Loop formulation of Supersymmetric Yang-Mills Quantum Mechanics

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Motivation

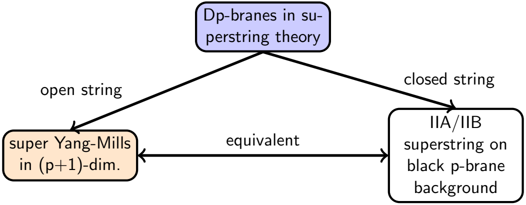
Motivation for the fermion loop formulation

- Possibility to control fermion sign problem:
  - e.g. for $\mathcal{N} = 16$ SUSY YM QM,
  - fermion contribution decomposes into fermion sectors,
  - each sector has definite sign
- New way to simulate fermions (including gauge fields):
  - local fermion algorithm,
  - works for massless fermions,
  - no critical slowing down
- Interesting physics:
  - testing gauge/gravity duality,
  - thermodynamics of black holes

Dualities, black holes and all that

Gauge/gravity duality conjecture:

- $U(N)$ gauge theories as a low energy effective theory of N D-branes
- Dimensionally reduced large-N super Yang-Mills might provide a nonperturbative formulation of the string/M-theory
- Connection to black p-branes allows studying black hole thermodynamics through strongly coupled gauge theory:



Continuum Model

- Start from $\mathcal{N} = 1$ SYM in $d = 4$ (or 10) dimensions
- Dimensionally reduce to 1-dim. $\mathcal{N} = 4$ (or 16) SYM QM:

$$S = \frac{1}{g^2} \int_0^\beta dt \, \mathrm{Tr} \left\{ (D_t X_i)^2 - \frac{1}{2} [X_i, X_j]^2 + \overline{\psi} D_t \psi - \overline{\psi} \sigma_i [X_i, \psi] \right\}$$

  - covariant derivative $D_t = \partial_t - i[A(t), \cdot]$,
  - time component of the gauge field $A(t)$,
  - spatial components become bosonic fields $X_i(t)$ with $i = 1, \ldots, d-1$,
  - anticommuting fermion fields $\overline{\psi}(t)$, $\psi(t)$,
  - σ_i are the γ-matrices in d dimensions
- all fields in the adjoint representation of SU(N)

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  - covariant derivative $D_t = \partial_t - i[A(t), \cdot]$,
  - time component of the gauge field $A(t)$,
  - spatial components become bosonic fields $X_i(t)$ with $i = 1, 2, 3$ (for $\mathcal{N} = 4$),
  - anticommuting fermion fields $\overline{\psi}(t)$, $\psi(t)$, (complex 2-component spinors for $\mathcal{N} = 4$)
  - σ_i are the γ-matrices in d dimensions (Pauli matrices for $\mathcal{N} = 4$)
- all fields in the adjoint representation of SU(N)

Lattice regularisation

- Discretise the bosonic part:

$$S_B = \frac{1}{g^2} \sum_{t=0}^{L_t-1} \mathrm{Tr} \left\{ D_t X_i(t) D_t X_i(t) - \frac{1}{2} \left[X_i(t), X_j(t)\right]^2 \right\}$$

with $D_t X_i(t) = U(t) X_i(t+1) U^\dagger(t) - X_i(t)$

- Use Wilson term for the fermionic part,

$$S_F = \frac{1}{g^2} \sum_{t=0}^{L_t-1} \mathrm{Tr} \left\{ \overline{\psi}(t) D_t \psi(t) - \overline{\psi}(t) \sigma_i \left[X_i(t), \psi(t)\right] \right\} \, ,$$

since

$$\partial^{\mathcal{W}} = \frac{1}{2}(\nabla^+ + \nabla^-) \pm \frac{1}{2} \nabla^+ \nabla^- \quad \overset{d=1}{\Longrightarrow} \quad \nabla^\pm$$

Lattice regularisation

- Specifically, we have in uniform gauge $U(t) = U$

$$S_F = \frac{1}{2g^2} \sum_{t=0}^{L_t-1} \left[-\overline{\psi}^a_\alpha(t) W^{ab}_{\alpha\beta} \psi^b_\beta(t+1) + \overline{\psi}^a_\alpha(t) \Phi^{ac}_{\alpha\beta}(t) \psi^c_\beta(t) \right]$$

  where $W^{ab}_{\alpha\beta} = 2\delta_{\alpha\beta} \otimes \mathrm{Tr}\{T^a U T^b U^\dagger\}$.

- Φ is a $2(N^2-1) \times 2(N^2-1)$ Yukawa interaction matrix:

$$\Phi^{ac}_{\alpha\beta}(t) = (\sigma_0)_{\alpha\beta} \otimes \delta^{ac} - 2\,(\sigma_i)_{\alpha\beta} \otimes \mathrm{Tr}\{T^a [X_i(t), T^c]\}$$

- Determinant reduction techniques give:

$$\det \mathcal{D}_{p,a} = \det \left[\prod_{t=0}^{L_t-1} (\Phi(t) W^\dagger) \mp 1 \right]$$

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- Determinant reduction techniques give:
  (for finite density $\mu \neq 0$)

$$\det \mathcal{D}_{p,a} = \det \left[\prod_{t=0}^{L_t-1} (\Phi(t) W^\dagger) \mp e^{-\mu L_t} \right]$$

Hopping expansion

- Hopping expansion of the fermion Boltzmann factor:

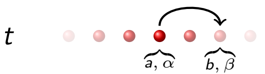
$$\exp(-S_F) \propto \prod_{t,a,b,\alpha,\beta} \left[\sum_{m^{ab}_{\alpha\beta}(t)=0}^{1} \left(-\Phi^{ab}_{\alpha\beta}(t) \overline{\psi}^a_\alpha(t) \psi^b_\beta(t) \right)^{m^{ab}_{\alpha\beta}(t)} \right] \times \prod_{t,a,\alpha} \left[\sum_{h^a_\alpha(t)=0}^{1} \left(\overline{\psi}^a_\alpha(t) \psi^a_\alpha(t+1) \right)^{h^a_\alpha(t)} \right]$$

- Grassmann integration:
  - every $\overline{\psi}^a_\alpha(t)\psi^a_\alpha(t)$ needs to be saturated,
  - yields local constraints on occupation numbers $h^a_\alpha(t)$ and $m^{ab}_{\alpha\beta}(t)$
- Represent each $\overline{\psi}^a_\alpha(t)\psi^a_\alpha(t)$ by ● and $h^a_\alpha(t), m^{ab}_{\alpha\beta}(t)$ by $\longrightarrow$:

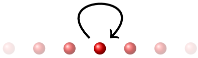
  only closed, oriented fermion loops survive
- Each fermion loop picks up a factor (-1)

Hopping expansion building blocks

- Non-temporal (flavour or colour) hops $m^{ab}_{\alpha\beta}(t) = 1$:

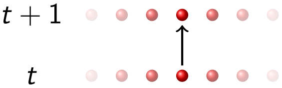


weight: $\Phi^{ab}_{\alpha\beta}(t)$



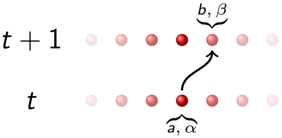
weight: $\Phi^{aa}_{\alpha\alpha}(t)$

- Temporal hops $h^{a}_{\alpha}(t) = 1$ (only forward!)



weight: 1

- Gauge links allow flavour non-diagonal temporal hops:

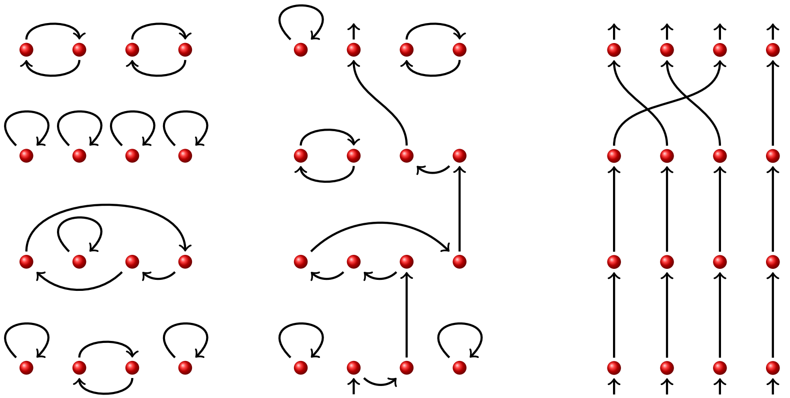


weight: $\delta_{\alpha\beta} \cdot \mathrm{Tr}\left[T^a U T^b U^\dagger\right]$

Fermion sectors

- Configurations can be classified according to the number of propagating fermions n_f:

$n_f = 0$ $\qquad$ $n_f = 1$ $\qquad \cdots \qquad$ $n_f = 2(N^2 - 1)$



Fermion sectors

- Propagation of fermions described by transfer matrices $T_{n_f}(t)$
- Fermion contribution to the partition function is simply

$$Z_{n_f} = \mathrm{Tr}\left[\prod_{t=0}^{L_t-1} T_{n_f}(t)\right]$$

and the full contribution with periodic b.c. is

$$Z_p = Z_0 - Z_1 \pm \ldots + Z_{2(N^2-1)} = \sum_{n_f=0}^{2(N^2-1)} (-1)^{n_f}\, \mathrm{Tr}\left[\prod_{t=0}^{L_t-1} T_{n_f}(t)\right]$$

- Size of T_{n_f} is given by the number of states in sector n_f:

$$\#\text{ of states} = \binom{2(N^2-1)}{n_f}$$

Fermion sectors

- Propagation of fermions described by transfer matrices $T_{n_f}(t)$
- Fermion contribution to the partition function is simply

$$Z_{n_f} = \mathrm{Tr}\left[\prod_{t=0}^{L_t-1} T_{n_f}(t)\right]$$

and the full contribution with antiperiodic b.c. is

$$Z_a = Z_0 + Z_1 + \ldots + Z_{2(N^2-1)} = \sum_{n_f=0}^{2(N^2-1)} \mathrm{Tr}\left[\prod_{t=0}^{L_t-1} T_{n_f}(t)\right]$$

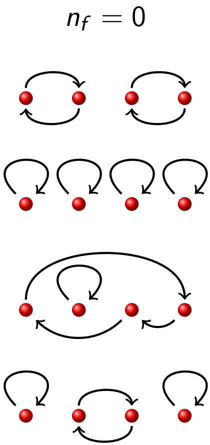
- Size of T_{n_f} is given by the number of states in sector n_f:

$$\#\text{ of states} = \begin{pmatrix} 2(N^2-1) \\ n_f \end{pmatrix}$$

Fermion sector $n_f = 0$

- Fermion sector $n_f = 0$ is simple:
  - $T_0(t)$ is a 1×1 matrix
  - $T_0(t) = \det \Phi(t)$
  - all signs from fermion loops taken into account
  - fermion contribution factorises completely:

$$Z_0 = \prod_{t=0}^{L_t - 1} \det \Phi(t)$$



Fermion sector $n_f = 2(N^2 - 1)$

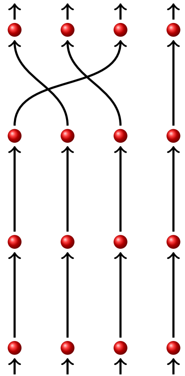
- Fermion sector $n_f = 2(N^2 - 1) \equiv n_f^{\max}$ is even simpler:
  - $T_{n_f^{\max}}(t) = 1$
  - including the gauge link:

    $$T_{n_f^{\max}}(t) = \det\left[\sigma_0 \otimes W\right] = 1$$

  - all signs from fermion loops taken into account
  - fermion contribution is trivial:

    $\Rightarrow$ quenched sector

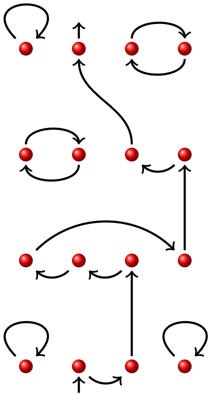
$n_f = 2(N^2 - 1)$



Fermion sector $n_f = 1$

- Fermion sector $n_f = 1$ less simple:
  - $T_1(t)$ is $[2(N^2-1)]^2$ matrix
  - $$\begin{aligned}(T_1)_{ij} &= \det \Phi\big|_{\Phi_{ki}=\delta_{kj},\Phi_{jk}=\delta_{ik}} \\ &= \det \Phi^{\not{j}\not{i}}\end{aligned}$$
  - including the gauge link:
    $$\left(T_1^U\right)_{ij} = \det[(\sigma_0 \otimes W)^{\not{j}\not{i}}]$$
  - all signs taken into account
  - fermion contribution:
    $$Z_1 = \prod_{t=0}^{L_t-1} \mathrm{Tr}\left[T_1(t) \cdot T_1^U\right]$$

$n_f = 1$



Fermion sector $n_f \geq 1$

- Z_1 not necessarily positive
- Generic fermion sector $n_f > 1$ increasingly more complicated:
  - transfer matrices become large,
  - matrix elements determined by permanents
- Sectors with many states may be simulated with worm algorithm:
  - boson bond formulation is also available

Conclusions

- Fermion loop formulation yields decomposition of fermion determinant into fermion sectors
- Each fermion sector described by transfer matrices
- $n_f = 0, 1$ and $n_f^{\max}$ implemented:
  - numerical results in reach,
  - sign problem for $n_f = 1$?
- Extension to $\mathcal{N} = 16$ SYM QM:
  - in principle straightforward,
  - but need $\overline{\psi} D_t \psi$,
  - no notion of n_f for Majorana in $d = 0$