Zero modes of Overlap fermions, instantons and monopoles

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Motivation

- **Our goal of this study is to show the relation among Chiral symmetry, instantons and monopoles.**

- A number of studies show, for example, by Instanton liquid model (E. V. Shuryak), and also by simulations, the relation between Chiral symmetry breaking and instantons.
- Moreover, there are a lot of studies showing the relation between the instantons and monopoles, for example, A. Hart and M. Teper, V. Bornyakov and G. Schierholz, M. N. Chernodub and F. V. Gubarev in Abelian gauge.

- However, it has been difficult to directly show the relation between Chiral symmetry and monopoles by simulations, because **Chiral symmetry,** for example, Wilson fermions, **is already broken in Chiral limit by discretizations.**

Introductions

How to show the relations?

1. Overlap Fermions

We generate quenched configurations (for Wilson gauge action), and construct Overlap operator from gauge links (hep-lat/ 0212012, and a Doctoral thesis by V. Weinberg). We show that the square of the topological charges of the Overlap operator is the number of instantons by analytical computations.

2. Additional monopoles

We'd like to show the quantitative relation between the number of instantons and monopoles. **Therefore, we directly add monopoles and anti-monopoles with charges to the configurations.** We use a technique by the University of Pisa group (A. D Giacomo, et al. Phys. Rev. D 56 (1997) 6816, Phys. Rev. D 61 (2000), 034503, C. Bonati, et al. Phys. Rev. D 85 (2012) 065001).

Introductions

3. Measuring the additional monopoles

How to confirm whether we can successfully add the monopoles or not?
We use techniques (DIK collaboration, Phys. Rev. D 70 (2004) 074511, A. Bode, et al., hep-lat9312006). By measuring the length of the monopole loops and monopoles density **we confirm that we are successfully adding monopoles to the configurations.**

4. Zero modes, instantons and monopoles

We would like to find the relation among the zero modes, instantons and monopoles using the Overlap fermions as a powerful tool.
We add the one monopole and one anti-monopole with charges by the monopole creation operator, and we count the number of zero modes of the configurations. **We find that the number of zero modes increases by the monopole charges.**

The final goal:

We will show the relation between the monopoles and Chiral symmetry.

Contents of my talk

1. Overlap fermions

- Zero modes, topological charge and susceptibility
- The number of zero modes and instantons

2. Additional monopoles

- The monopole creation operator

3. Detecting the monopoles

- Monopole loops, clusters, and density of additional monopoles

4. Zero modes, instantons and monopoles

- Zero modes, instantons, and monopole charges

Ginsparg-Wilson relation

Chiral transformations

$$\psi \to \psi' = \exp^{i\theta\gamma_5}\psi$$

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}\exp^{i\theta\gamma_5}$$

Fermion bilinear from of Lagrangian in continuum

$$\mathscr{L} = \bar{\psi} D \psi$$

If Lagrangian holds the Chiral symmetry, the relation

$$\gamma_5 D + D\gamma_5 = 0.$$

But, there is Nelsen-Ninomiya theorem, so, the relation will be

$$\gamma_5 D + D\gamma_5 = O(a), \text{ a: lattice spacing.}$$

From computations of renomalization group (P. H. Ginsparg and K. G. Wilson Phys. Rev. D25 (1982) 2649), the relation is

$$\gamma_5 D + D\gamma_5 = aDR\gamma_5 D: \text{ Ginsparg-Wilson relation}$$

Multiplying G-W relation by operator D^{-1},

$$\gamma_5 D^{-1} + D^{-1}\gamma_5 = aR\gamma_5.$$

This formula show us that the propagator D^{-1} is O(a) and local.

Overlap operator

Overlap operator is defined by N. Neuberger (Phys. Lett. B427 (1998) 353) as follows.

$$D = \frac{1}{Ra}\left[1 + \frac{A}{\sqrt{A^\dagger A}}\right], \; A = -M_0 + aD_W$$

A condition to a doubler of Overlap fermions: $0 < M_0 < 2$

D_W is the mass less Wilson fermion operator (r = 1).

$$D_W = \frac{1}{2}[\gamma_\mu(\nabla_\mu^* + \nabla_\mu) - a\nabla_\mu^*\nabla_\mu]$$

$$[\nabla_\mu \psi](n) = \frac{1}{a}[U_{n,\mu}\psi(n+\hat{\mu}) - \psi(n)],$$

$$[\nabla_\mu^* \psi](n) = \frac{1}{a}[\psi(n) - U_{n-\hat{\mu},\mu}^\dagger \psi(n-\hat{\mu})]$$

Overlap Fermions in sumulations

How to compute Overlap operator in numerical simulations?

D_w is the mass less Wilson Dirac operator.
M is Wilson's hopping term.
H_W is Hermitian Wilson Dirac operator.

$$D(0) = \frac{\rho}{a}\left[1 + \frac{D_W}{\sqrt{D_W^\dagger D_W}}\right], \quad D_W = M + \frac{\rho}{a}, \ (\rho = 1.4)$$

$$\frac{D_W}{\sqrt{D_W^\dagger D_W}} = sgn(D_W) = \gamma_5 sgn(H_W), \ H_W = \gamma_5 D_W$$

Almost all computations are for this term!!
Chebyshev polynomials approximation and Arnoldi method by ARPACk

L. Giusti, et al. Com. Phys. Comm. 153 (2003) 31, etc

Simulation details

- The action is for Wilson gauge action
- O(200) ~ O(500) configurations are generated each parameter (β, and Volume)
- Constructing the Overlap Dirac operator from gauge links of the Wilson gauge action
- Resolving eigenvalue problems using the subroutines ARPACK
- Saving O(80) pairs of eigenvalues and eigenvectors, and analyzing the pair of modes

These numerical techniques are already introduced by some groups, for example, **L. Giusti, et al. Com. Phys. Comm. 153 (2003) 31**, and the Doctoral thesis by V. Weinberg, etc.

Simulation parameters

17 parameters

β	a/r_0	V	V/r_0^4	N_{conf}
5.789	0.279	12^4	126	200
5.812	0.266	10^4	50.0	520
		14^4	192	240
		16^4	327	228
5.846	0.248	12^4	78.9	200
		16^4	250	260
5.864	0.240	14^4	126	287
5.904	0.222	12^4	50.0	515
		16^4	158	268
5.926	0.213	14^4	78.9	200
5.989	0.190	14^4	50.0	286
6.000	0.186	12^4	25.0	280
		14^4	46.3	437
		$12^3 \times 24$	50.0	423
		16^4	78.9	304
		18^4	126	218
6.068	0.166	16^4	50.0	287

Three different Physical volumes

$\boxed{V/r_0^4 = 50.0}$

$\boxed{V/r_0^4 = 78.9}$

$\boxed{V/r_0^4 = 126}$

We use an analytic function from S. Necco, at al. Nucl. Phys. B622 (2002) 328 and compute the lattice spacing in all of our simulations.

Observables in simulations

The definition of the spectrum density of non zero modes:

$$\rho(\lambda,\ V) = \frac{1}{V}\langle \sum_{\lambda} \delta(\lambda - \bar{\lambda}) \rangle$$

λ_{imp} : Eigenvalues of improved D(0)

$$D^{\mathrm{imp}}(0) = \left(1 - \frac{a}{2\rho} D(0)\right)^{-1} D(0)$$

The number of **zero modes**

n_+ : The number of zero modes has + chirality.

n_- : The number of zero modes has - chirality.

The number of **instantons**

n_+ : The number of instantons has + charge.

n_- : The number of instantons has - charge.

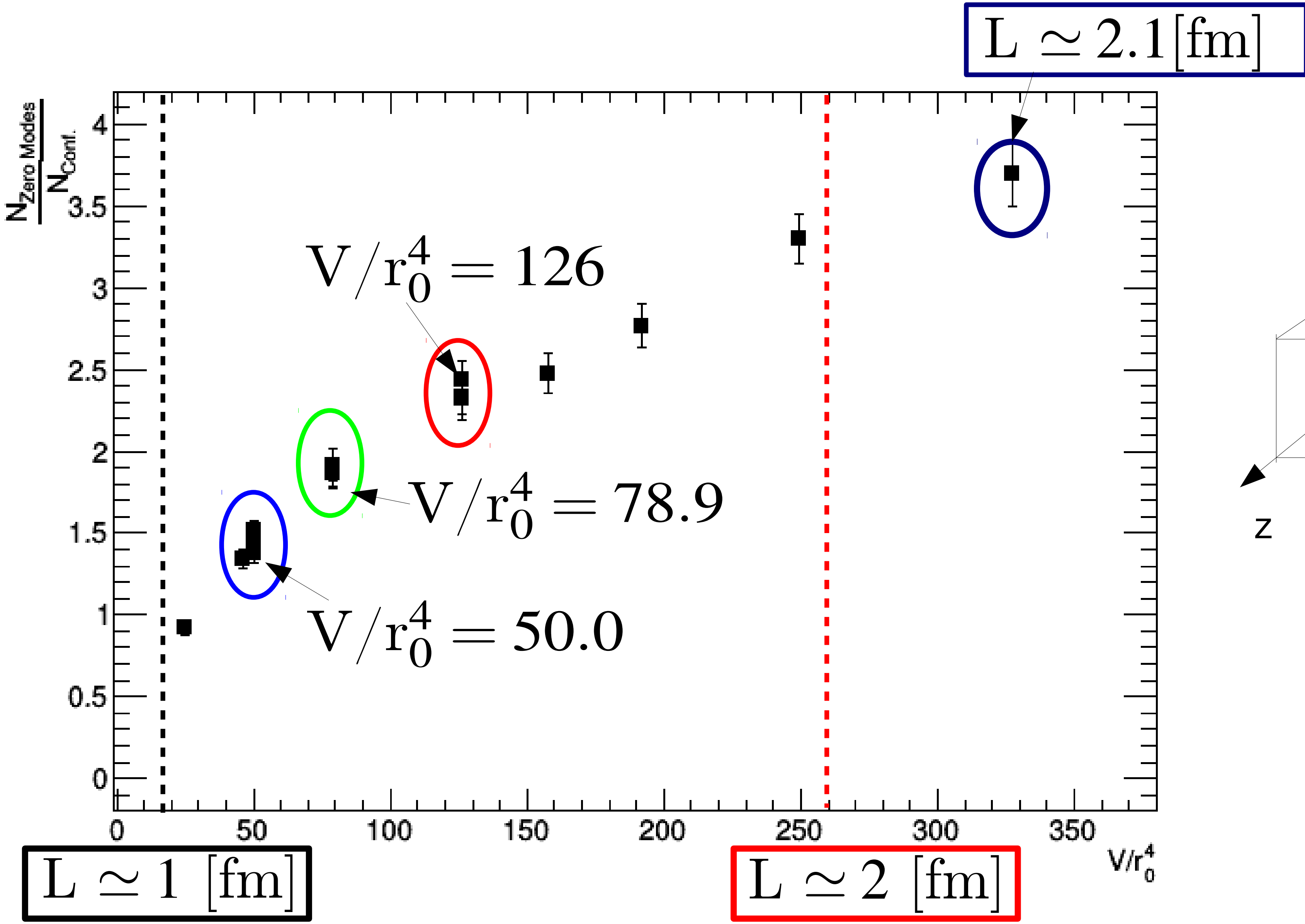
Topological charge: $Q = n_+ - n_-$

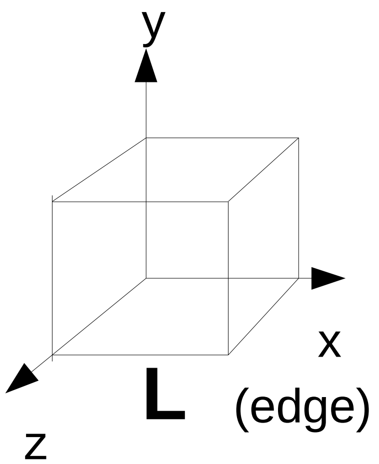
Atiyah–Singer index theorem: $\mathrm{index}(D) = n_+ - n_- \to \text{anomaly}$

Topological susceptibility:

$$\chi / r_0^4 \equiv \frac{\langle Q^2 \rangle r_0^4}{V}$$

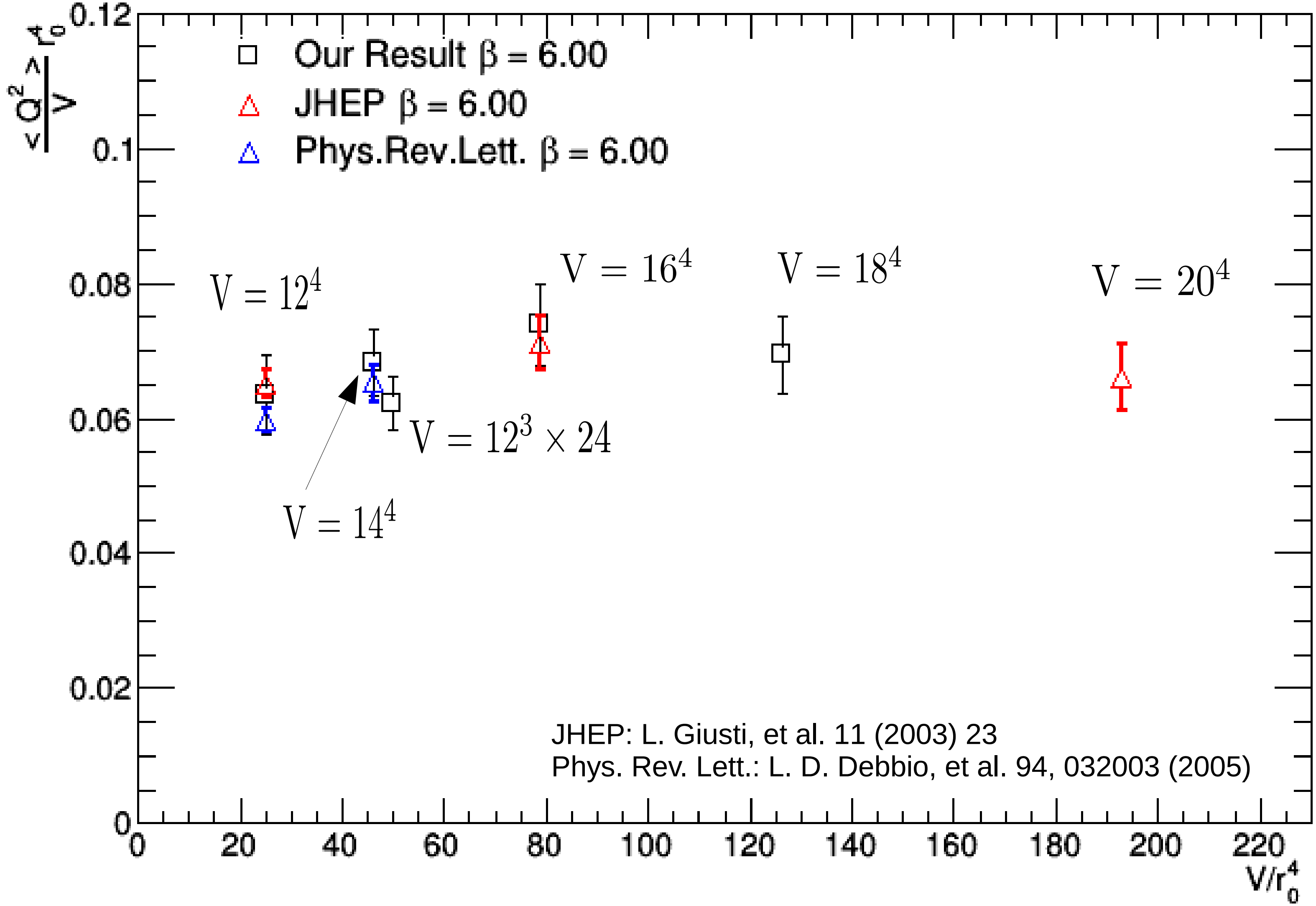
Number of Zero modes





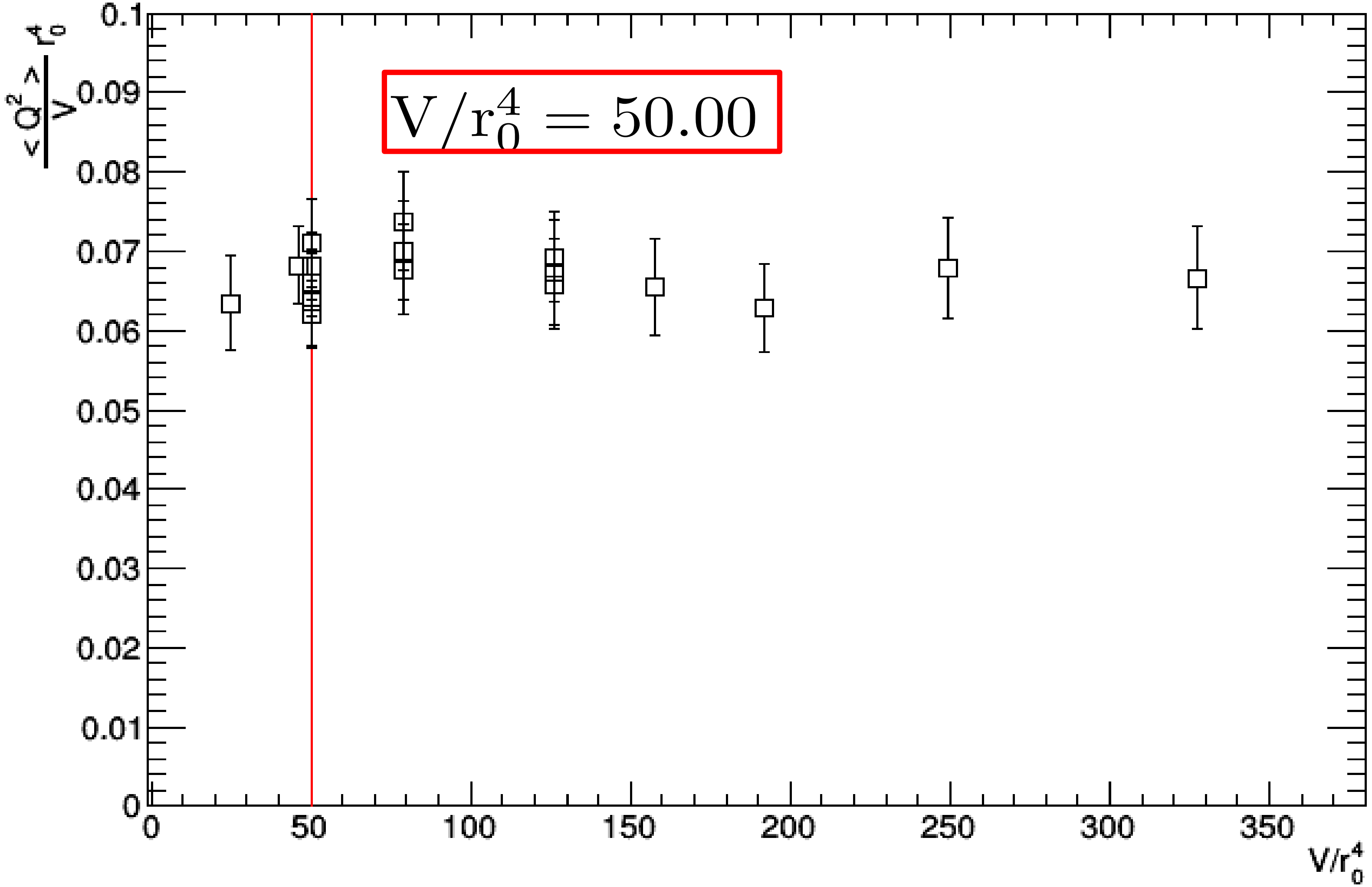
Topological Susceptibility

The physical volume dependence $\beta = 6.00.$



Topological Susceptibility

To see Topological susceptibility in continuum limit, I fix one physical volume.

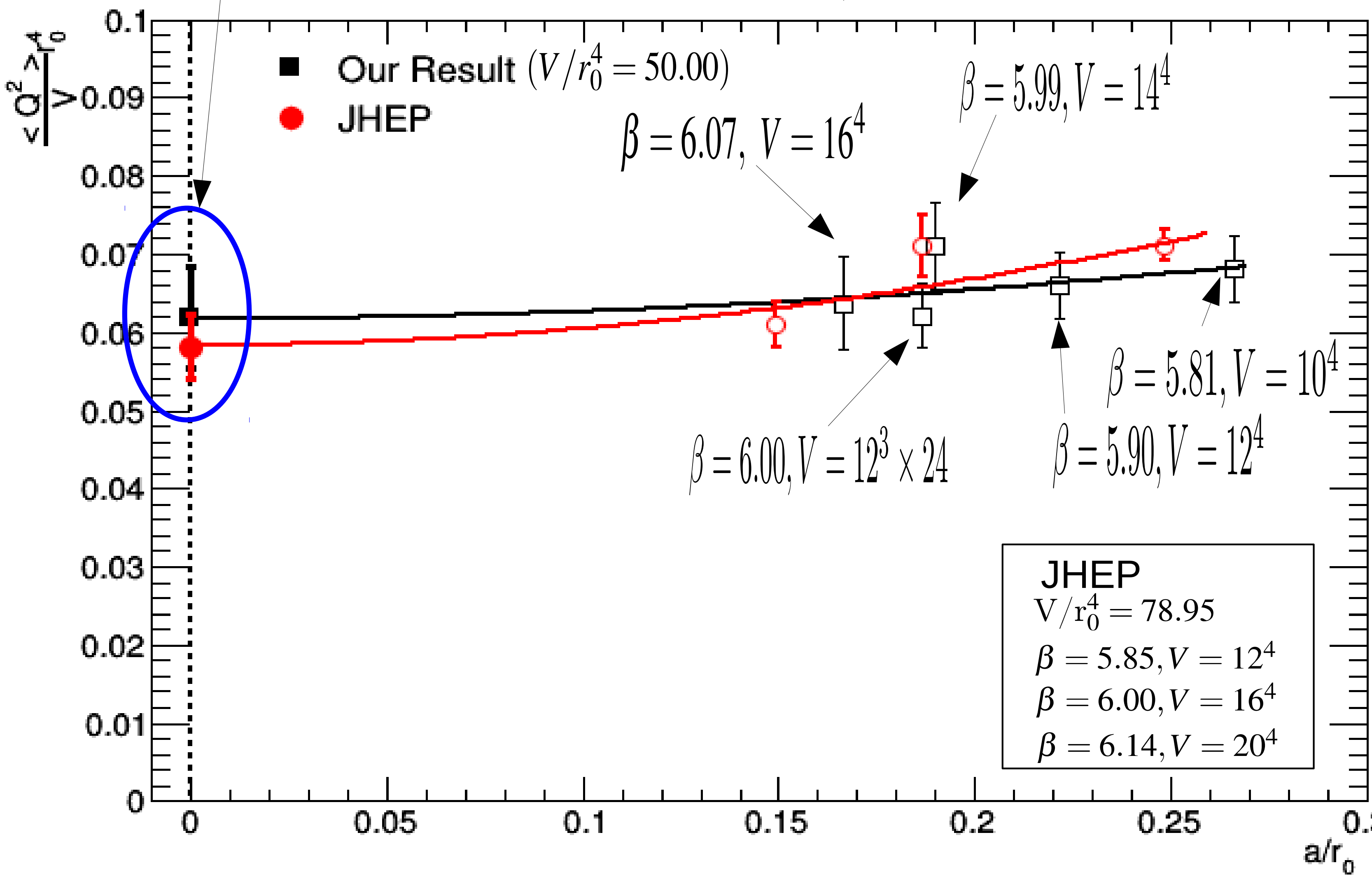


Topological Susceptibility in continuum limit

Continuum limit

Fiting a function: $\dfrac{\langle Q^2\rangle r_0^4}{V} = A + B(a/r_0)^2$

JHEP: L. Giusti, et al. 11 (2003) 23



Topological Susceptibility in continuum limit

Our Result: $(\chi = 1.92(7) \times 10^2 \ [MeV])^4$

L. D. Debbio, et al., PRL 94, 032003 (2005):

$$(\chi = 1.91(5) \times 10^2 \ [MeV])^4$$

Taking $F_k = 160(2)\ [MeV]$ as an experimental input.

G. Veneziano, Nucl. Phys. B159, 213 (1979), and
E. Witten, Nucl. Phys. B156, 269 (1979):

$$\frac{F_\pi}{6}(m_\eta^2 + m_{\eta'}^2 - 2m_K^2)|_{exp} \simeq (1.80 \times 10^2 \ [MeV])^4$$

Number of Zero modes

The definition of the topological charge $Q = n_+ - n_-$. n_+ and n_- are the number of the zero modes. **We never observed n_+ and n_- in the same configurations simultaneously up to L = 2.1 [fm]!**

What is the zero mode that we observed?

We suppose that we observe the "net" number of zero modes in simulations. Thus, we observe "Topological charge Q" as the zero modes 0, or $N_\pm$. The zero modes in our simulations:

$$\text{"Net" number of zero modes } 0,\ +N_+,\ \text{or } -N_- = Q$$

$$\frac{\langle N_{Zero}^2 \rangle}{V} = \frac{\langle Q^2 \rangle}{V} = constant$$

$$N_{Zero} \equiv \begin{cases} N_+ & n_+ - n_- > 0 \\ 0 & n_+ - n_- = 0 \\ N_- & n_+ - n_- < 0 \end{cases}$$

The number of instantons

How to derive the number of Instantons from Q?

Computations by A. Di Giacomo:

He supposes that large volume V filled with instantons n_+ and anti-instantons n_-. $\langle n_+\rangle = \langle n_-\rangle = \frac{N}{2} = \rho_i V,\ \rho_i:$ Instanton density

The probabilities of instantons and anti-instantons would be the Poisson distribution.

$$\begin{cases} \mathrm{P}(n_+) = \frac{1}{n_+!}\left(\frac{N}{2}\right)^{n_+} \mathrm{e}^{\frac{-N}{2}} \\ \mathrm{P}(n_-) = \frac{1}{n_-!}\left(\frac{N}{2}\right)^{n_-} \mathrm{e}^{\frac{-N}{2}} \end{cases}$$

A probability function of the topological charge (our zero modes) is

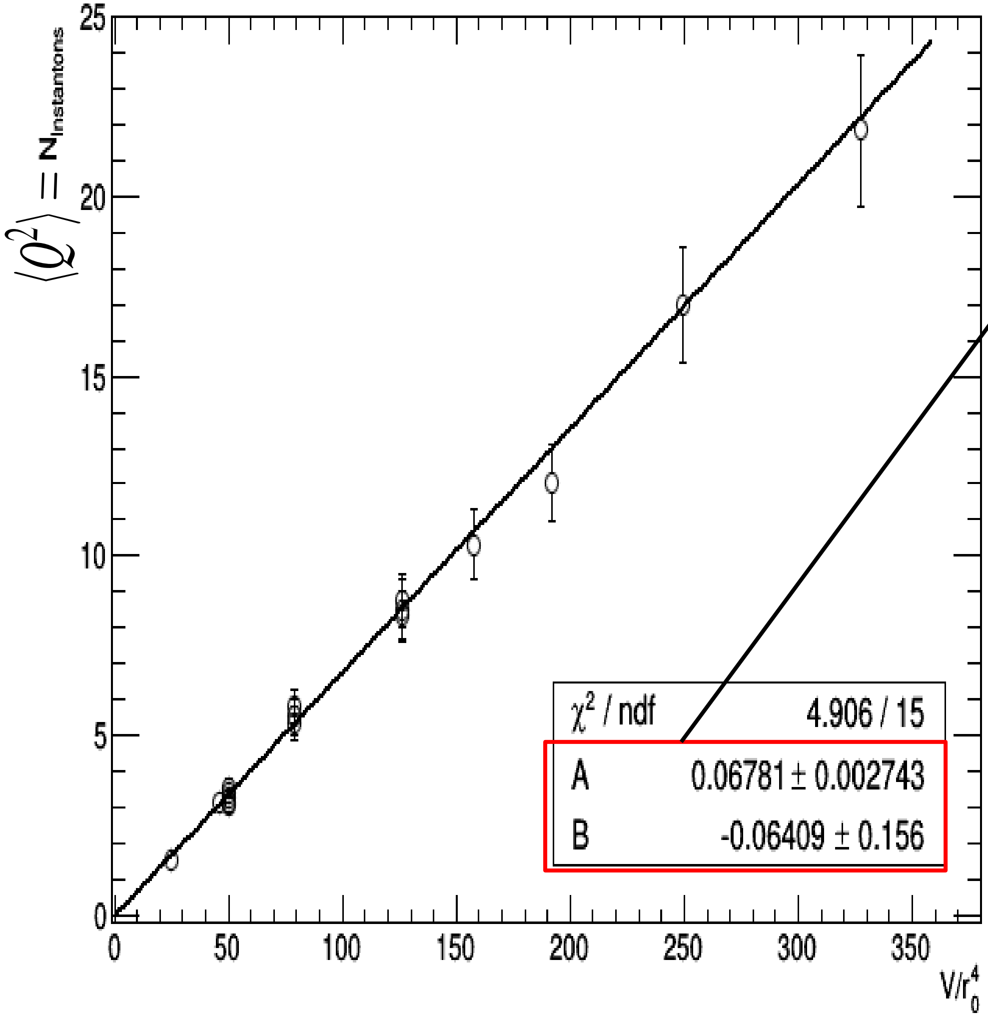
$$\mathrm{P}(Q) \simeq \frac{1}{\sqrt{2N\pi}}\mathrm{e}^{\frac{-Q^2}{2N}}, Q = n_+ - n_-,\ n_+ = n_- + Q, (Q \ll 1, \text{ and } N \gg 1).$$

Modified Bessel function is used.

The number of instantons is $N = \langle Q^2\rangle$.

The instanton density

$$y = Ax + B$$



Results

$$y = 2\rho_i r_0^4 * V/r_0^4 + B$$

$$A = 2\rho_i r_0^4$$

Good!

$$A = 6.8(3) \times 10^{-2},\ \boxed{B \simeq 0}$$

Thus, instanton density:

$$\underline{\rho_i = 8.2(3) \times 10^{-4}\ [\mathrm{GeV}^4]}$$

$$(r_0 = 0.5\ [\mathrm{fm}])$$

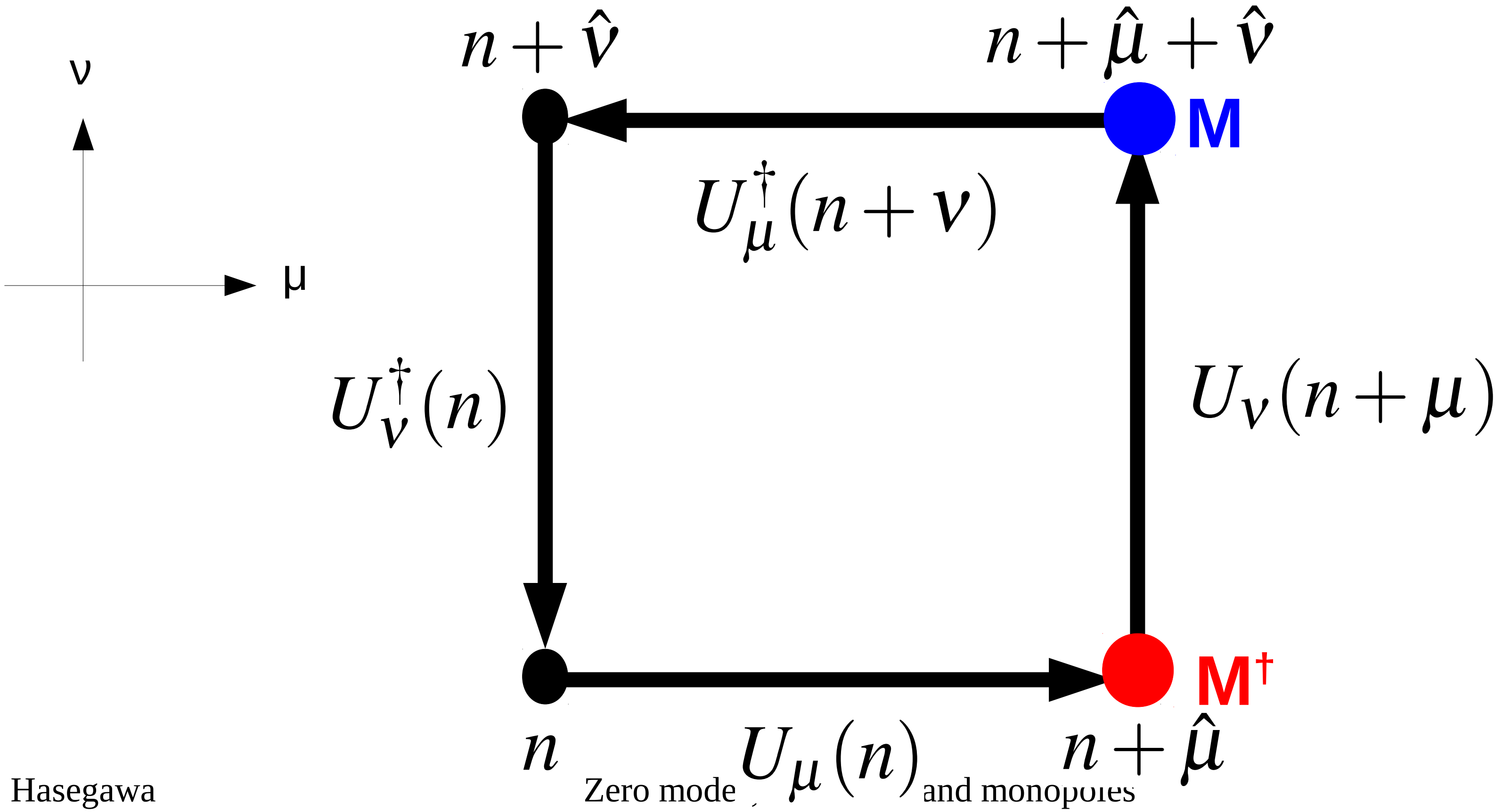
Instanton liquid by E. V. SHURYAK

Nucl. Phys. B203 (1982) 93-115

$$\underline{n_c = 8 \times 10^{-4}\ [\mathrm{GeV}^4]}$$

The monopole creation operator

Plaquette action,
A. Di Giacomo, et al. (Phys. Rev. D 85 (2012) 065001)



The monopole creation operator

In this study we use a monopole creation operator following a paper of A. Di Giacomo, et al. (Phys. Rev. D 85 (2012) 065001).

$$S+\Delta S=\sum_{n,\mu<\nu}\mathrm{Re}\left(1-\bar{\Pi}_{\mu\nu}(n)\right)$$

$$\bar{\Pi}_{i0}(t,\vec{n})=\frac{1}{\mathrm{Tr}[I]}\mathrm{Tr}\Big[U_i(t,\vec{n})M_i^{\dagger}(\vec{n}+\hat{i})$$

$$\times U_0(t,\vec{n}+\hat{i})M_i(\vec{n}+\hat{i})U_i^{\dagger}(t+1,\vec{n})U_0^{\dagger}(t,\vec{n})\Big]$$

$$M_i(\vec{n})=\exp\left(iA_i^0(\vec{n}-\vec{x})\right),\ (i=x,\ y,\ z)$$

$(i)\ n_z-z\geqq 0$

$$\begin{pmatrix}A_x^0\\A_y^0\\A_z^0\end{pmatrix}=\begin{pmatrix}\frac{m_c}{2gr}\frac{\sin\phi(1+\cos\theta)}{\sin\theta}\lambda_3\\-\frac{m_c}{2gr}\frac{\cos\phi(1+\cos\theta)}{\sin\theta}\lambda_3\\0\end{pmatrix}$$

$(ii)\ n_z-z<0$

$$\begin{pmatrix}A_x^0\\A_y^0\\A_z^0\end{pmatrix}=\begin{pmatrix}-\frac{m_c}{2gr}\frac{\sin\phi(1-\cos\theta)}{\sin\theta}\lambda_3\\\frac{m_c}{2gr}\frac{\cos\phi(1-\cos\theta)}{\sin\theta}\lambda_3\\0\end{pmatrix}$$

(r,ϕ,θ) : Spherical coordinates system

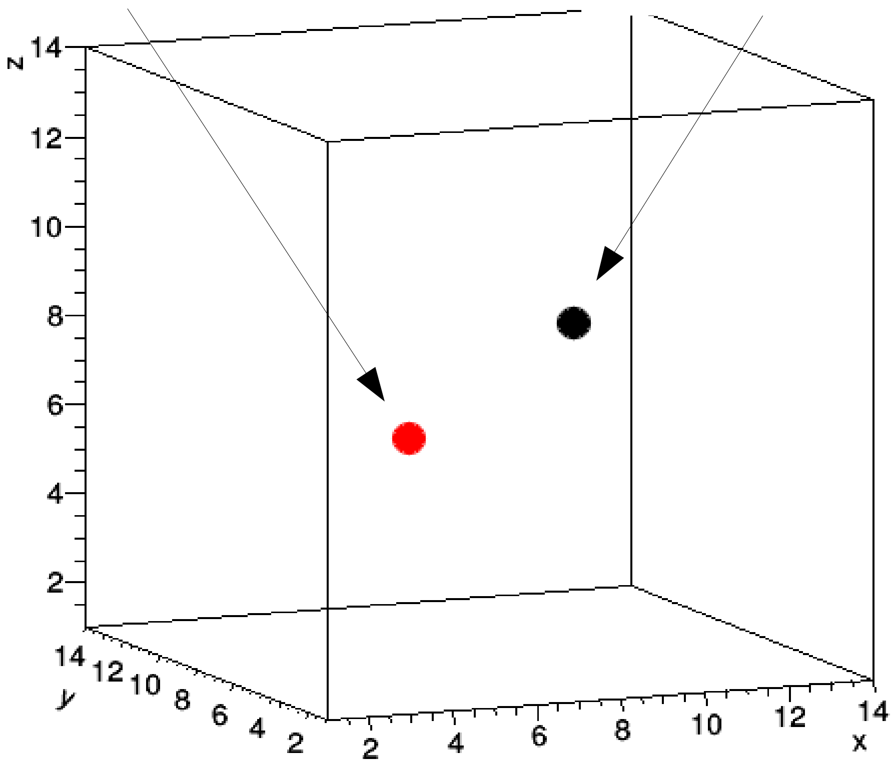
$g=\sqrt{\frac{6}{\beta}}$: Electric charge in SU(3)

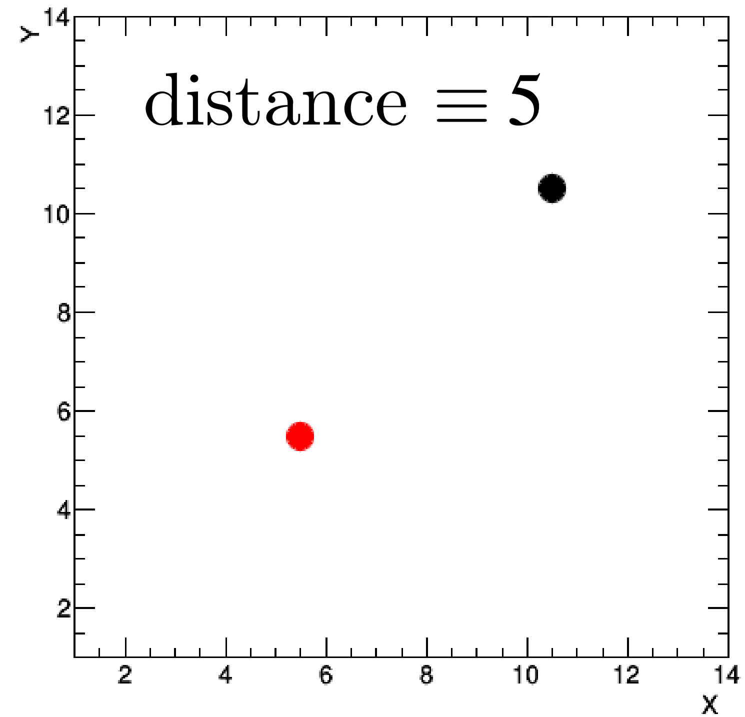
$\boxed{m_c \text{ : Monopole charge}}$

Place of monopoles

Anti-monopole (X_2, Y_2, Z_2)

Monopole (X_1, Y_1, Z_1)





The monopoles are added at T = 7.

Measuring monopoles

- **The monopole currents after the Abelian projection are defined as follows:**

S. Kitahara, et al. Nucl. Phys. B 533 (1998) 576
M. I. Polikarpov, et al. Phys. Lett. B 316 (1993) 333
F. Brandstaeter, et al. Phys. Lett. B 272 (1991) 319

$$k_\mu^i(s) = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial_\nu n_{\rho\sigma}^i(s+\nu)$$

- **The computation of monopole loops is defined in a paper (A. Bode, et al. Hep-lat/9312006), and discussions of the large and small monopole clusters in papers (S. Kitahara, et al. Prog. Theor. Phys. 93 (1995) 1, A. Hart, et al. Phsy. Rev. D 58 (1998) 014504)**

- **The total length of monopole loops is defined as follows:**

$$L_{\mathrm{mon}}/r_0 \equiv \frac{1}{12}\sum_i \sum_{s,\mu} |k_\mu^i(s)|/r_0$$

- **The monopole density is defined as follows:**

$$\rho r_0^3 = \frac{1}{12V}\sum_i \sum_{s,\nu} |k_\nu^i(s)| r_0^3$$

Simulation details for adding monopoles

- We added one monopole with +charges and one anti-monopole with -charges (the total charge is zero) to the configurations, and perform MA gauge fixing using by the simulated annealing algorithm.
- After the Abelian projection we measure:

(1) Total length of the monopole loops

(2) Long monopole loops

(3) Monopole density

DESY-ITEP-Kanazawa collaboration, Phys Rev D 70 (2004) 074511

Simulation parameters

The study of the additional monopoles.

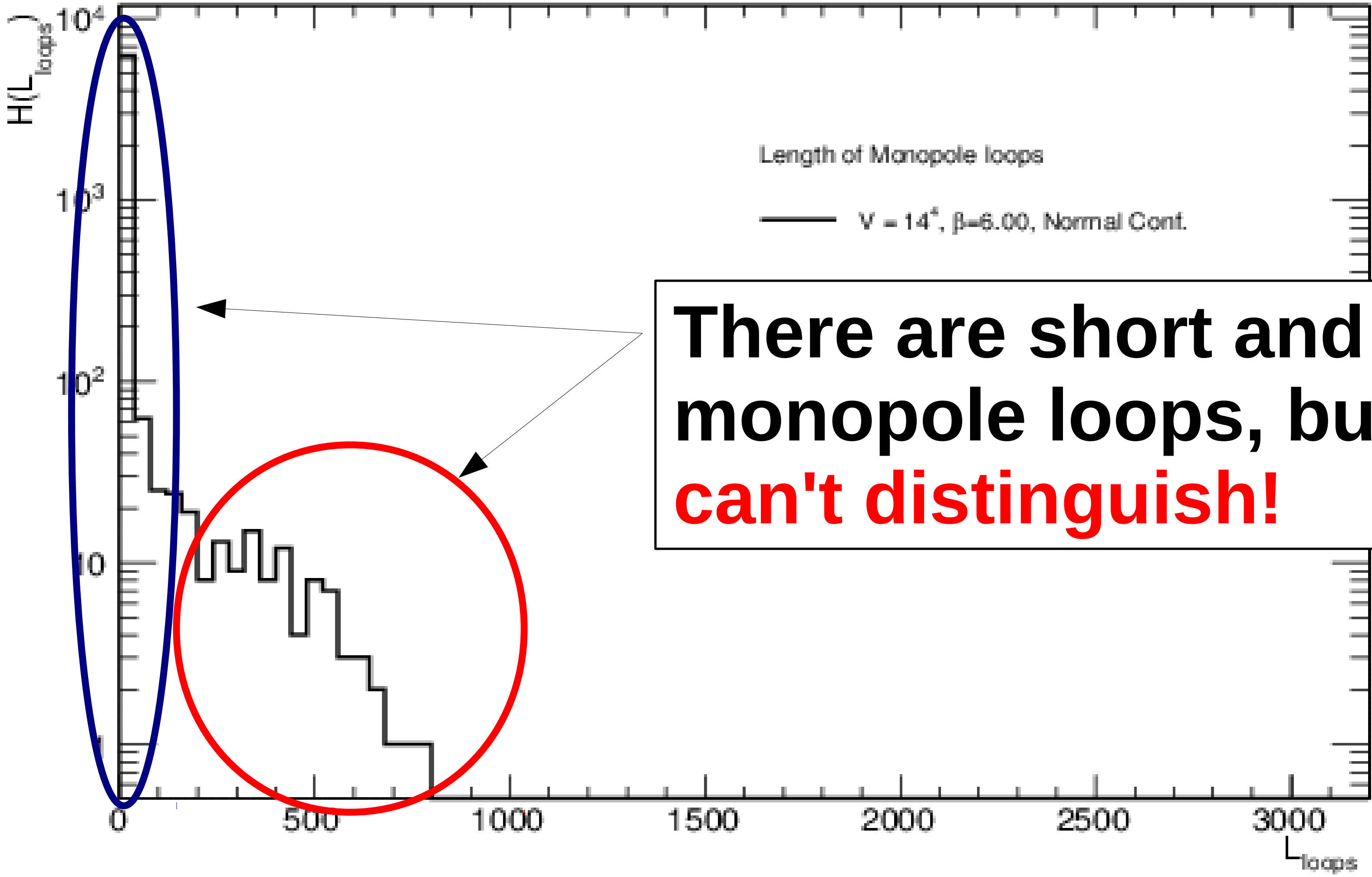
We generate configurations have additional monopoles. The parameters are listed in the table below.

β	V	N_{pairs} & $N_{charges}$	Distance	$N_{Conf.}$
6.00	14^4	(1, 0)	3	30
		(1, 1)	4	30
		(1, 2)	5	30
		(1, 3)	7	30
		(1, 4)	7	30

β = 6.00 and V = 14^4 lattice doesn't have the dependence on the physical volume. Therefore, we add the monopoles to this lattice.

The monopole clusters

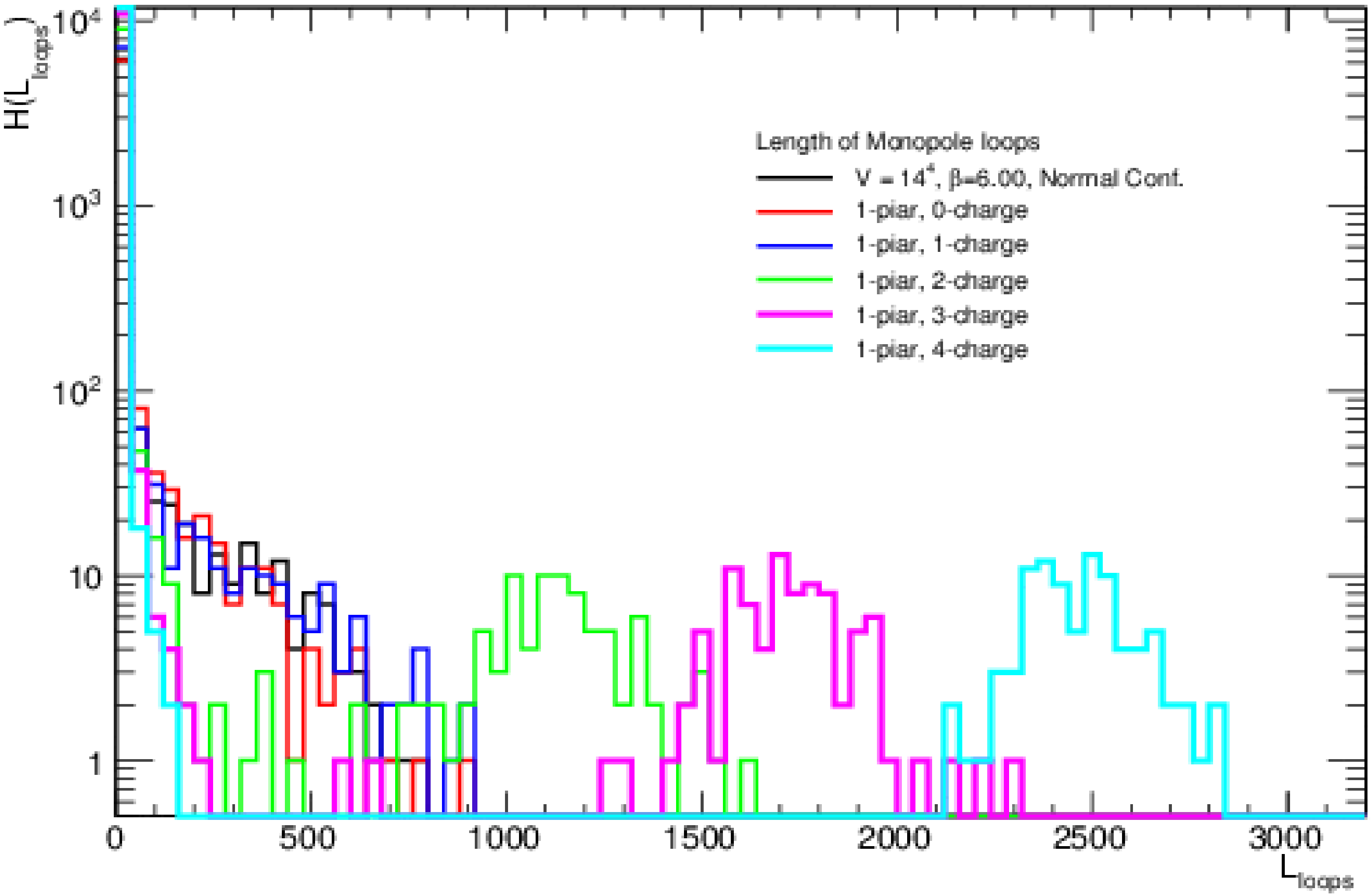
Normal configurations $\beta = 6.00$, $V = 14^4$



There are short and long monopole loops, but we can't distinguish!

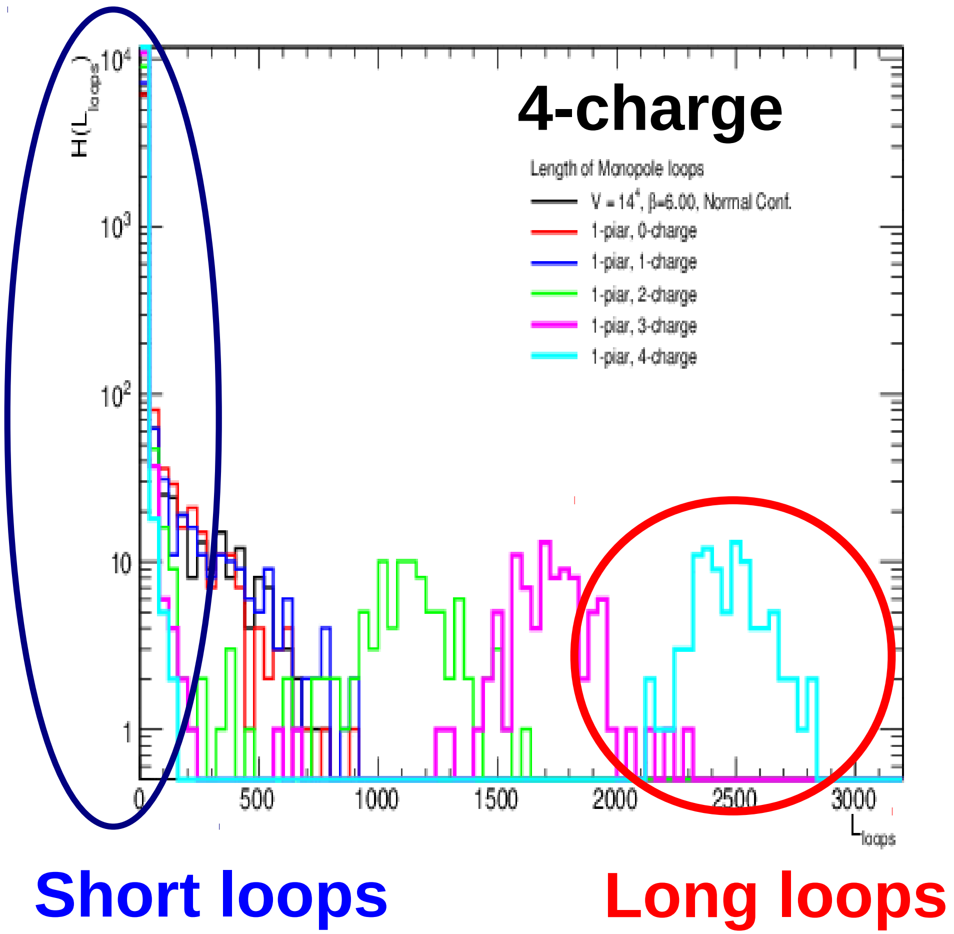
The additional monopole clusters

1-pair of monopoles with 0~4-charge



The additional monopole clusters

We add one monopole and one anti-monopole, and change charges of monopoles



The monopole creation operator makes **the long monopole loops!!**

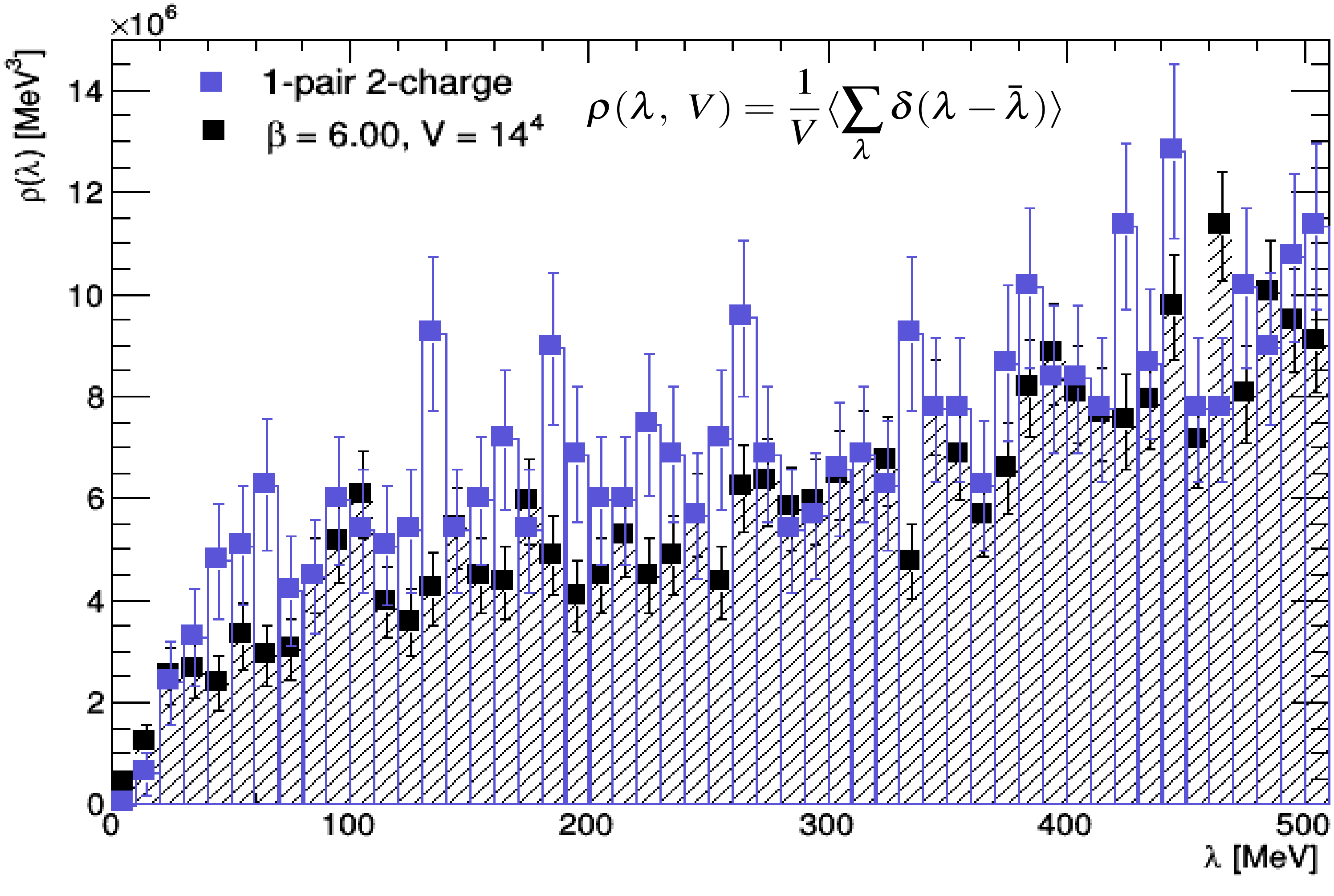
Simulation parameters for Overlap fermions

We generate configurations adding monopoles for the studying of instantons and additional monopoles. The parameters are listed in the table below.

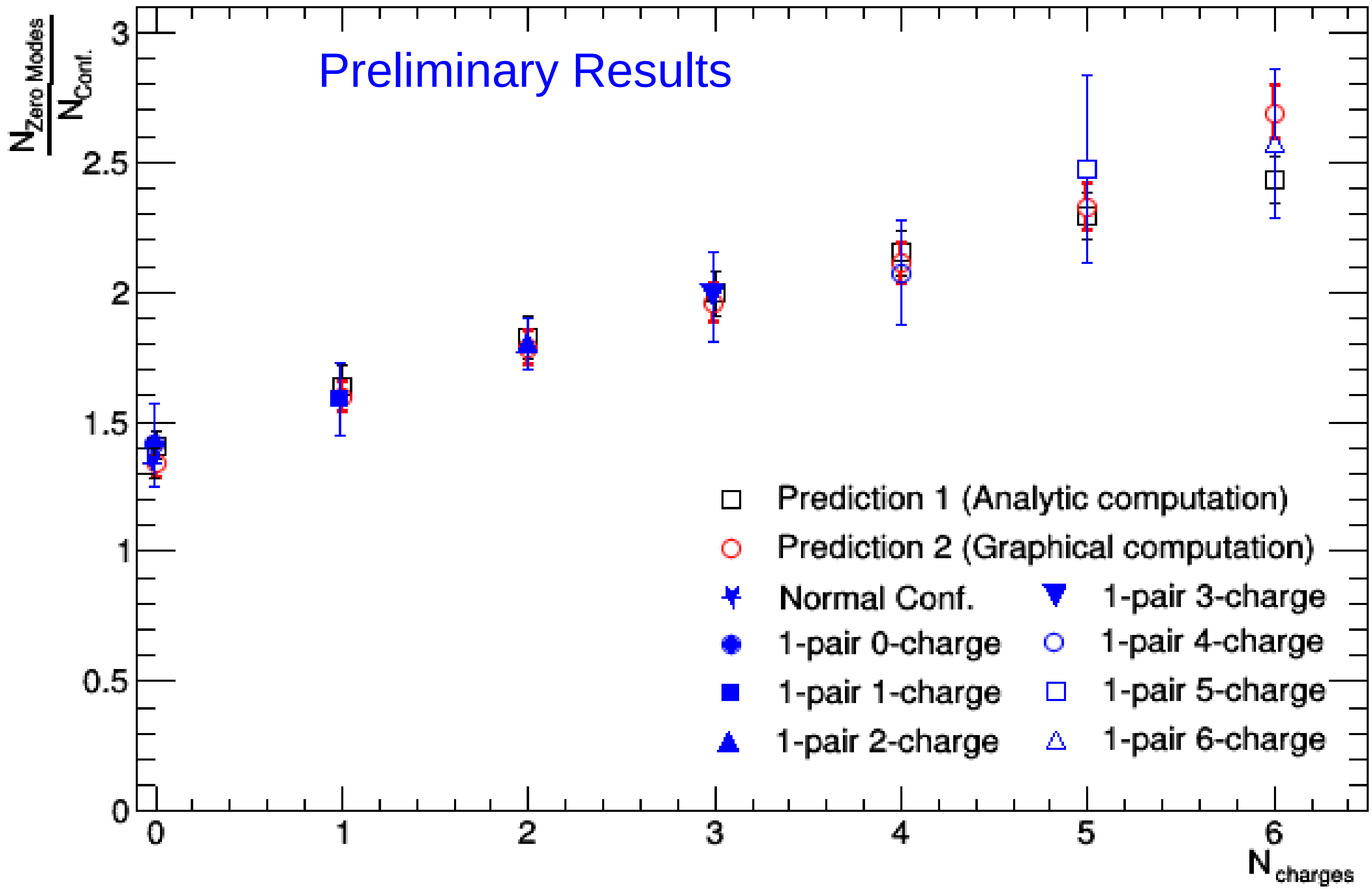
β	V	N_{pairs} & $N_{charges}$	Distance	$N_{Conf.}$
6.00	14^4	(1, 0)	3	51
		(1, 1)	6	89
		(1, 2)	5	191
		(1, 3)	8	60
		(1, 4)	8	87
		(1, 5)	8	30
		(1, 6)	8	30

Spectrum density of Overlap fermions

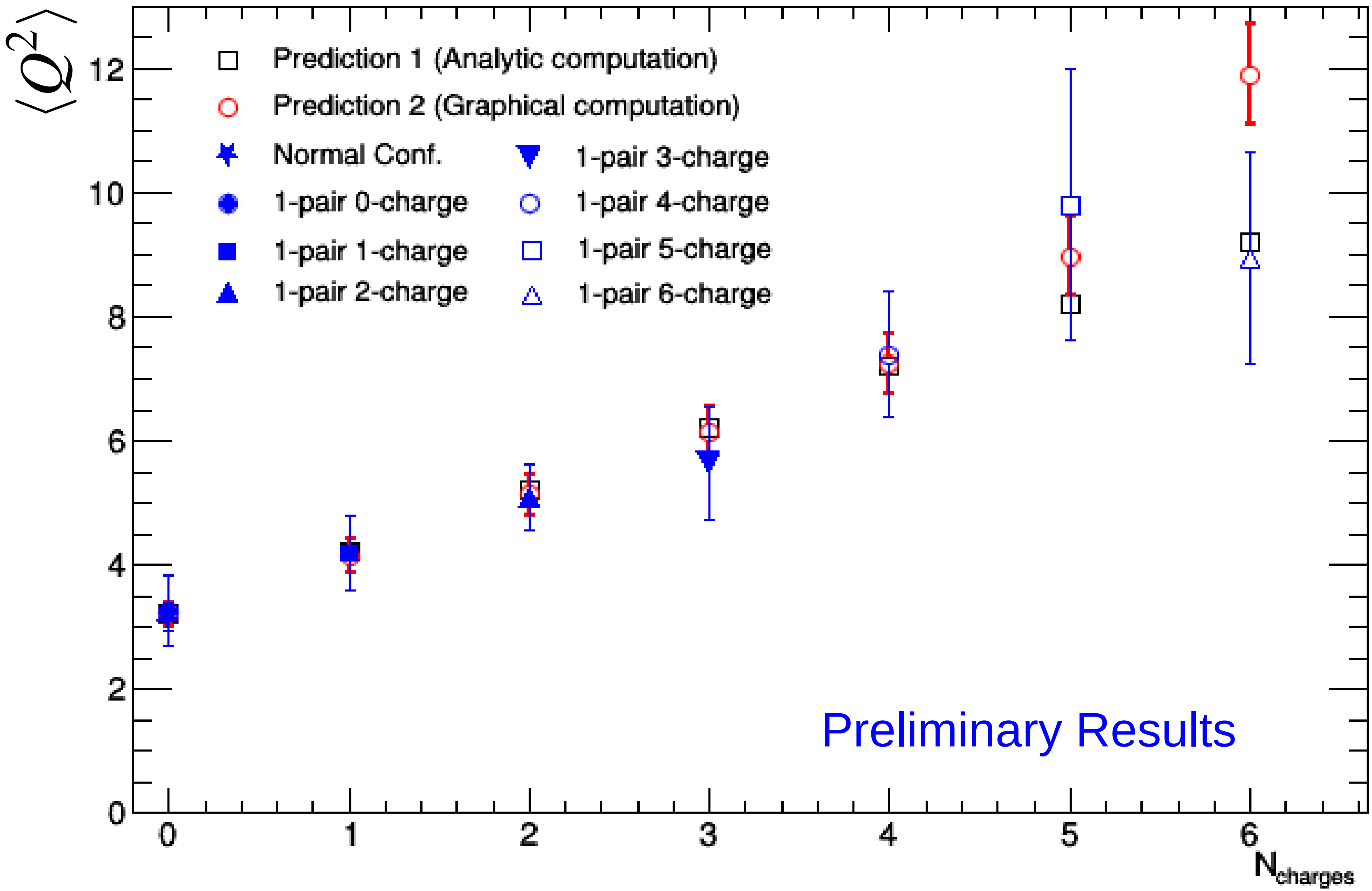
The spectrum density of non-zero modes.
In case of 1-pair of monopoles with 2-charge. N_c=176



The number of zero modes and monopole charges



The square of topological charges and monopole charges



Summary of my talk

I presented these contents:

- The number of zero modes, and topological susceptibility of Overlap fermions
- The number of instantons and zero modes
- The studies of additional monopoles
- The relation between the number of zero modes and monopoles

Conclusions

- I showed that the number of instantons was proportional to physical volume. We compared the instanton density with Instanton liquid model.
- We confirmed that monopoles and anti-monopoles were successfully added to configurations by the monopole creation operator
- The number of zero modes and $\langle Q^2 \rangle$ increased with the monopoles charges

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- I really appreciate helps to **F. Pucci,** M. Ilgenfritz, Y. Nakamura, K. Ishiguro, T. Sekido, and T. Suzuki.
- I thank to University of Parma, University of Bielefeld, and University of Kanazawa for their hospitality.
- All of the simulations have been performed on SX8 and SX9 at RCNP and CMC in University of Osaka, and SR16000 at YITP in University of Kyoto. We thank to all institutes for their technical supports and computer time.

Thank you for listening!