate at similar lattice spacings and look at the static potential.

- Compare to high precision results obtained with the Wilson action.

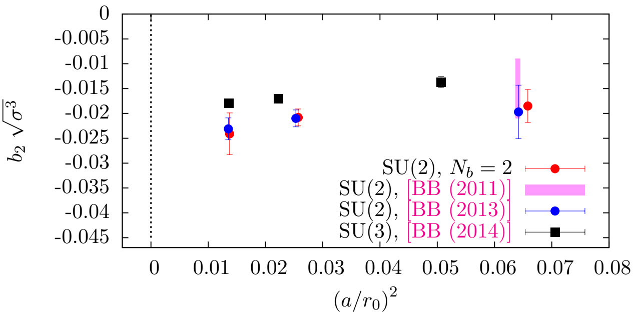
  [BB, PoS EPS-HEP (2013)]

- Here we use the prediction for the potential of an effective string theory for the flux tube as a method to look at its subleading properties.
  $\Rightarrow$ There are two non-universal parameters, σ and $\bar{b}_2$ (boundary coeff.).
- An agreement of $\bar{b}_2$ means that the potential is identical up to 4-5 significant digits!

Results for $\bar{b}_2$

First result: $\sqrt{\sigma}\, r_0$ is equivalent in both theories!

Results for $\bar{b}_2$:



$\Rightarrow$ All results are in excellent agreement!

Finite T properties

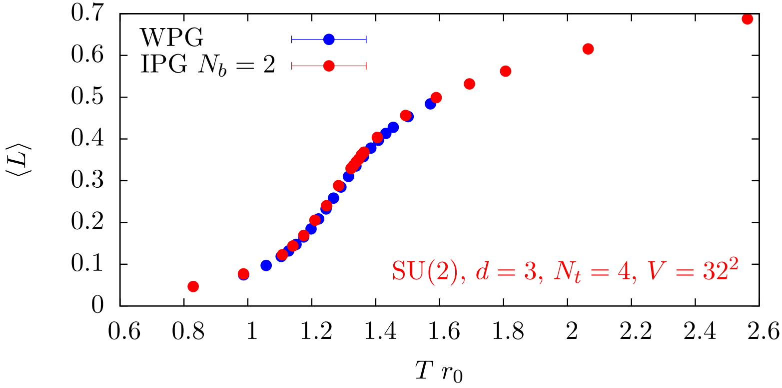
For $T = 0$ quantities comparison looks good!

So what about the finite temperature transition?

- For SU(2) and $d = 3$:
  Second order phase transition in the $2d$ Ising universality class.
  [Engels *et al*, NPPS 53 (1997)]
- We will test this at $N_t = 4$ first!
  $\Rightarrow \quad N_t = 6$ is in progress.
- Scale setting via r_0 and the mapping obtained at $T = 0$.

Phase transition at $N_t = 4$

Polyakov loop expectation value:



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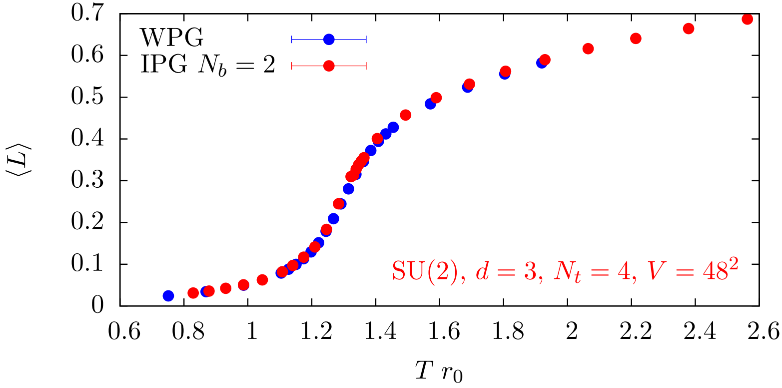

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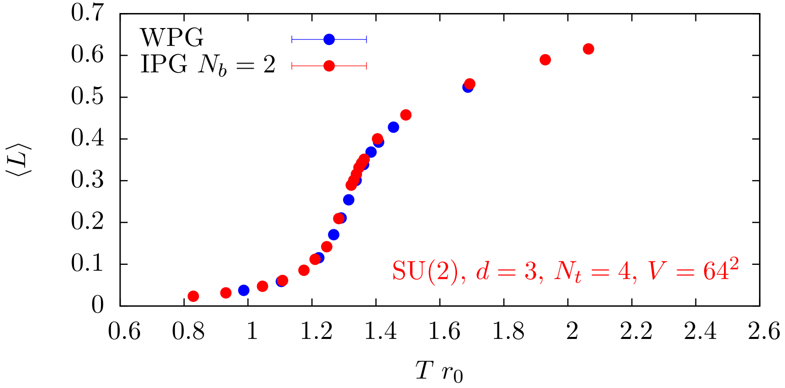

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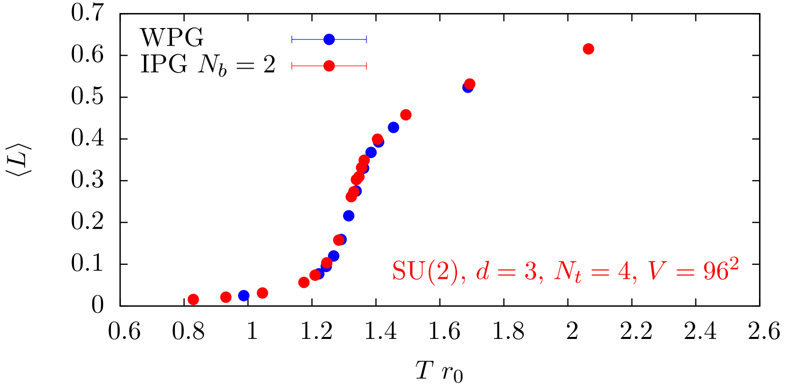
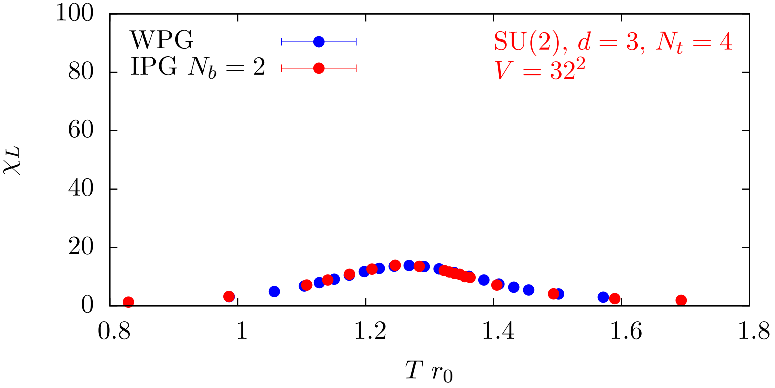


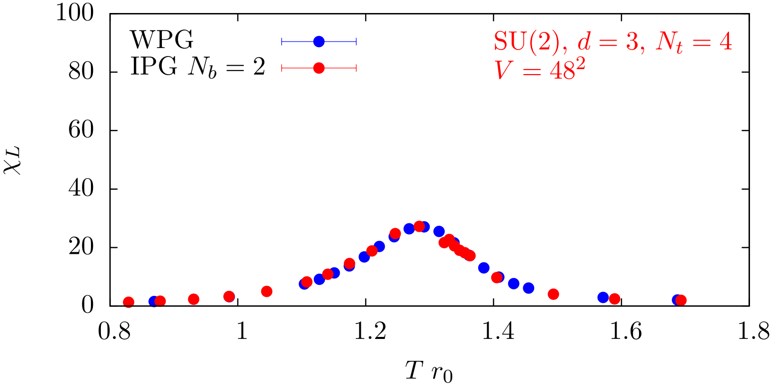
Phase transition at $N_t = 4$

Polyakov loop susceptibility:



Phase transition at $N_t = 4$

Polyakov loop susceptibility:



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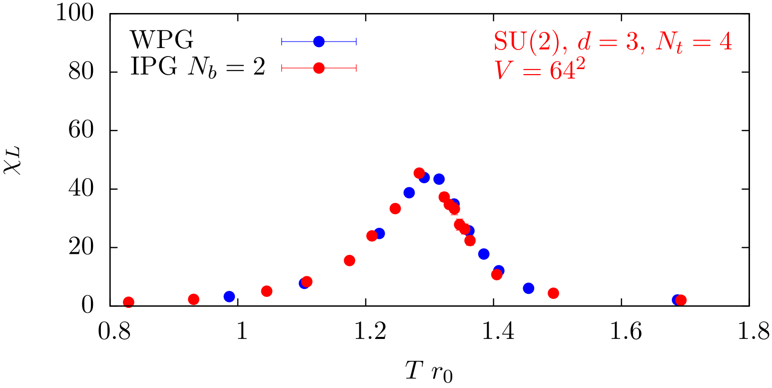
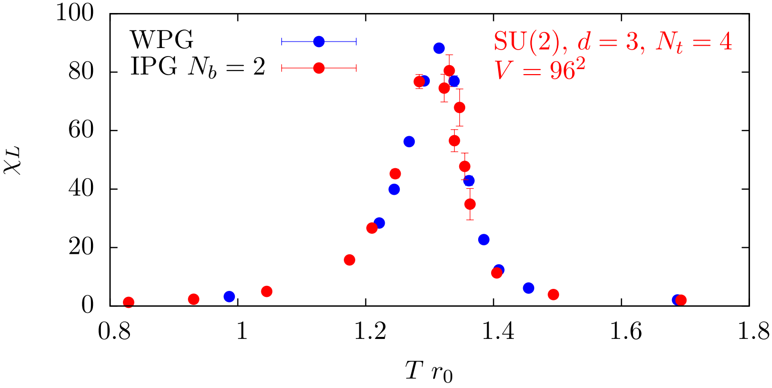


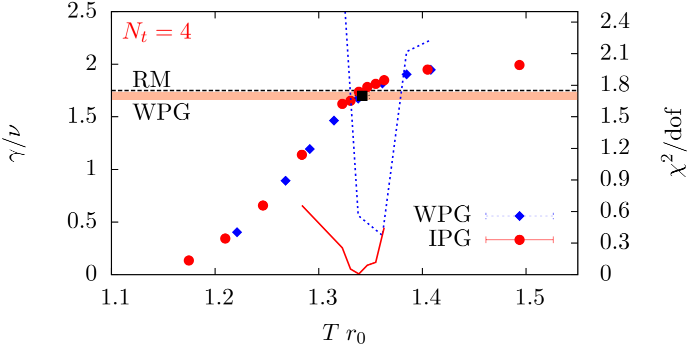
Phase transition at $N_t = 4$

Polyakov loop susceptibility:



Phase transition at $N_t = 4$

Fit: $\ln(\chi_L) = C + \gamma/\nu \, \ln(N_s)$



Result for critical exponents: $\gamma/\nu = 1.74(2)(9)$

Black point: $\gamma/\nu = 1.70(4)$ (WPG) [Engels *et al*, NPPS 53 (1997)]

4. Dual representation

The bosonic version

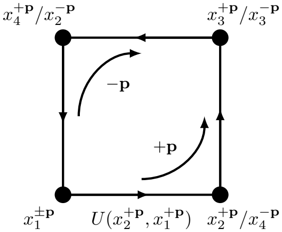
Now: Why is this weight factor interesting?

Bosonisation of the determinant: [Budczies, Zirnbauer, math-ph/0305058]

$$\omega_{\mathrm{BZ}}[U] = \prod_p \left| \det \left(m_{\mathrm{BZ}}^4 - U_p \right) \right|^{-2N_b} = \int [d\phi] \; \exp \left\{ -S_{\mathrm{BZ}}[\phi, \bar{\phi}, U] \right\}$$

$$S_{\mathrm{BZ}}[\phi, \bar{\phi}, U] = \sum_{b=1}^{N_b} \sum_{\pm \mathbf{p}} \sum_{j=1}^{4} \left[m_{\mathrm{BZ}} \, \bar{\phi}_{b,\mathbf{p}}(x_j) \phi_{b,\mathbf{p}}(x_j) - \bar{\phi}_{b,\mathbf{p}}(x_{j+1}) \, U(x_{j+1}, x_j) \, \phi_{b,\mathbf{p}}(x_j) \right]$$

- ϕ are complex scalar fields
- **p**: index for oriented plaquette
- Scalar fields carry plaquette index **p**.
  $\Rightarrow$ Propagate only locally opposite to the plaquette orientation.
- Gauge field only couples to bosons.
  $\Rightarrow$ Can be modified more easily!
- N_b defines the number of boson fields.



Modified version

Problem: This action is complex!

Solution: Rewrite determinant weight factor:

$$\omega_{\mathrm{BZ}}[U] \sim \prod_p \left[\det\left(m_{\mathrm{BZ}}^4 - U_p\right)\ \det\left(m_{\mathrm{BZ}}^4 - U_p^\dagger\right)\right]^{-N_b}$$

$$\sim \prod_p \left[\det\left(\tilde{m} - \left\{U_p + U_p^\dagger\right\}\right)\right]^{-N_b}$$

Now bosonize this determinant:

$\Rightarrow$ Real action:

$$S_B[\phi, \bar{\phi}, U] = \sum_{b=1}^{N_b} \sum_p \sum_{j=1}^{4} \left[m\, \bar{\phi}_{b,p}(x_j)\phi_{b,p}(x_j) - \bar{\phi}_{b,p}(x_{j+1})\, U(x_{j+1}, x_j)\, \phi_{b,p}(x_j) - \bar{\phi}_{b,p}(x_j)\, U(x_j, x_{j+1})\, \phi_{b,p}(x_{j+1})\right]$$

Here: $\tilde{m} = m_{\mathrm{BZ}}^4 + m_{\mathrm{BZ}}^{-4}$ and $\tilde{m} = m^4 - 4\, m^2 + 2.$

Integration over gauge fields

First step: Integration over the gauge degrees of freedom.

Rewrite the partition function as a product of Itzykson-Zuber integrals:

$$Z = \int d[\phi]\, \mathcal{F}[\phi,\bar{\phi}] \prod_{x,\mu} \int dU_\mu(x)\, e^{\frac{1}{2}\mathrm{Tr}\left[U_\mu(x)\ \mathcal{V}_\mu(x)[\phi,\bar{\phi}] + U_\mu^\dagger(x)\ \mathcal{V}_\mu^\dagger(x)[\phi,\bar{\phi}]\right]}$$

With $\quad \mathcal{F}[\phi,\bar{\phi}] = \exp\left\{ -\sum_{b=1}^{N_b} \sum_p \sum_{j=1}^{4} m\, \bar{\phi}_{b,p}(x_j)\phi_{b,p}(x_j) \right\}$

and $\quad \mathcal{V}_\mu(x)[\phi,\bar{\phi}] = 2\sum_{b=1}^{N_b} \sum_{\nu\neq\mu} \Big[\phi_{b,\bar{p}(x,\mu,\nu)}(x_{\bar{j}(\mu,\nu,0,1)})\bar{\phi}_{b,\bar{p}(x,\mu,\nu)}(x_{\bar{j}(\mu,\nu,0,0)})$

$$+\phi_{b,\bar{p}(x-\hat{\nu},\mu,\nu)}(x_{\bar{j}(\mu,\nu,1,1)})\bar{\phi}_{b,\bar{p}(x-\hat{\nu},\mu,\nu)}(x_{\bar{j}(\mu,\nu,1,0)})\Big]$$

Integration over gauge fields – IZ integrals

Need to solve integrals $\quad \mathcal{I} = \int dU \; e^{\mathrm{Tr}[U\,\mathcal{V} + U^\dagger\;\mathcal{V}^\dagger]}$.

For U(N_c) they are known. [e.g. Brower, Rossi, Tan, PRD23 (1981)]

For SU(N_c): $\quad \Rightarrow \quad \mathcal{I} \sim \dfrac{1}{\Delta(\lambda^2)} \sum_{\xi=0}^{\infty} \varepsilon_\xi \; \cos(\xi\;\varphi) \; \det\left(A_\xi(\lambda)\right)$

- ε_ξ: Neumann's factor; $\varepsilon_\xi = \begin{cases} 1 & \text{for} \xi = 0 \\ 2 & \text{for} \xi > 0 \end{cases}$
- φ: Phase of the determinant $\det(\mathcal{V})$
- λ_i^2: eigenvalues of the $N_c \times N_c$ matrix $\frac{1}{4}\mathcal{V}\mathcal{V}^\dagger$
- $\Delta(\lambda^2)$: Vandermonde determinant
- $A_\xi(\lambda)$: $N_c \times N_c$ matrix; $(A_\xi(\lambda))_{ij} = \lambda_i^{j-1} \; I_{\xi+j-1}(\lambda_i)$
  with $I_m(z)$ modified Bessel function of the first kind (and $z \in \mathbb{R}$).

$\Rightarrow$ Looks difficult, but the sum in $\mathcal{I}$ converges numerically very fast.

Full QCD

Now consider also fermionic fields, e.g. with a staggered type action:

$$S_F = \sum_x \Big\{ \sum_\mu \Big[\bar{\psi}(x)\alpha_\mu(x)\, U_\mu(x)\psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu})\tilde{\alpha}_\mu(x)\, U_\mu^\dagger(x)\psi(x) \Big] + m_q \bar{\psi}(x)\psi(x) \Big\}$$

Most promising idea: Expand weight factor $\exp(-S_F)$ in grassmann variables.

- Introduce dual variables $b_{\mu,ab}(x)$, $b^\dagger_{\mu,ab}(x)$ and $n_a(x)$.
- Integral over grassmann fields leads to constraints for those variables.
- Integrate out the gauge fields.

Resulting dual partition function:

$$Z_{\text{dual}} = \sum_{(\vec{b},\vec{b}^\dagger,\vec{n})} \mathbb{I}_{(\vec{b},\vec{b}^\dagger,\vec{n})}\, m_q^{N_m} \int [d\bar{\phi}][d\phi]\, \mathcal{F}(\phi,\bar{\phi}) \prod_{x,\mu} w(b(x,\mu), b^\dagger(x,\mu), \partial A)\, \mathcal{I}_\mu(x,\phi,\bar{\phi})$$

Problem: The dual theory has a sign problem!

Summary and Perspectives

- We have investigated a possible alternative discretisation of continuum pure gauge theory.
- While for $d = 2$ it can be shown that the theory has the correct continuum limit this is not guaranteed if $d > 2$.
- Numerical tests show good agreement with simulations using Wilson's gauge action, both for $T = 0$ and $T \neq 0$.
- In its original formulation with auxiliary boson fields the theory has a sign problem. $\Rightarrow$ We introduced a modified version without sign problem.
- Pass to a dual theory via a direct integration over gauge fields:
  - Leads to a theory formulated in terms of auxiliary bosonic fields.
  - When fermions are include one can expand the action in grassmann variables and integrate over the fermionic degrees of freedom and the gauge fields.
  - However, the resulting dual representation has a sign problem.
  - Is it possible to find a formulation without sign problem?
- Explore other analytical methods ...

Thank you for your attention!