Density of States Method for the Effective Center Model of QCD

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Basic idea
- At finite density lattice QCD faces the complex phase problem ⇒ alternative methods!
- Approach: We explore the density of states method [1]
- Model theory: The effective center model of QCD
- Crosscheck: Exact results from the dual representation [2]

Effective center model of QCD
The theory is described by the action:

\[ S[P] = - \tau \sum_{\alpha} \left( \sum_{i} \left( P_{\alpha}^{+} P_{\alpha} + \text{c.c.} - 1 \right) + \kappa e^{\mu}(P_{\alpha} - 1) + \kappa e^{-\mu}(P_{\alpha}^* - 1) \right), \]

\[ P_{\alpha} \in \mathbb{Z}_2 = \{1, e^{i2\pi/3}, e^{-i2\pi/3} \}, \]

\[ \tau = 0.13, \tau = 0.16, \tau = 0.19 \]

\[ V = 16 \]

\[ \kappa = 0.01 \]

Rewriting the action
\[ N_+, N_- \text{ is the number of spins } = 1, e^{\pm 2\pi/3} \text{ and } \Delta N = N_+ - N_- \]

\[ S[P] = -\tau \sum_{\alpha} \sum_{i} \left( P_{\alpha}^{+} P_{\alpha} + \text{c.c.} - 1 \right) - 3\kappa \cosh(\mu)(N_0[P] - V) 
- i\sqrt{3}\kappa \sinh(\mu)\Delta N[P] \]

Partition sum: \[ Z = \sum_{\{P\}} e^{-S[P]} \cos(\sqrt{3}\kappa \sinh(\mu)\Delta N[P]) \]

Density of states
Definition of a weighted density of states:

\[ \rho(d) = \sum_{\{P\}} e^{-S[P]} \delta(d - \Delta N[P]) \]

Rewriting the partition sum in terms of \( \rho(d) \):

\[ Z = \sum_{d} \rho(d) \cos(\sqrt{3}\kappa \sinh(\mu)d) \]

Calculating the density of states
Spline ansatz in the interval \( d \in [d_0, d_0 + \delta d] \), \( \delta d \in \mathbb{N} \),

\[ \rho_{d_0}(d) = \exp \left[ \Delta \zeta(d_0)(d - d_0) + \zeta(d_0 - \delta d) \delta d \right], \]

with \( \Delta \zeta(d_0) = \zeta(d_0) - \zeta(d_0 - \delta d) \).

Computing the coefficients using restricted Monte Carlo updates [1]:

\[ \langle \zeta(d) \rangle = \frac{1}{N} \sum_{d=d_0}^{d_0+\delta d} \rho_{d_0}(d) e^{-\zeta d} \]

with \( \Delta d = d - d_0 - \delta d/2 \). Using an iterative approach:

\[ \Delta \zeta^{(n+1)}(d_0) = \Delta \zeta^{(n)}(d_0) + \frac{6}{\delta d(1+\delta d/2)} \langle \zeta(d) \rangle \]

Convergence of the iteration:

Reconstruction of \( \rho(d) \)
By using the symmetry

\[ \rho(d) = \rho(-d) \]

and demanding

\[ \zeta(-\delta d) = 0 \Rightarrow \rho(0) = 1 \]

we get the complete density of states:

\[ \log[\rho(d)] = \sum_{d=0}^{\Delta N} \left( (d - n\delta d) \Delta \zeta(n\delta d) + \delta d \sum_{m=0}^{n-1} \Delta \zeta(m\delta d) \right) \times \begin{cases} 1, & d \in [n, n+1)\delta d \\ 0, & \text{otherwise} \end{cases} \]

Examples for \( \log(\rho) \):

Observables
The expectation value of the observable \( \mathcal{O} \) is given by

\[ \langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{d=-\Delta N}^{\Delta N} \rho(d) \cos \left( \sqrt{3}\kappa \sinh(\mu)d \right) \mathcal{O}(d) \]

We choose the observable

\[ \frac{1}{V} (M - M^*) = \frac{1}{V} \left( \sum_{d} (P_{\alpha} - P_{\alpha}^*) \right) = \frac{1}{V} \frac{\partial \log Z}{\partial (\kappa \sinh(\mu))} \]

and the corresponding susceptibility \( \chi_{M - M^*} \).

Comparison to the reference data from dual simulations (dotted lines):

References