Scalar QED$_2$ with a θ-Term

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Introduction

- Studying **θ-vacua on the lattice** is difficult due to the notorious **sign problem**, similar to lattice simulations at non-zero chemical potential.
- By now, it is well known that such problems – at least in abelian theories – can be solved by mapping the system to a **dual representation**.
- In this exploratory study we investigate the θ-vacuum structure of 2 dimensional scalar QED, i. e. the **scalar version of the Schwinger model**.
- The **aim of this study** is
  - first, to find a suitable expression for the **topological charge** on the lattice that allows for dualization,
  - second, to obtain a **real, non-negative representation** of the partition sum, in order to make Monte Carlo simulations feasible and
  - third, to study the **dependence of observables on the θ-angle**.

Scalar Quantum Electrodynamics with a θ-Term

We study the system given by

$$Z(\kappa,\lambda,\beta,\theta) = \int \mathcal{D}[U]\,\mathcal{D}[\phi]\; e^{-S_G[U]-S_M[U,\phi]-i\,\theta\,Q[U]} \,.$$

With the gauge part

$$S_G[U] = -\frac{\beta}{2}\sum_x \Big[U_p(x) + U_p^*(x)\Big] \,,$$

and matter part

$$S_M[U,\phi] = \sum_x \Big[\kappa|\phi_x|^2 + \lambda|\phi_x|^4 - \sum_\nu \big(\phi_x^* U_{x,\nu}\phi_{x+\hat\nu} + \phi_x U^*_{x,\nu}\phi^*_{x+\hat\nu}\big)\Big] \,.$$

The **(integer-valued) topological charge** is given in the continuum by

$$Q = \frac{1}{4\pi}\int d^2x\; \epsilon_{\mu\nu}F_{\mu\nu}(x) = \frac{1}{2\pi}\int d^2x\; F_{12}(x) \,.$$

On the (2 dimensional) lattice one can derive the following expression

$$Q[U] = \frac{1}{4\pi i}\sum_x \Big[U_p(x) - U_p^*(x)\Big] \,,$$

since $U_p(x) = e^{\,i\,\left(A_1(x)+A_2(x+\hat1)-A_1(x+\hat2)-A_2(x)\right)} = 1 + iF_{12}(x) + \ldots$

Dual Representation

The **sign problem is basis dependent**, this means for most systems it is – at least in principle – possible to express the partition function in terms of **new degrees of freedom**, where the partition sum consists of real, non-negative terms only and each term can be assigned a probability weight.

We follow [1], where the matter part was treated and obtain

$$Z(\kappa,\lambda,\beta,\theta) = \sum_{\{l,\bar l,p,\bar p\}} \prod_{x,\nu}\frac{1}{(|l_{x,\nu}|+\bar l_{x,\nu})!\;\bar l_{x,\nu}!}\prod_x P(n_x)\;\delta\Big(\sum_\nu\big[l_{x,\nu}-l_{x-\hat\nu,\nu}\big]\Big)$$
$$\prod_x \frac{\eta^{(|p_x|+p_x)/2+\bar p_x}\;\bar\eta^{(|p_x|+p_x)/2-\bar p_x}}{(|p_x|+\bar p_x)!\;\bar p_x!}$$
$$\prod_x \delta\big(p_x - p_{x-\hat2} + l_{x,1}\big)\;\delta\big(p_{x-\hat1} - p_x + l_{x,2}\big) \,.$$

$$\eta \equiv \frac{\beta}{2} - \frac{\theta}{4\pi}\,, \qquad \bar\eta \equiv \frac{\beta}{2} + \frac{\theta}{4\pi}\,, \qquad P(n_x) \equiv \int_0^\infty dr\; r^{n_x+1}\,e^{-\kappa r^2-\lambda r^4} \,.$$

Four new **integer-valued fields** are introduced, which replace the original complex-valued fields. The l- and $\bar l$-variables are associated with the matter fields ϕ and live on the links of the lattice, the p- and $\bar p$-variables are associated with the gauge fields U and are attached to the lattice sites. They assume the following values

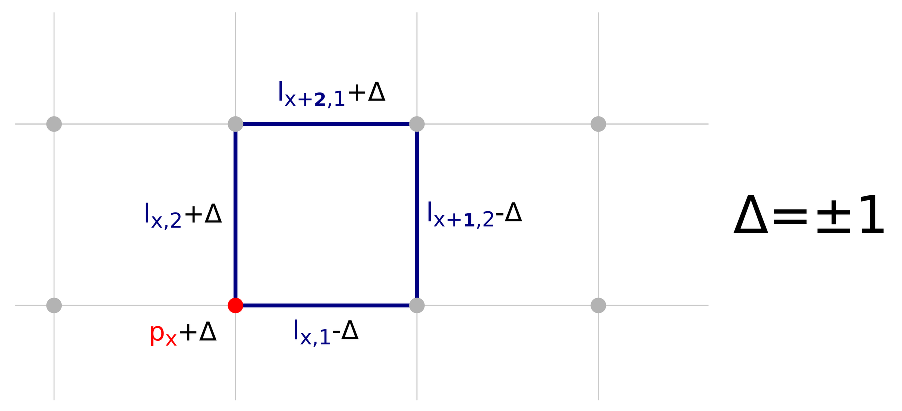
$$l_{x,\nu},\; p_x \in \mathbb{Z}\,, \qquad \bar l_{x,\nu},\; \bar p_x \in \mathbb{N}_0 \,.$$

Due to the **Kronecker deltas (from the $\mathrm{U}(1)$ integration)** the l- and p-fields are constrained and have to be updated accordingly.

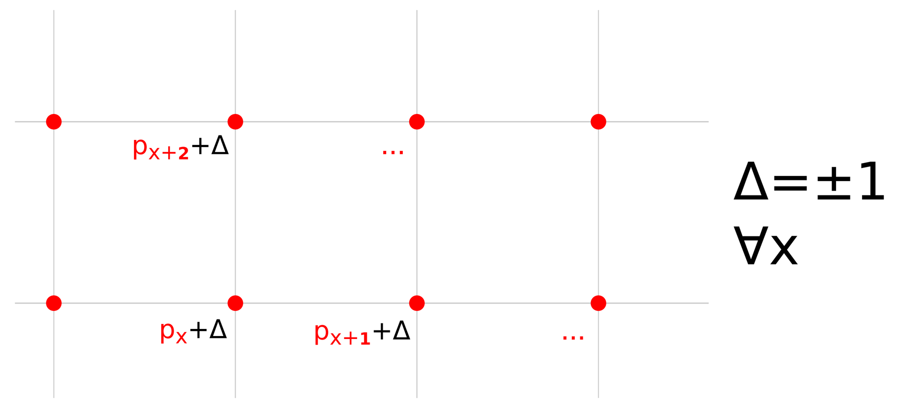
Update Scheme

In order to obey the constraints imposed by the Kronecker deltas, we update the p- and l-fields with the following ergodic procedure:

Local mixed plaquette update:



Global pure gauge update:

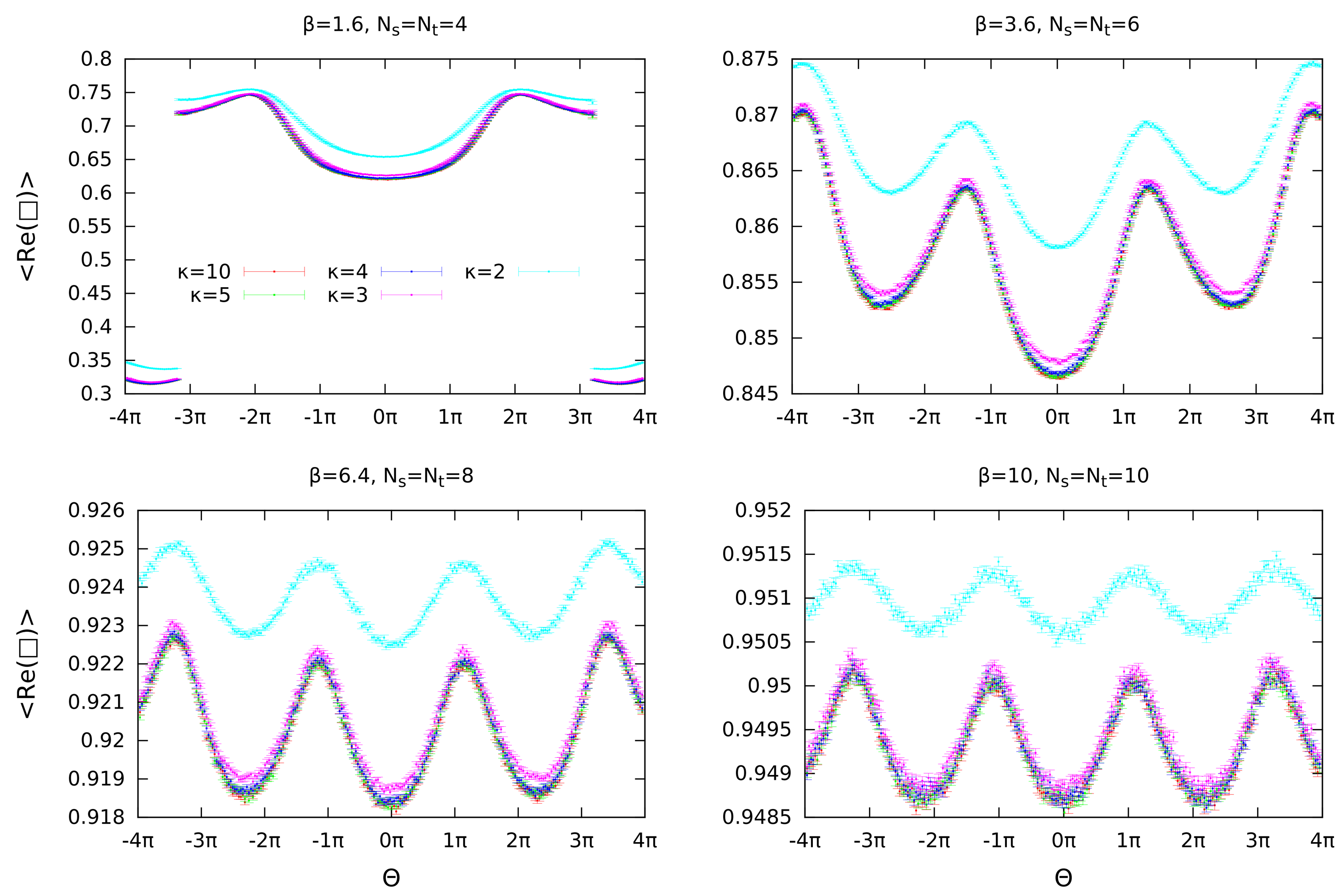


Numerical Results (preliminary)

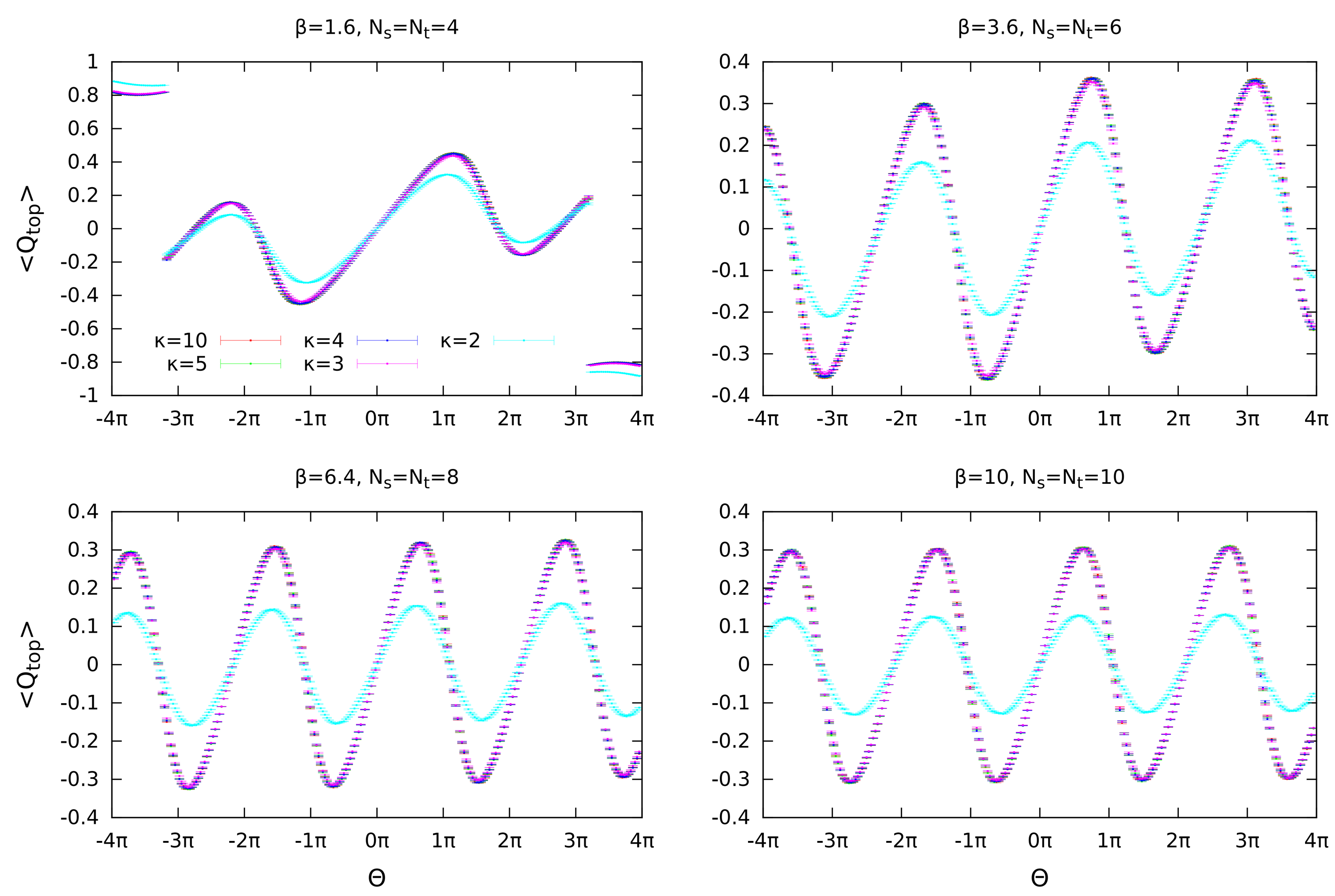
As one would expect from a topological quantity, the **topological charge shows a very sensitive dependence on the lattice spacing** and the continuum limit has to be taken with care. Here this amounts to

$$\beta \to \infty \qquad V \to \infty \qquad \text{with} \qquad \frac{\beta}{V} = \text{const.}$$

Plaquette: Development towards the continuum limit



Topological Charge: Development towards the continuum limit



References

[1] Y. Delgado Mercado, C. Gattringer and A. Schmidt, Comput. Phys. Commun. **184** (2013) 1535