Renormalization of Flavor Singlet and Nonsinglet Fermion Bilinear Operators

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Sphinx [Cyprus, 5th century B.C.]
Metropolitan Museum of Art, NY

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Overview

- Renormalization of fermion bilinears $\mathcal{O} = \bar{\psi}\Gamma\psi$ on the lattice, where
$$\Gamma = \hat{1},\ \gamma_5,\ \gamma_\mu,\ \gamma_5\,\gamma_\mu,\ \gamma_5\,\sigma_{\mu\,\nu}$$
- We consider both flavor singlet and nonsinglet operators ($\bar{\psi}\lambda^a\Gamma\psi$)
- Action: Symanzik gluons, Wilson/clover fermions, stout links

  [Includes twisted mass actions, SLiNC action]
- Calculation of the fermion self energy, up to two loops: $\Sigma^L_\psi(q, a_{\rm L})$
  Calculation of the 2-pt Green's functions, up to two loops: $\Sigma^L_\Gamma(q, a_{\rm L})$
- Calculation of the lattice two-loop renormalization functions:
  $Z^{L,Y}_\psi$, $Z^{L,Y}_\Gamma$ ($Y : RI'$ and $\overline{MS}$ schemes)
  - Results computed in an arbitrary covariant gauge
  - Generalization to fermionic fields in an arbitrary representation
- By-product: Quark mass multiplicative renormalization, $Z^{L,RI'}_m$
- Prerequisites: Calculation of 1-loop renormalization functions:
  [For Gluons A, Ghosts c, Gauge parameter α, Coupling constant g]
  $Z^{L,RI'}_A$, $Z^{L,RI'}_c$, $Z^{L,RI'}_\alpha$, $Z^{L,RI'}_g$

Motivation

- Studies of hadronic properties using Lattice QCD rely on:
  Matrix elements / Correlation functions of fermion composite operators.
  Variety of operators: “ultra-local” / extended bilinears, 4-fermi operators.
  Proper renormalization: indispensable for extraction of physical results.
- Non-perturbative estimates: preferred (but feasible? precise? often...).
  In all cases, a comparison with perturbation theory is desirable.
- Two-loop calculations of composite fermion operators thus far:
  Only $\bar{\psi}\Gamma\psi$ with Wilson gluons / clover fermions.
  [A. Skouroupathis, HP : PRD76 (2007) 094514, PRD79 (2009) 094508]
  Also results in Stochastic PT: F. DiRenzo et al.
- Z_m : Essential ingredient in computation of quark masses.
- Prototype for the renormalization of operators such as:
  - $\bar{\psi}\Gamma\, D^{\mu}\psi$ (appearing in Hadron Structure Functions)
  - $(\bar{s}\,\Gamma_1\, d)\,(\bar{s}\,\Gamma_2\, d)$ (appearing in $\Delta S = 2$ transitions, etc.)
- Generalize to other actions (Symanzik gluons, Stout links).
- Singlet vs. nonsinglet renormalization: Appears first at 2 loops.
  Non-perturbative determination is difficult. [QCDSF, talk by H. Perlt]
  Relevant for studies of $\eta - \eta'$ mesons, etc.

Lattice Action: Symanzik Gluons, Clover Fermions

- Complete discretized action: $S = S_G + S_F^W + S_{SW}$, where:

$$S_F^W = -\frac{a^4}{2}\sum_{n,\mu}\left[\frac{1}{a}\overline{\Psi}(\vec{n})\Big((r-\gamma_\mu)\tilde{U}_{\vec{n},\,\vec{n}+\hat{\mu}}\Psi(\vec{n}+\hat{\mu}) + (r+\gamma_\mu)\tilde{U}^\dagger_{\vec{n}-\hat{\mu},\,\vec{n}}\Psi(\vec{n}-\hat{\mu}) - \Big(2r+\frac{Ma}{2}\Big)\Psi(\vec{n})\Big)\right]$$

$\tilde{U}_{\vec{n},\,\vec{n}+\hat{\mu}}$: Stout links

$$S_{SW} = -a^5\sum_n c_{SW}\,\overline{\Psi}(\vec{n})\,\frac{1}{4}\,\sigma_{\mu\nu}\,\hat{G}_{\mu\nu}(\vec{n})\,\Psi(\vec{n})$$

and: $\hat{G}_{\mu\nu}(\vec{n}) = \frac{1}{8a^2}\Big[Q_{\mu\nu}(\vec{n}) - Q_{\nu\mu}(\vec{n})\Big]$,

$$\begin{aligned} Q_{\mu\nu} = \quad & U_{\vec{n},\,\vec{n}+\hat{\mu}}\,U_{\vec{n}+\hat{\mu},\,\vec{n}+\hat{\mu}+\hat{\nu}}\,U_{\vec{n}+\hat{\mu}+\hat{\nu},\,\vec{n}+\hat{\nu}}\,U_{\vec{n}+\hat{\nu},\,\vec{n}} \\ & + U_{\vec{n},\,\vec{n}+\hat{\nu}}\,U_{\vec{n}+\hat{\nu},\,\vec{n}+\hat{\nu}-\hat{\mu}}\,U_{\vec{n}+\hat{\nu}-\hat{\mu},\,\vec{n}-\hat{\mu}}\,U_{\vec{n}-\hat{\mu},\,\vec{n}} \\ & + U_{\vec{n},\,\vec{n}-\hat{\mu}}\,U_{\vec{n}-\hat{\mu},\,\vec{n}-\hat{\mu}-\hat{\nu}}\,U_{\vec{n}-\hat{\mu}-\hat{\nu},\,\vec{n}-\hat{\nu}}\,U_{\vec{n}-\hat{\nu},\,\vec{n}} \\ & + U_{\vec{n},\,\vec{n}-\hat{\nu}}\,U_{\vec{n}-\hat{\nu},\,\vec{n}-\hat{\nu}+\hat{\mu}}\,U_{\vec{n}-\hat{\nu}+\hat{\mu},\,\vec{n}+\hat{\mu}}\,U_{\vec{n}+\hat{\mu},\,\vec{n}} \end{aligned}$$

c_{SW} : Free parameter

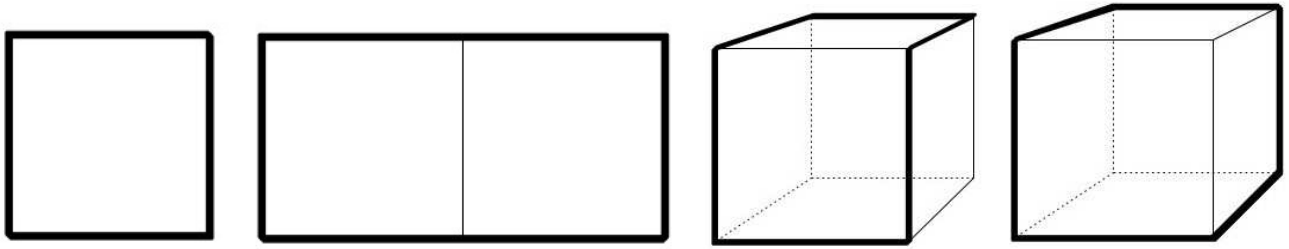
Stout Links

- $\tilde{U}_{\vec{n},\vec{n}+\hat{\mu}} = e^{iQ_{\hat{\mu}}(\vec{n})} U_{\vec{n},\vec{n}+\hat{\mu}}$
- $$Q_{\hat{\mu}}(\vec{n}) = \frac{\omega}{2i}\Big[V_{\hat{\mu}}(\vec{n})U^{\dagger}_{\vec{n},\vec{n}+\hat{\mu}} - U_{\vec{n},\vec{n}+\hat{\mu}}V^{\dagger}_{\hat{\mu}}(\vec{n}) - \frac{1}{N_c}\mathrm{Tr}\big(V_{\hat{\mu}}(\vec{n})U^{\dagger}_{\vec{n},\vec{n}+\hat{\mu}} - U_{\vec{n},\vec{n}+\hat{\mu}}V^{\dagger}_{\hat{\mu}}(\vec{n})\big)\Big]$$
- $$V_{\hat{\mu}}(\vec{n}) = \sum_{\hat{\nu}\neq\hat{\mu}}\big(U_{\vec{n},\vec{n}+\hat{\nu}}U_{\vec{n}+\hat{\nu},\vec{n}+\hat{\mu}+\hat{\nu}}U_{\vec{n}+\hat{\mu}+\hat{\nu},\vec{n}+\hat{\mu}} + U_{\vec{n},\vec{n}-\hat{\nu}}U_{\vec{n}-\hat{\nu},\vec{n}+\hat{\mu}-\hat{\nu}}U_{\vec{n}+\hat{\mu}-\hat{\nu},\vec{n}+\hat{\mu}}\big)$$
- ω : Free parameter

Gluon Action

$$S_G = \frac{2}{g^2}\left[c_0 \sum_{\text{Plaq}} \text{Re}\,\text{Tr}\left(1 - U_{\text{Plaq}}\right) + c_1 \sum_{\text{Rect}} \text{Re}\,\text{Tr}\left(1 - U_{\text{Rect}}\right) + c_2 \sum_{\text{Chair}} \text{Re}\,\text{Tr}\left(1 - U_{\text{Chair}}\right) + c_3 \sum_{\text{Parall}} \text{Re}\,\text{Tr}\left(1 - U_{\text{Parall}}\right)\right]$$

where: $c_0 + 8c_1 + 16c_2 + 8c_3 = 1$



Plaquette Rectangle Chair Parallelogram

Action	c_0	c_1	c_2	c_3
Wilson	1	0	0	0
Tree-Level Symanzik	$5/3$	$-1/12$	0	0
Iwasaki	3.648	-0.331	0	0
DBW2	12.2688	-1.4086	0	0

Common sets of values for Symanzik coefficients

Preliminaries

- Denoting all bare quantities in the Lagrangian with the subscript "o", the corresponding renormalized quantities read:

$$A^a_{\mu\,\circ} = \sqrt{Z_A}\, A^a_\mu, \qquad c^a_\circ = \sqrt{Z_c}\, c^a, \qquad \psi_\circ = \sqrt{Z_\psi}\, \psi$$
$$g_\circ = \mu^\epsilon\, Z_g\, g, \qquad \alpha_\circ = Z_\alpha^{-1}\, Z_A\, \alpha$$

  where μ is the mass scale introduced to ensure the coupling constant has the correct dimensionality in d dimensions and $d = 4 - 2\epsilon$.

- The RI' renormalization scheme is defined by imposing renormalization conditions on matrix elements at a scale $\bar{\mu}$, where (just as in $\overline{MS}$):

$$\bar{\mu} = \mu\,(4\pi/e^{\gamma_{\rm E}})^{1/2}$$

- Mass-independent renormalization scheme $\Longrightarrow$ All renormalization functions are calculable at vanishing renormalized mass:

$$m_\circ \to m_{\rm cr} = m_1\, g_\circ^2 + \mathcal{O}(g_\circ^4)$$

- Free parameters:
  Coupling constant g_0, number of flavors N_f, number of colors N_c, clover coefficient c_{SW}, stout coefficient ω, gauge parameter α.

- Numerical results for specific values of Symanzik parameters c_i .

Definition of 2-point Green's Functions

- Fermion and ghost self energy in Euclidean space:

$$\Sigma_\psi(q, a_{\rm L}) = i\not{q} + m_\circ + \mathcal{O}(g_\circ^2) \quad , \quad \Sigma_c(q, a_{\rm L}) = q^2 + \mathcal{O}(g_\circ^2)$$

- Gluon propagator with radiative corrections:

$$G^L_{\mu\,\nu}(q, a_{\rm L}) = \frac{1}{q^2}\left[\frac{\delta_{\mu\,\nu} - q_\mu q_\nu/q^2}{\Pi_T(a_{\rm L}\, q)} + \alpha_\circ\, \frac{q_\mu q_\nu/q^2}{\Pi_L(a_{\rm L}\, q)}\right]$$

$$\left(\Pi_{T,L}(a_{\rm L}\, q) = 1 + \mathcal{O}(g_\circ^2)\right)$$

- $\mathcal{O}_\Gamma$, 2-point amputated Green's functions:

$$\Sigma_S(qa_{\rm L}) = \hat{1}\, \Sigma_S^{(1)}(qa_{\rm L})$$

$$\Sigma_P(qa_{\rm L}) = \gamma_5\, \Sigma_P^{(1)}(qa_{\rm L})$$

$$\Sigma_V(qa_{\rm L}) = \gamma_\mu\, \Sigma_V^{(1)}(qa_{\rm L}) + \frac{q^\mu \not{q}}{q^2} \Sigma_V^{(2)}(qa_{\rm L})$$

$$\Sigma_{AV}(qa_{\rm L}) = \gamma_5\gamma_\mu\, \Sigma_{AV}^{(1)}(qa_{\rm L}) + \gamma_5\, \frac{q^\mu \not{q}}{q^2} \Sigma_{AV}^{(2)}(qa_{\rm L})$$

$$\Sigma_T(qa_{\rm L}) = \gamma_5\, \sigma_{\mu\,\nu} \Sigma_T^{(1)}(qa_{\rm L}) + \gamma_5\, \frac{\not{q}\,(\gamma_\mu q_\nu - \gamma_\nu q_\mu)}{q^2} \Sigma_T^{(2)}(qa_{\rm L})$$

$$\left(\Sigma_\Gamma^{(1)}(qa_{\rm L}) = 1 + \mathcal{O}(g_\circ^2), \qquad \Sigma_\Gamma^{(2)}(qa_{\rm L}) = \mathcal{O}(g_\circ^2)\right)$$

The RI′ Renormalization Scheme

- Renormalization conditions for the fermion and ghost self-energy:

$$\lim_{a_{\rm L}\to 0}\left[Z_\psi^{L,RI'}(a_{\rm L}\bar\mu)\,{\rm tr}\left(\Sigma_\psi(q,a_{\rm L})\not{q}\right)/(4i\,q^2)\right]_{q^2=\bar\mu^2}=1$$

$$\lim_{a_{\rm L}\to 0}\left[Z_c^{L,RI'}(a_{\rm L}\bar\mu)\frac{\Sigma_c(q,a_{\rm L})}{q^2}\right]_{q^2=\bar\mu^2}=1$$

- Renormalization conditions for Z_A and Z_α :

$$\lim_{a_{\rm L}\to 0}\left[Z_A^{L,RI'}(a_{\rm L}\bar\mu)\,\Pi_T(a_{\rm L}q)\right]_{q^2=\bar\mu^2}=\lim_{a_{\rm L}\to 0}\left[Z_\alpha^{L,RI'}(a_{\rm L}\bar\mu)\,\Pi_L(a_{\rm L}q)\right]_{q^2=\bar\mu^2}=1$$

- Renormalization conditions for Z_g, on the lattice:

$$\lim_{a_{\rm L}\to 0}\left[Z_\psi^{L,RI'}\,(Z_A^{L,RI'})^{1/2}\,Z_g^{L,RI'}\,G_{A\bar\psi\psi}(q,a_{\rm L})\right]_{q^2=\bar\mu^2}=G^{\rm finite}_{A\bar\psi\psi}$$

or, equivalently:

$$\lim_{a_{\rm L}\to 0}\left[Z_c^{L,RI'}\,(Z_A^{L,RI'})^{1/2}\,Z_g^{L,RI'}\,G_{A\bar c c}(q,a_{\rm L})\right]_{q^2=\bar\mu^2}=G^{\rm finite}_{A\bar c c}$$

where the gluon-fermion-antifermion (gluon-ghost-antighost) 1PI vertex function $G^{\rm finite}_{A\bar\psi\psi}$ ($G^{\rm finite}_{A\bar c c}$) is required to be the same as the one stemming from the continuum (dimensional regularization, $\overline{\rm MS}$).

The RI′ Renormalization Scheme

- We define renormalized operators by:

$$\mathcal{O}_\Gamma^{RI'} = Z_\Gamma^{L,RI'}(a_{\rm L}\bar{\mu})\,\mathcal{O}_{\Gamma\,\circ} \qquad (\mathcal{O}_{\Gamma\,\circ} = \bar{\psi}\,\Gamma\,\psi)$$

- Renormalization conditions for $\mathcal{O}_\Gamma$:

$$\lim_{a_{\rm L}\to 0}\left[Z_\psi^{L,RI'}\; Z_\Gamma^{L,RI'}\;\Sigma_\Gamma^{(1)}(qa_{\rm L})\right]_{q^2=\bar{\mu}^2} = 1$$

$\Sigma_\Gamma^{(2)}$ is not included.

- Alternative scheme, more appropriate for nonpertubative renormalization:

$$\lim_{a_{\rm L}\to 0}\left[Z_\psi^{L,RI'}\; Z_\Gamma^{L,RI'}\;\frac{{\rm tr}(\Gamma\Sigma_\Gamma(qa_{\rm L}))}{{\rm tr}(\Gamma\Gamma)}\right]_{q^2=\bar{\mu}^2} = 1$$

- Difference between two schemes (for V, A, T):
  - Finite
  - Deducible from lower loop calculation (plus continuum results).

Conversion to the $\overline{\mathrm{MS}}$ Scheme

- Conversion of the coupling constant and of the gauge parameter to $\overline{MS}$:

$$g_{RI'} = g_{\overline{MS}} \quad , \quad \alpha_{RI'} = \left(Z_A^{L,\overline{MS}} / Z_A^{L,RI'} \right) \alpha_{\overline{MS}}$$

- Ratios of Z's must be regularization independent. Having computed the renormalization functions in RI', we can construct their $\overline{MS}$ counterparts using conversion factors ($DR \equiv$ Dimensional Regularization):

$$C_i(a,\alpha) \equiv \frac{Z_i^{L,RI'}}{Z_i^{L,\overline{MS}}} = \frac{Z_i^{DR,RI'}}{Z_i^{DR,\overline{MS}}}, \qquad \text{where } i : A,\ c,\ \psi,\ S,\ V,\ T.$$

- In the case of the scalar, vector and tensor operators, the renormalization functions, $Z_\Gamma^{L,\overline{MS}}$ can be obtained by: $\quad Z_\Gamma^{L,\overline{MS}} = Z_\Gamma^{L,RI'} / C_\Gamma(g,\alpha)$

- For pseudoscalar and axial vector: In order to satisfy Ward identities, extra finite factors $Z_5^P(g)$ and $Z_5^{AV}(g)$, calculable in DR, are required:

$$\mathcal{O}_P = Z_5^P(g)\, Z_P^{DR,\overline{MS}}\, \mathcal{O}_P^\circ \quad , \quad \mathcal{O}_{AV}^{\overline{MS}} = Z_5^{AV}(g)\, Z_{AV}^{DR,\overline{MS}}\, \mathcal{O}_{AV}^\circ$$

$$Z_P^{L,\overline{MS}} = Z_P^{L,RI'} / \left(C_S\, Z_5^P \right) \quad , \quad Z_{AV}^{L,\overline{MS}} = Z_{AV}^{L,RI'} / \left(C_V\, Z_5^{AV} \right)$$

- Z_5 differ for flavor singlet and nonsinglet.

Conversion Factors

$$C_A(g,\alpha) = 1 + \frac{g^2}{36(16\pi^2)}\left[\left(9\alpha^2 + 18\alpha + 97\right) N_c - 40 N_f\right]$$

$$C_c(g,\alpha) = 1 + \frac{g^2}{16\pi^2} N_c$$

$$C_\psi(g,\alpha) = 1 - \frac{g^2}{16\pi^2} c_F\,\alpha + \frac{g^4}{8\,(16\pi^2)^2} c_F\Big[\left(8\alpha^2 + 5\right) c_F + 14\,N_f - \left(9\alpha^2 - 24\zeta(3)\,\alpha + 52\alpha - 24\zeta(3) + 82\right) N_c\Big]$$

$$C_S(g,\alpha) = 1 + \frac{g^2}{16\pi^2} c_F\,(\alpha + 4) + \frac{g^4}{24\,(16\pi^2)^2} c_F\Big[\left(24\alpha^2 + 96\alpha - 288\zeta(3) + 57\right) c_F + 166\,N_f - \left(18\alpha^2 + 84\alpha - 432\zeta(3) + 1285\right) N_c\Big]$$

$$Z_5^P(g) = 1 - \frac{g^2}{16\pi^2}(8\,c_F) + \frac{g^4}{(16\pi^2)^2}\left(\frac{2}{9} c_F\,N_c + \frac{4}{9} c_F\,N_f\right)$$

($c_F = (N_c^2 - 1)/(2\,N_c)$; $\zeta(x)$: Riemann's zeta function)

Conversion Factors

$$C_V(g,\alpha) = 1 + \mathcal{O}(g^8)$$

$$C_T(g,\alpha) = 1 + \frac{g^2}{16\pi^2}\, c_F\, \alpha + \frac{g^4}{216\,(16\pi^2)^2}\, c_F \left[\left(216\alpha^2 + 4320\zeta(3) - 4815\right)\, c_F \right.$$
$$\left. -626\, N_f + \left(162\alpha^2 + 756\alpha - 3024\zeta(3) + 5987\right)\, N_c \right]$$

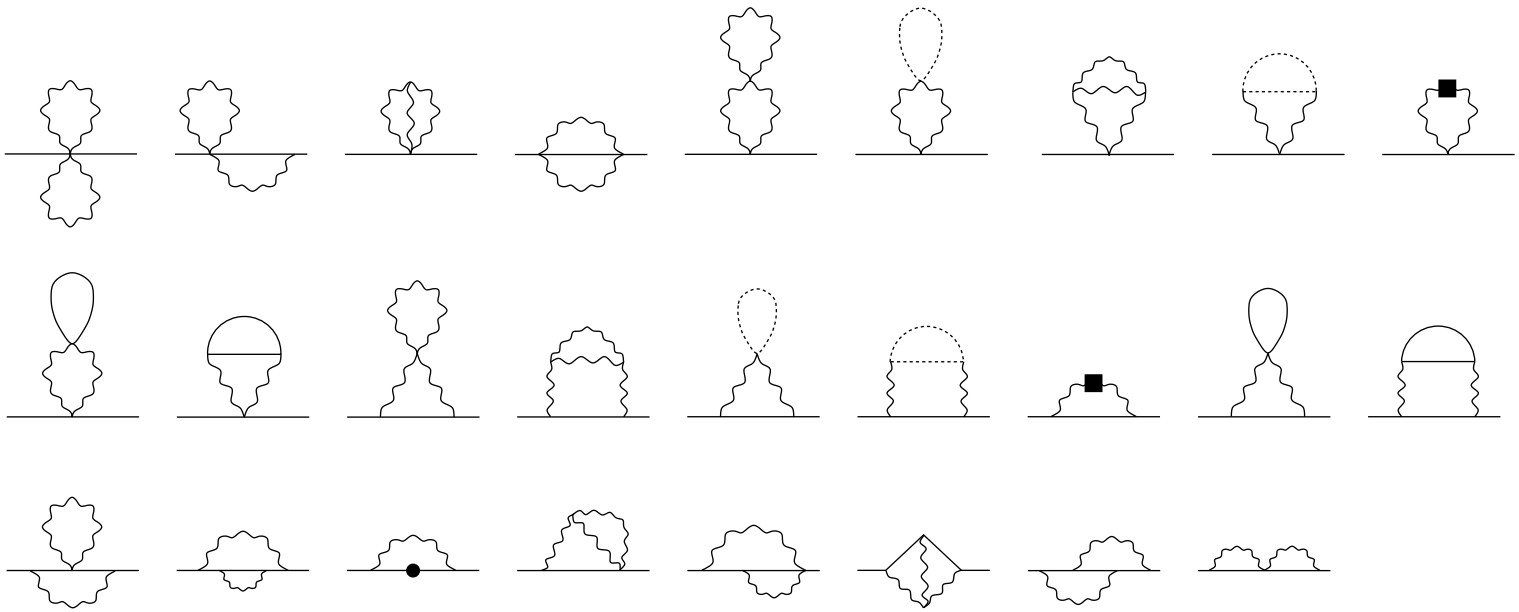
$$Z_5^{AV,s}(g) \;= 1 - \frac{g^2}{16\pi^2}\,(4\,c_F) + \frac{g^4}{(16\pi^2)^2}\left(22\,c_F^2 - \frac{107}{9}\,c_F\,N_c + \frac{31}{18}\,c_F\,N_f\right)$$

$$Z_5^{AV,ns}(g) = 1 - \frac{g^2}{16\pi^2}\,(4\,c_F) + \frac{g^4}{(16\pi^2)^2}\left(22\,c_F^2 - \frac{107}{9}\,c_F\,N_c + \frac{2}{9}\,c_F\,N_f\right)$$

[S. A. Larin, J. A. Gracey]

Feynman Diagrams

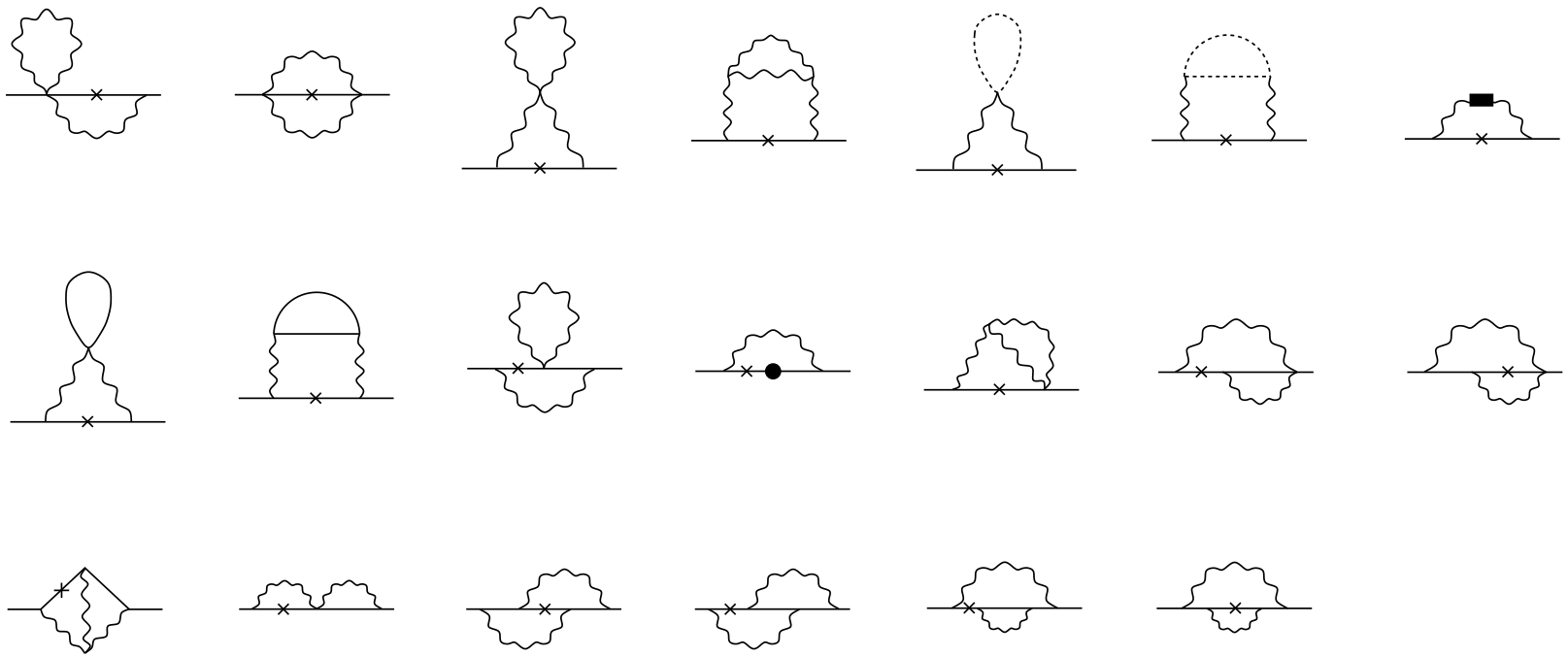
- 2-loop diagrams contributing to $\mathbf{Z}_{\psi}^{\mathbf{L},\mathbf{RI}'}$: [Z_A, Z_α, Z_c, Z_g needed to 1 loop]



- Wavy (solid, dotted) lines: gluons (fermions, ghosts). Solid boxes: vertices from measure part of the action. Solid circle: fermion mass counterterm.
- Several diagrams, evaluated individually, are IR divergent.
- Previously known with Wilson gluons / clover fermions. Differences?
  - No new $\ln^2(\bar{\mu} a_{\mathrm{L}})$ terms
  - Gluon propagator numerically inverted (more efficient)
  - Plethora of terms in vertices, due to stout links
  - e.g.: First diagram is no longer *back-of-the-envelope*: $>10,000$ terms

Feynman Diagrams

- Two-loop diagrams contributing to $\mathbf{Z}_{\mathbf{\Gamma}}^{\mathbf{L},\mathbf{RI}'}$ (flavor nonsinglet):

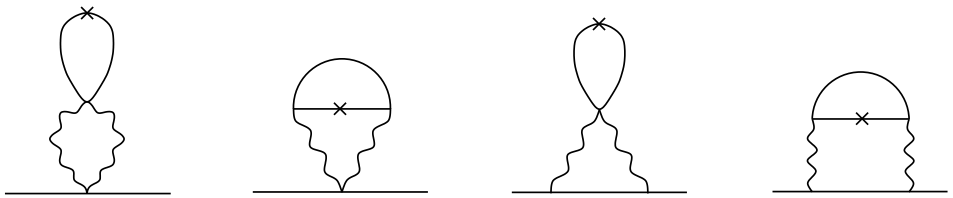


- Wavy (solid, dotted) lines: gluons (fermions, ghosts).
- Solid box: vertex from the measure part of the action.
- Solid circle: fermion mass counterterm.
- Crosses denote the Dirac matrices: $\hat{1}$, γ_5, γ_μ, $\gamma_5\,\gamma_\mu$, $\gamma_5\,\sigma_{\mu\,\nu}$.

Results

- One-loop results: Known for most actions we have considered.
- Two-loop results: Computation still in progress.
- Focus on the difference $\mathbf{Z}_{\mathbf{\Gamma}}^{\mathrm{singlet}} - \mathbf{Z}_{\mathbf{\Gamma}}^{\mathrm{non-singlet}}$:

  A two-loop effect, computation just completed.
- Two-loop diagrams contributing to $Z_{\Gamma}^{\mathrm{singlet}}$: $\left[\mathcal{O}_{singlet} \equiv \sum_f \bar{\psi}_f \Gamma \psi_f\right]$



- $\mathbf{Z}_{\mathbf{\Gamma}}^{\mathrm{singlet}} - \mathbf{Z}_{\mathbf{\Gamma}}^{\mathrm{non-singlet}} = \mathcal{O}(g_0^4) \Longrightarrow$ All other factors (Z_ψ , . . .) are set to 1.
- Similarly, conversion factors can be set to 1

  $\Longrightarrow$ Same difference in all renormalization schemes.
- The contribution of these diagrams to Z_P, Z_V, Z_T vanishes identically

  $\Longrightarrow$ Only Z_S and Z_{AV} are affected.

Results for the Scalar Operator

$$Z_S^{\rm singlet}(\bar{\mu} a_{\rm L}) - Z_S^{\rm non-singlet}(\bar{\mu} a_{\rm L})$$

$$= -\frac{g_0^4}{(4\pi)^4} c_F N_f \Big[\; \big(s_{00} + s_{01}\, \mathbf{c}_{SW} + s_{02}\, \mathbf{c}_{SW}^2 + s_{03}\, \mathbf{c}_{SW}^3 + s_{04}\, \mathbf{c}_{SW}^4\big)$$
$$+\big(s_{10} + s_{11}\, \mathbf{c}_{SW} + s_{12}\, \mathbf{c}_{SW}^2 + s_{13}\, \mathbf{c}_{SW}^3\big)\, \omega$$
$$+\big(s_{20} + s_{21}\, \mathbf{c}_{SW} + s_{22}\, \mathbf{c}_{SW}^2\big)\, \omega^2$$
$$+\big(s_{30} + s_{31}\, \mathbf{c}_{SW}\big)\, \omega^3$$
$$+ s_{40}\, \omega^4 \Big] + \mathcal{O}\big(g_0^6\big)$$

$[c_F = (N_c^2 - 1)/(2\, N_c)]$

- Numerical constants $\mathbf{s}_{i,j}$: Computed for various sets of values of the Symanzik coefficients.
- Computation done in general gauge. Result gauge independent. [As it should in $\overline{MS}$ and, to 2 loops, in all schemes.]
- For the scalar operator: The result is actually scale independent.

Results for the Scalar Operator

	Wilson	TL Symanzik	Iwasaki	DBW2
s_{00}	107.76(2)	76.29(1)	42.973(7)	17.031(5)
s_{01}	-82.27(1)	-69.01(1)	-49.356(8)	-25.620(7)
s_{02}	29.730(2)	26.178(1)	20.312(3)	11.825(3)
s_{03}	-3.4399(7)	-2.9533(5)	-2.2166(3)	-1.2488(5)
s_{04}	-2.2750(4)	-1.6403(3)	-0.8547(2)	-0.22630(6)
s_{10}	-1854.4(2)	-1107.0(1)	-444.69(4)	-85.808(5)
s_{11}	506.26(5)	364.01(3)	192.35(1)	54.210(4)
s_{12}	-95.42(2)	-70.94(1)	-40.162(6)	-13.003(2)
s_{13}	7.494(1)	5.356(1)	2.8030(4)	0.8093(1)
s_{20}	18317(2)	10081(1)	3511.3(4)	511.35(5)
s_{21}	-2061.8(2)	-1350.7(1)	-595.79(7)	-117.08(1)
s_{22}	202.75(7)	133.19(4)	59.25(2)	12.149(2)
s_{30}	-96390(10)	-50300(5)	-16185(2)	-2097.2(2)
s_{31}	3784.8(4)	2336.0(3)	925.6(1)	150.26(2)
s_{40}	213470(20)	106940(10)	32572(3)	3960.4(4)

- Errors stem from numerical integration over loop momenta

Results for the Axial Vector Operator

$$
\begin{aligned}
Z_{AV}^{\text{singlet}}(\bar{\mu}a_{\text{L}}) - Z_{AV}^{\text{non-singlet}}(\bar{\mu}a_{\text{L}}) \\
= -\frac{g_0^4}{(4\pi)^4} c_F N_f \Big[\; & \big(a_{00} + a_{01}\, \mathbf{c}_{SW} + a_{02}\, \mathbf{c}_{SW}^2 + a_{03}\, \mathbf{c}_{SW}^3 + a_{04}\, \mathbf{c}_{SW}^4\big) \\
& + \big(a_{10} + a_{11}\, \mathbf{c}_{SW} + a_{12}\, \mathbf{c}_{SW}^2 + a_{13}\, \mathbf{c}_{SW}^3\big)\, \omega \\
& + \big(a_{20} + a_{21}\, \mathbf{c}_{SW} + a_{22}\, \mathbf{c}_{SW}^2\big)\, \omega^2 \\
& + \big(a_{30} + a_{31}\, \mathbf{c}_{SW}\big)\, \omega^3 \\
& + a_{40}\, \omega^4 + 6 \ln(\bar{\mu}^2 a_{\text{L}}^2) \Big] + \mathcal{O}\big(g_0^6\big)
\end{aligned}
$$

- Numerical constants $a_{i,j}$: Depend on Symanzik coefficients.
- Again: Computation done in general gauge, result gauge independent.
- For the axial vector operator: Scale dependence (related to axial anomaly).
- Presence of $\gamma_5\, q^\mu \not{q}/q^2$ term in Green's function
  $\Rightarrow$ In alternative RI' scheme, add $(g_0^4/(4\pi)^4)\, c_F\, N_f$ to the above result.

Results for the Axial Vector Operator

	Wilson	TL Symanzik	Iwasaki	DBW2
a_{00}	2.051(2)	3.098(3)	5.226(4)	9.75(1)
a_{01}	-15.033(3)	-12.851(3)	-9.426(3)	-5.000(3)
a_{02}	-5.013(2)	-3.361(1)	-1.3526(7)	0.1040(2)
a_{03}	2.1103(3)	1.7260(1)	1.1251(2)	0.42834(6)
a_{04}	0.0434(2)	0.01636(1)	-0.01074(5)	-0.0171(2)
a_{10}	43.75(1)	36.66(1)	25.827(9)	12.576(8)
a_{11}	76.993(3)	57.190(3)	31.768(2)	9.550(2)
a_{12}	44.260(4)	29.363(2)	12.962(1)	2.3370(6)
a_{13}	-4.4660(6)	-3.3740(5)	-1.8710(2)	-0.50863(4)
a_{20}	-126.45(1)	-92.853(7)	-50.378(1)	-14.391(1)
a_{21}	-259.59(3)	-175.65(2)	-81.45(1)	-17.159(5)
a_{22}	-107.48(1)	-67.737(8)	-27.500(3)	-4.5270(5)
a_{30}	295.76(3)	198.78(2)	90.96(1)	18.6645(6)
a_{31}	400.05(5)	253.87(3)	104.74(1)	17.954(2)
a_{40}	-348.41(4)	-220.12(3)	-90.11(1)	-15.236(2)

Plots

- $Z_\Gamma^{\rm singlet}(\bar{\mu}a_{\rm L}) - Z_\Gamma^{\rm non-singlet}(\bar{\mu}a_{\rm L}) = -\frac{g_0^4}{(4\pi)^4} c_F\, N_f \cdot \zeta$

- ζ depends on:
  - c_{SW}
  - ω
  - Symanzik coefficients
  - $\bar{\mu}a_{\rm L}$ (axial vector case)

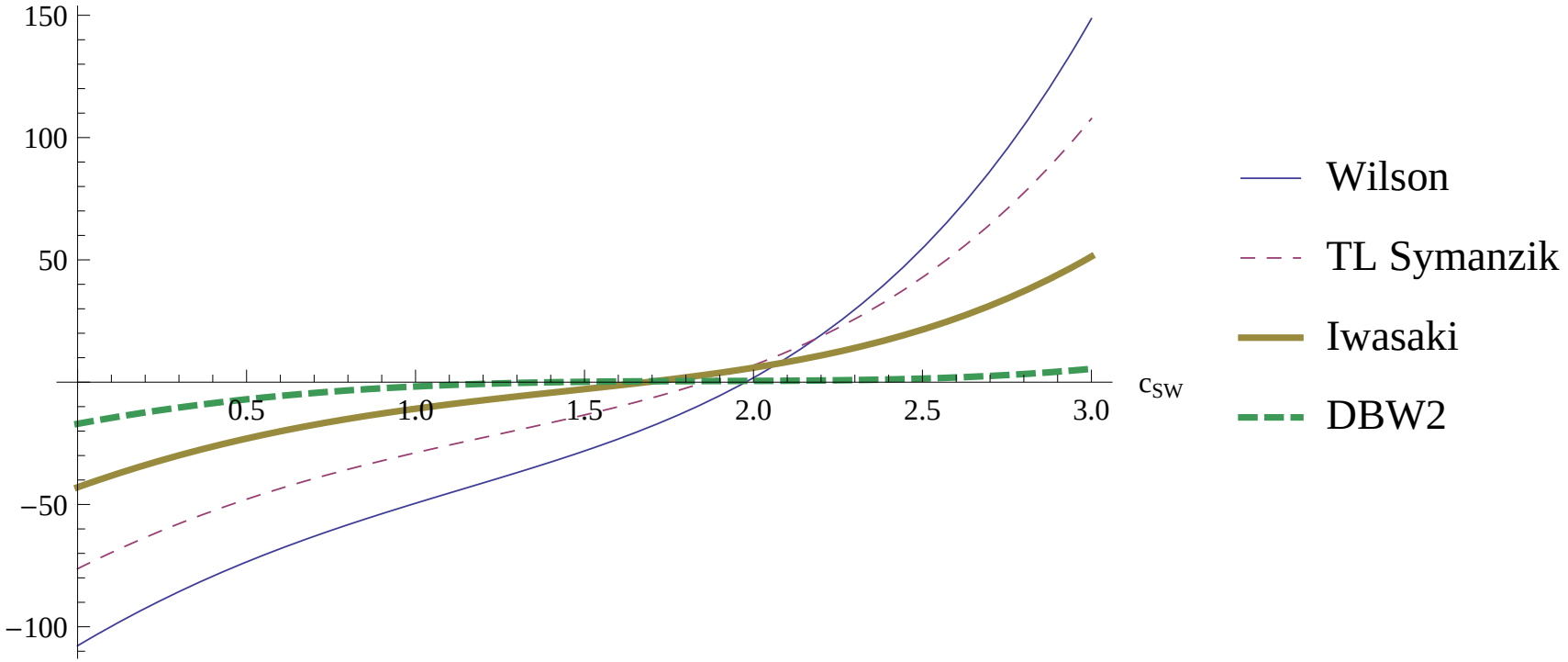
- Plot ζ at $\bar{\mu}a_{\rm L} = 1$, vs. $\mathbf{c}_{SW}$ or vs. ω,

  for specific values of Symanzik coefficients

- Plots for Scalar and Axial Vector

Plots (Scalar)

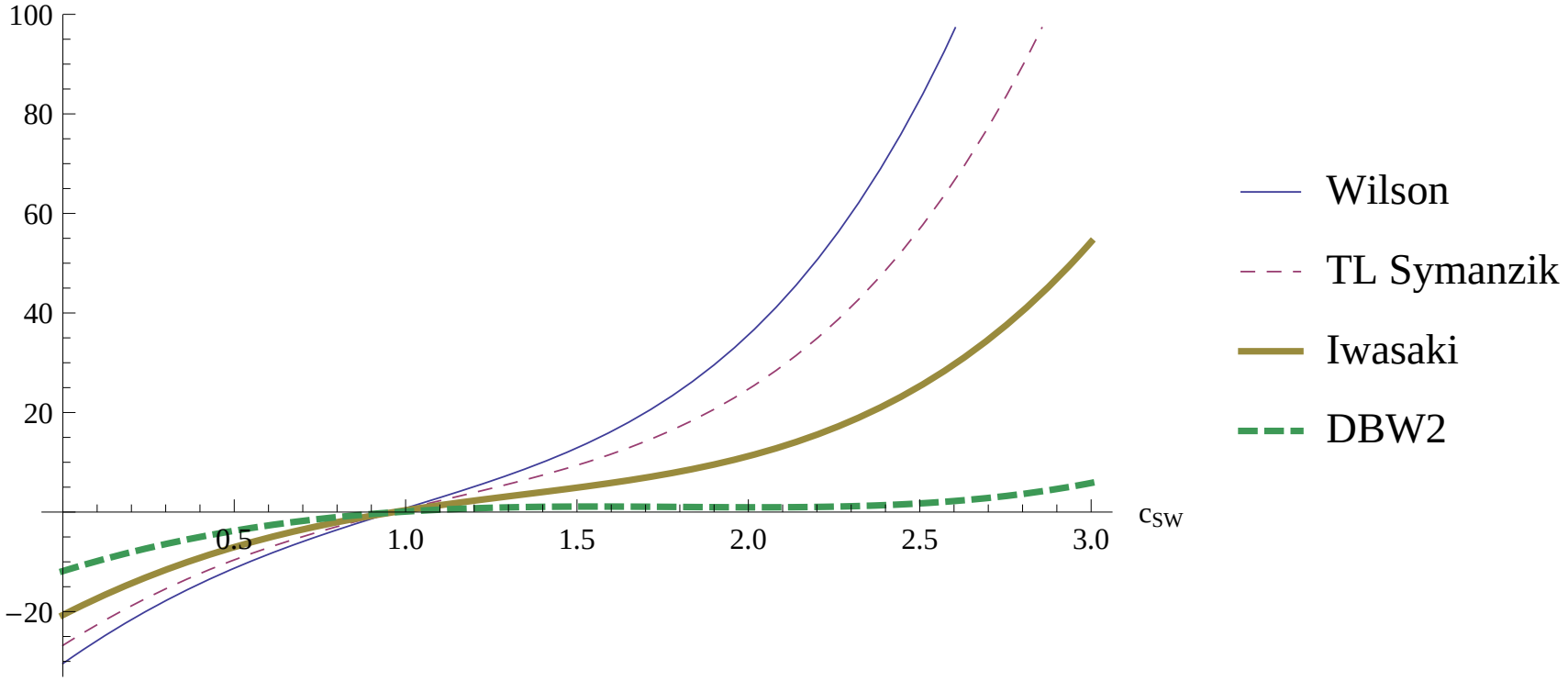
- ζ vs. $\mathbf{c}_{SW}$ $[\omega = 0]$



- $Z_\Gamma^{\text{singlet}}(\bar{\mu} a_{\text{L}}) - Z_\Gamma^{\text{non-singlet}}(\bar{\mu} a_{\text{L}}) = -\dfrac{g_0^4}{(4\pi)^4} c_F \, N_f \cdot \zeta$

Plots (Scalar)

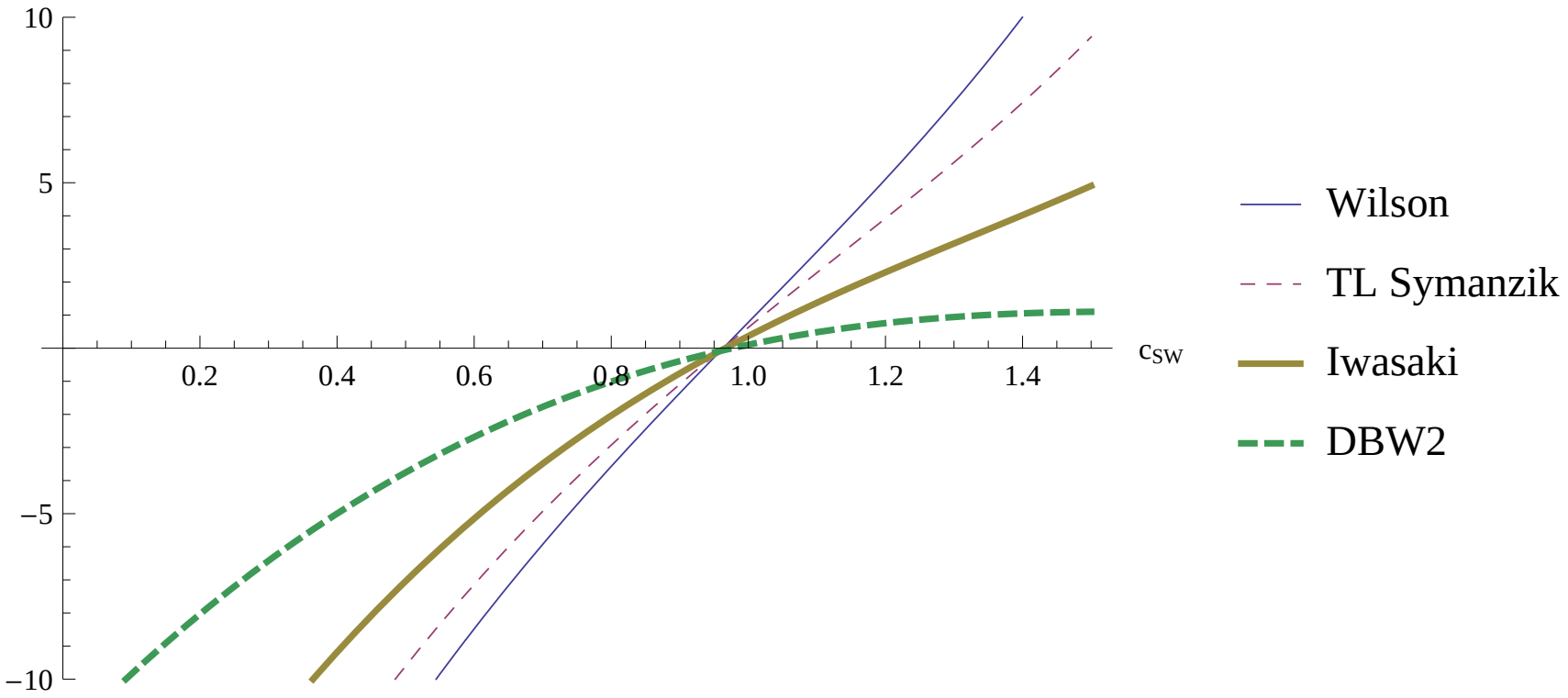
- ζ vs. $\mathbf{c}_{SW}$ $[\omega = 0.1]$



- $Z_\Gamma^{\text{singlet}}(\bar{\mu} a_{\text{L}}) - Z_\Gamma^{\text{non-singlet}}(\bar{\mu} a_{\text{L}}) = -\dfrac{g_0^4}{(4\pi)^4} c_F \, N_f \cdot \zeta$

Plots (Scalar)

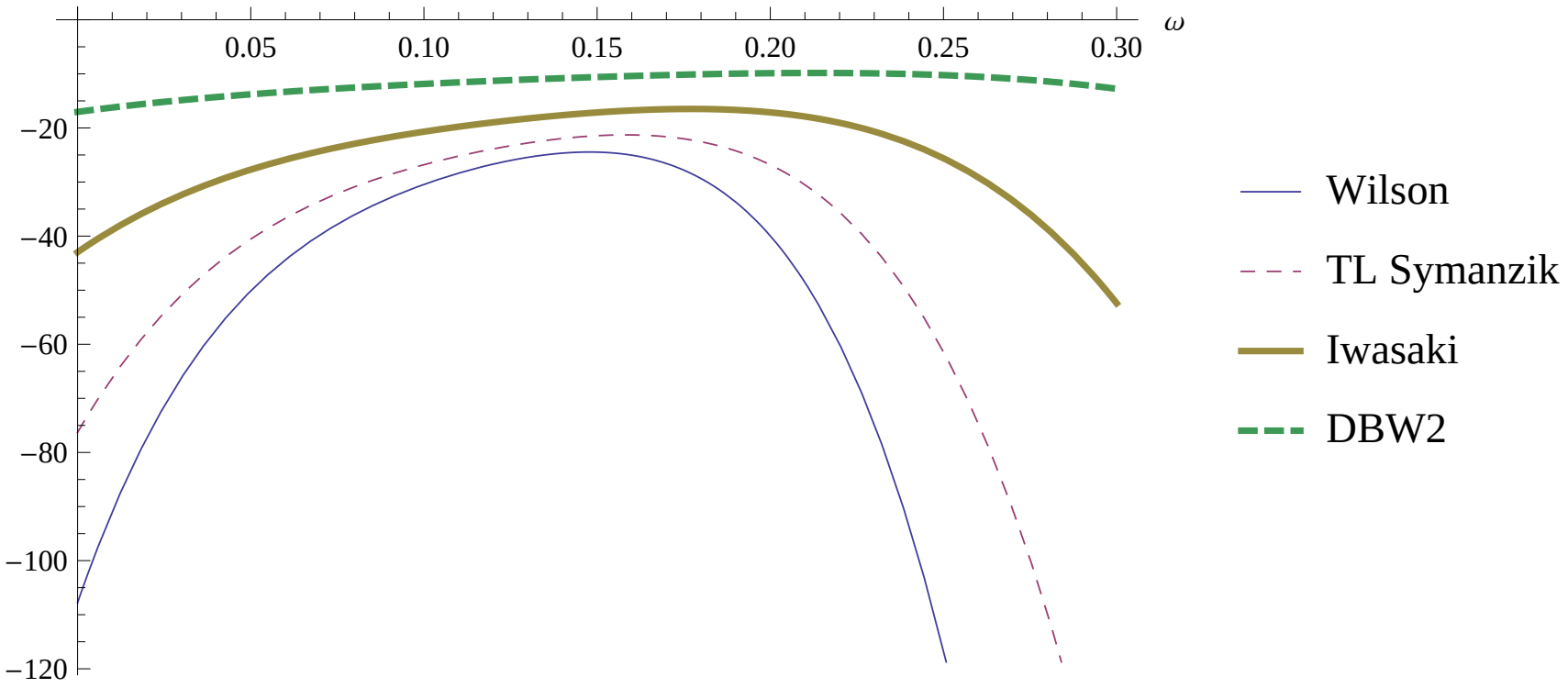
- ζ vs. $\mathbf{c}_{SW}$ [$\omega = 0.1$] Zoom-in



- $Z_\Gamma^{\text{singlet}}(\bar{\mu} a_{\text{L}}) - Z_\Gamma^{\text{non-singlet}}(\bar{\mu} a_{\text{L}}) = -\frac{g_0^4}{(4\pi)^4} c_F \, N_f \cdot \zeta$

Plots (Scalar)

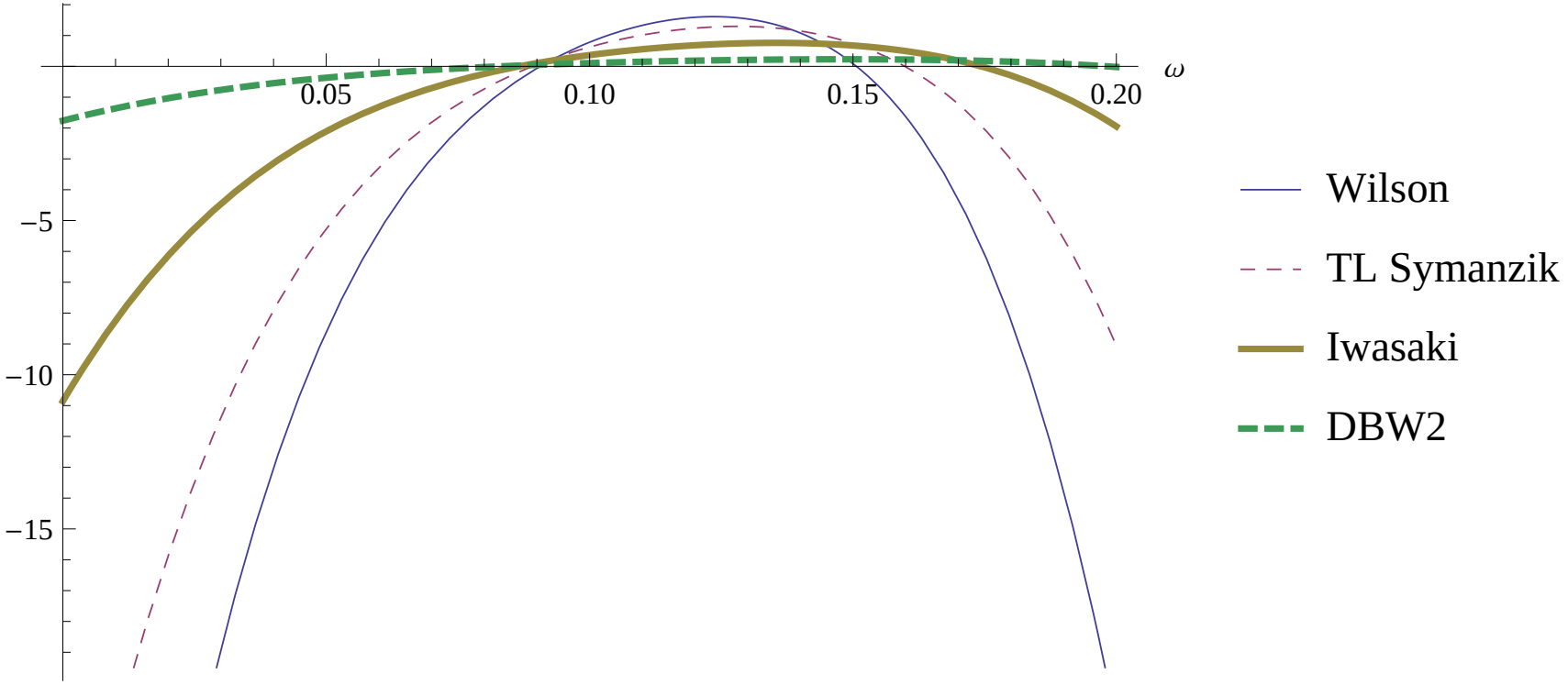
- ζ vs. ω [$\mathbf{c}_{SW} = 0$]



- $Z_\Gamma^{\text{singlet}}(\bar{\mu} a_{\text{L}}) - Z_\Gamma^{\text{non-singlet}}(\bar{\mu} a_{\text{L}}) = -\dfrac{g_0^4}{(4\pi)^4} c_F \, N_f \cdot \zeta$

Plots (Scalar)

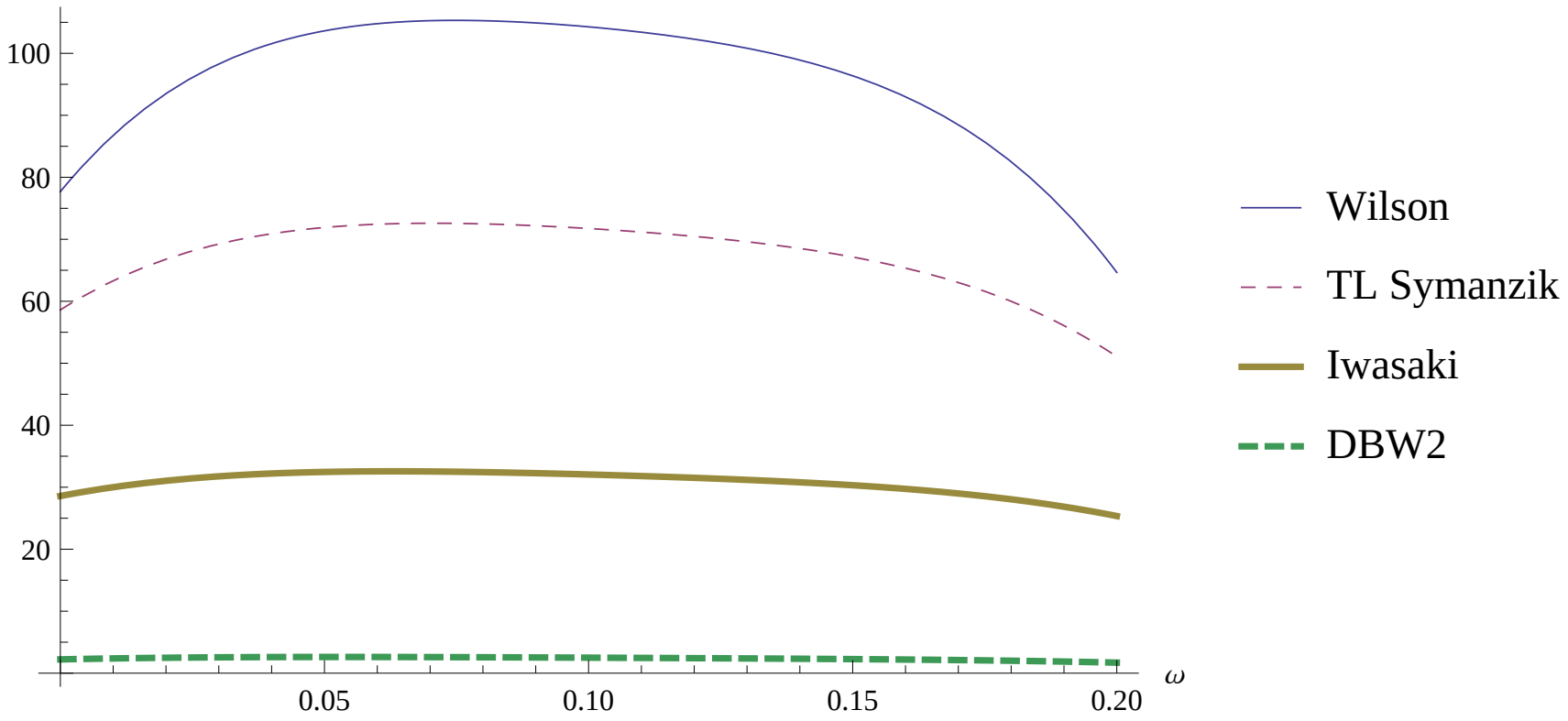
- ζ vs. ω [$\mathbf{c}_{SW} = 1.0$]



- $Z_\Gamma^{\text{singlet}}(\bar{\mu} a_{\text{L}}) - Z_\Gamma^{\text{non-singlet}}(\bar{\mu} a_{\text{L}}) = -\dfrac{g_0^4}{(4\pi)^4} c_F \, N_f \cdot \zeta$

Plots (Scalar)

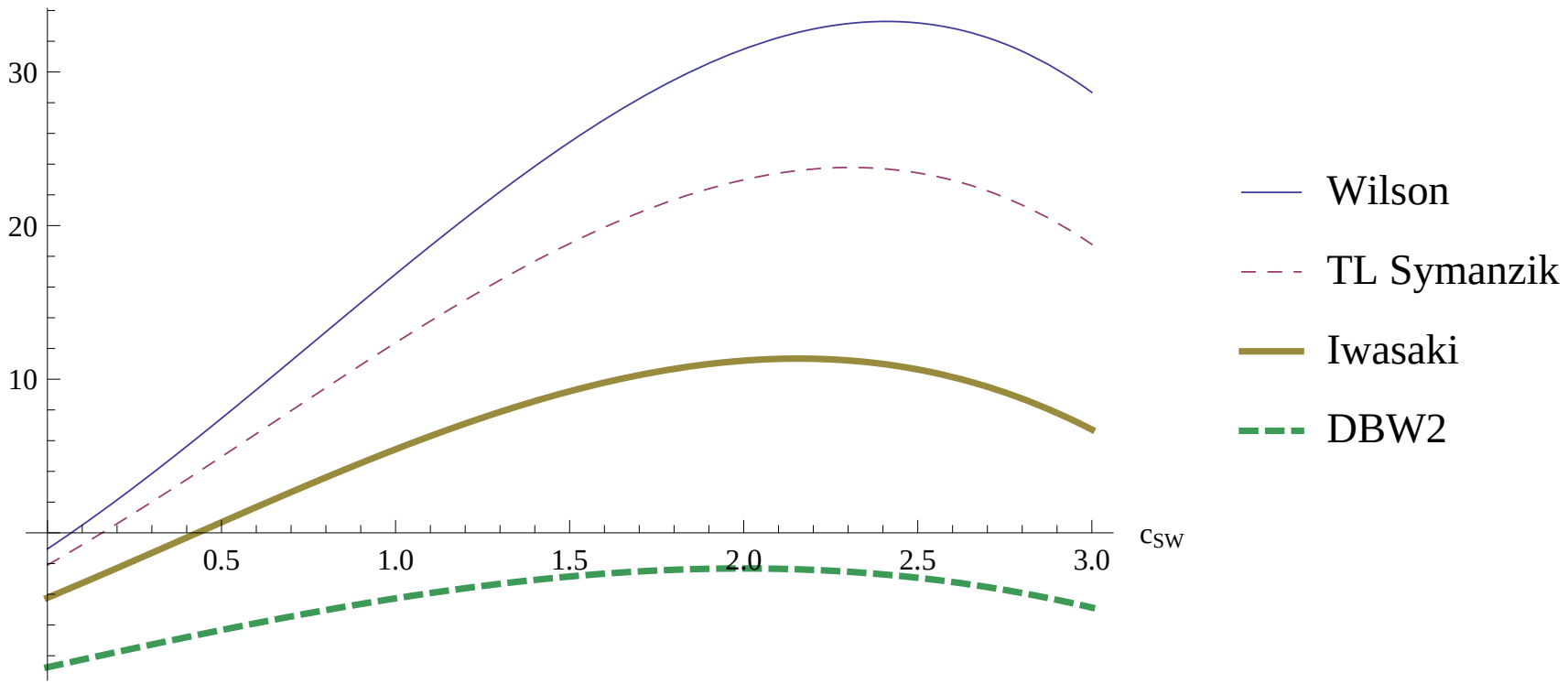
- ζ vs. ω [$\mathbf{c}_{SW} = 2.65$]



- $Z_\Gamma^{\text{singlet}}(\bar{\mu} a_{\text{L}}) - Z_\Gamma^{\text{non-singlet}}(\bar{\mu} a_{\text{L}}) = -\frac{g_0^4}{(4\pi)^4} c_F \, N_f \cdot \zeta$

Plots (Axial Vector)

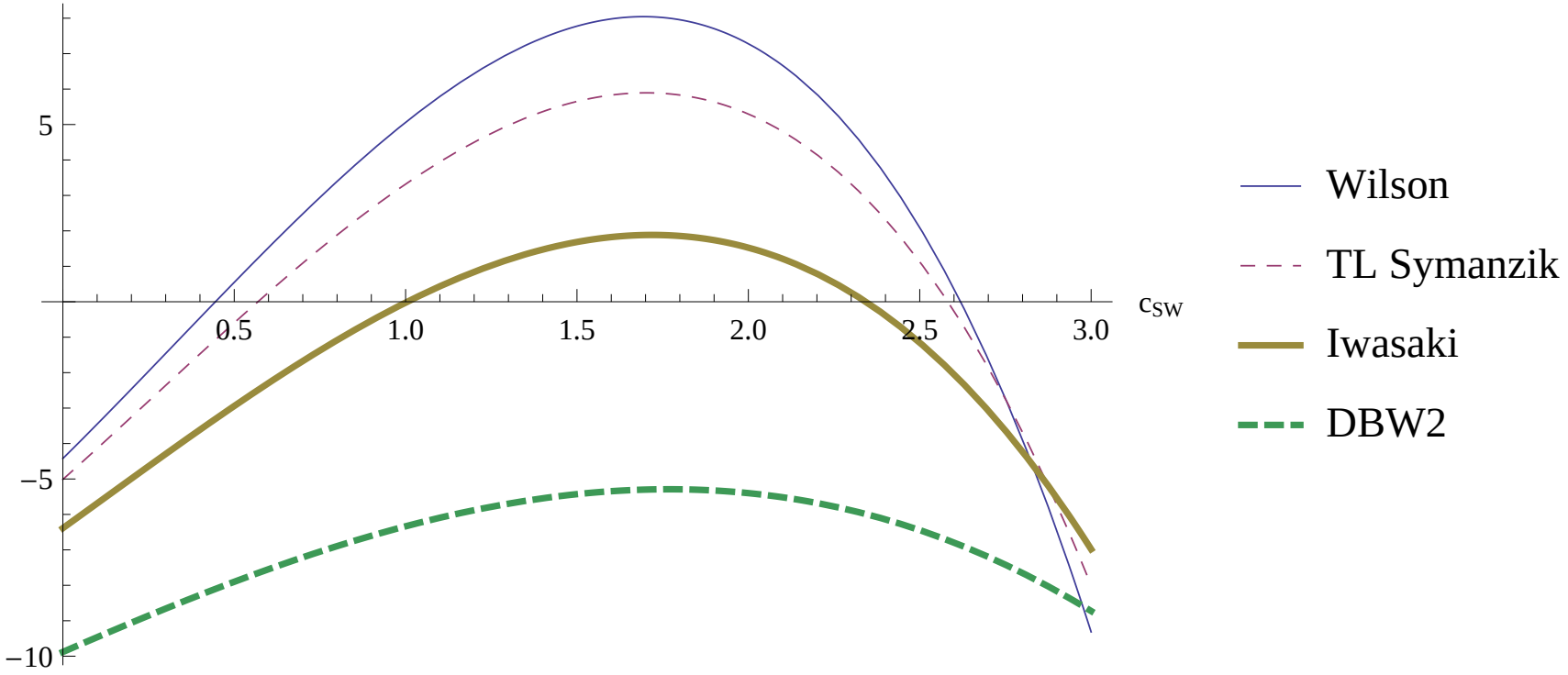
- ζ vs. $\mathbf{c}_{SW}$ $[\omega = 0]$



- $Z_\Gamma^{\text{singlet}}(\bar{\mu} a_{\text{L}}) - Z_\Gamma^{\text{non-singlet}}(\bar{\mu} a_{\text{L}}) = -\dfrac{g_0^4}{(4\pi)^4} c_F \, N_f \cdot \zeta$

Plots (Axial Vector)

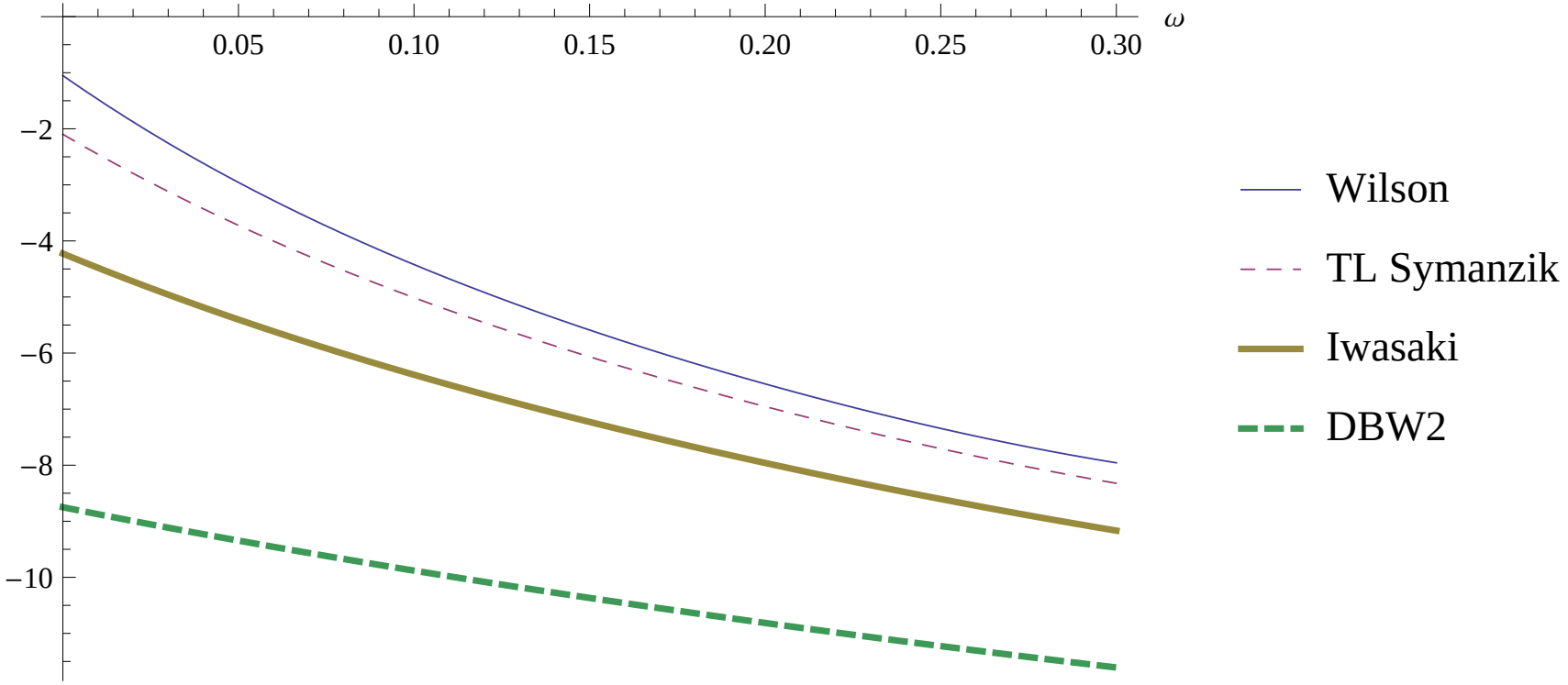
- ζ vs. $\mathbf{c}_{SW}$ $[\omega = 0.1]$



- $Z_\Gamma^{\text{singlet}}(\bar{\mu} a_{\text{L}}) - Z_\Gamma^{\text{non-singlet}}(\bar{\mu} a_{\text{L}}) = -\dfrac{g_0^4}{(4\pi)^4} c_F \, N_f \cdot \zeta$

Plots (Axial Vector)

- ζ vs. ω [$\mathbf{c}_{SW} = 0$]



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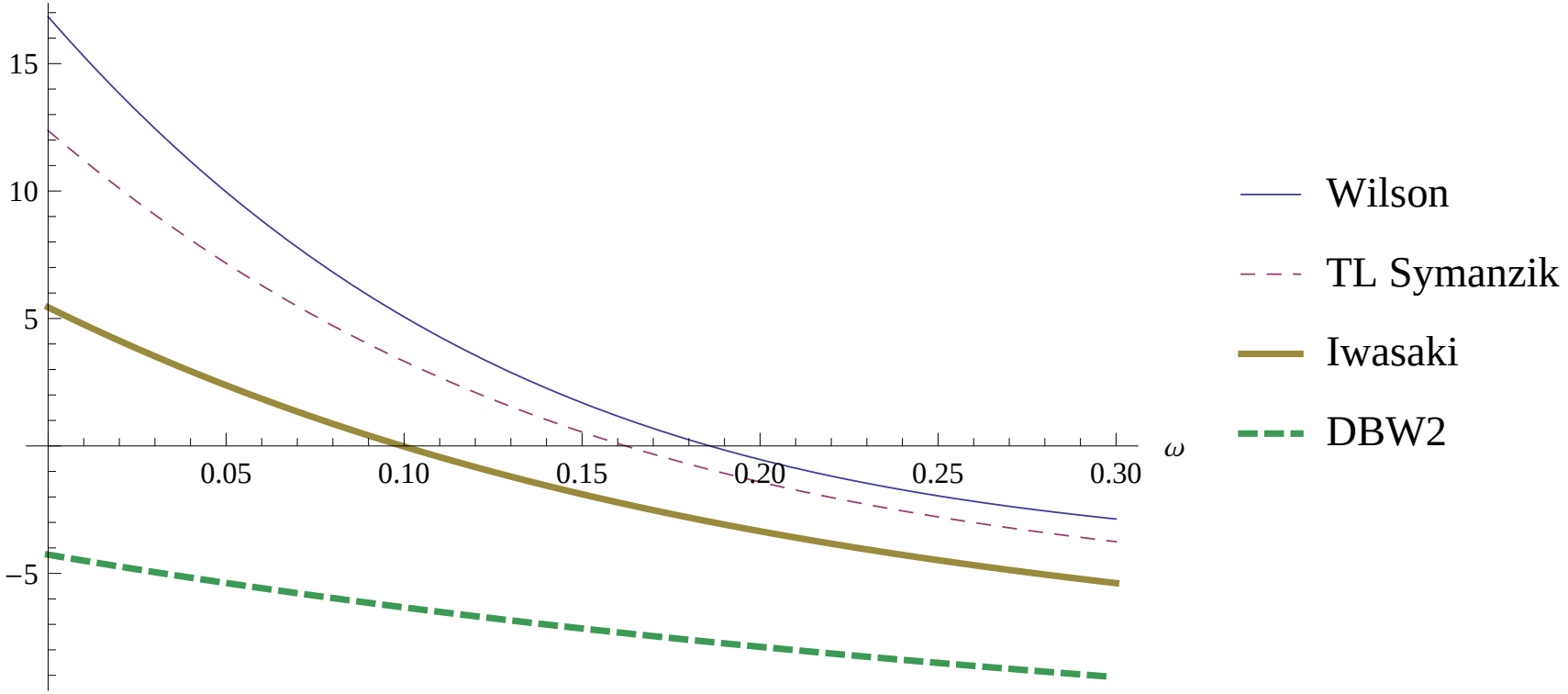
- ζ vs. ω [$\mathbf{c}_{SW} = 1.0$]



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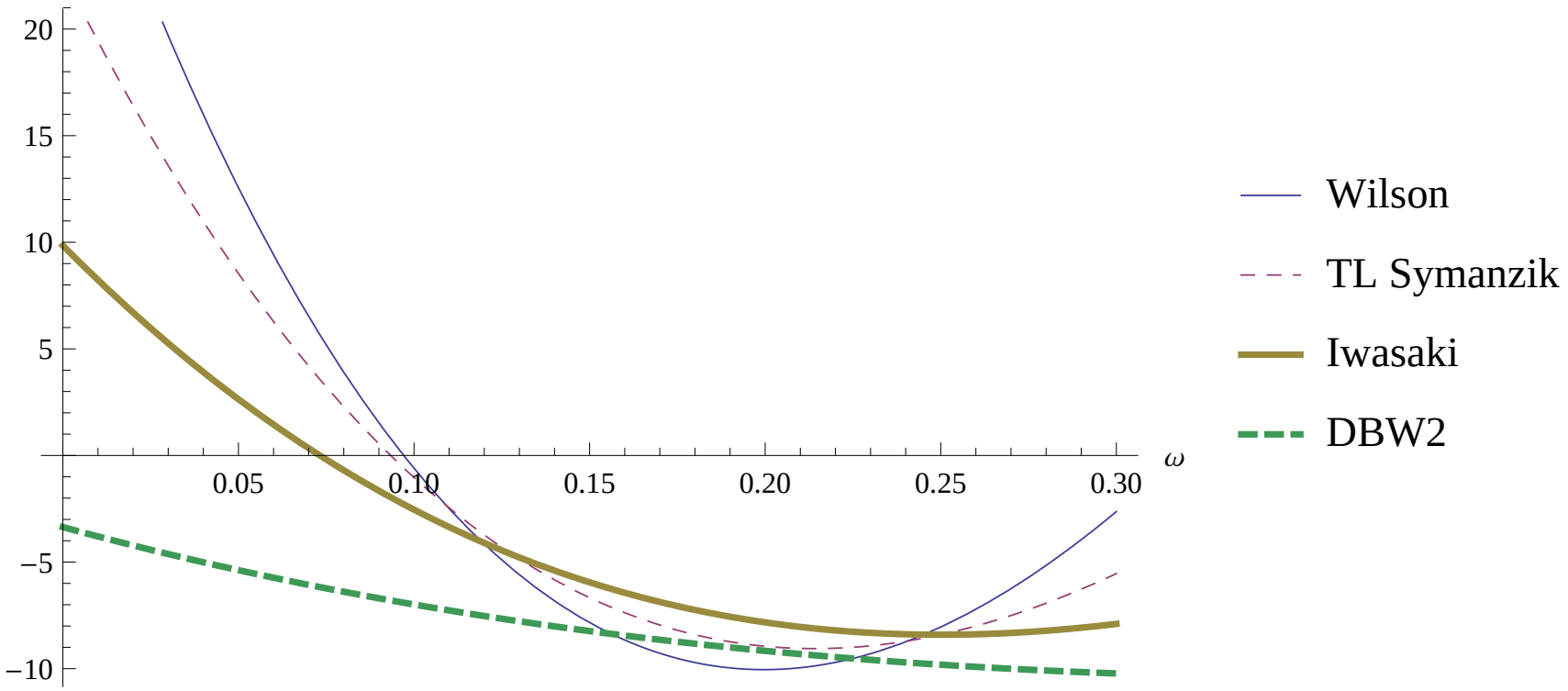
Plots (Axial Vector)

- ζ vs. ω [$\mathbf{c}_{SW} = 2.65$]



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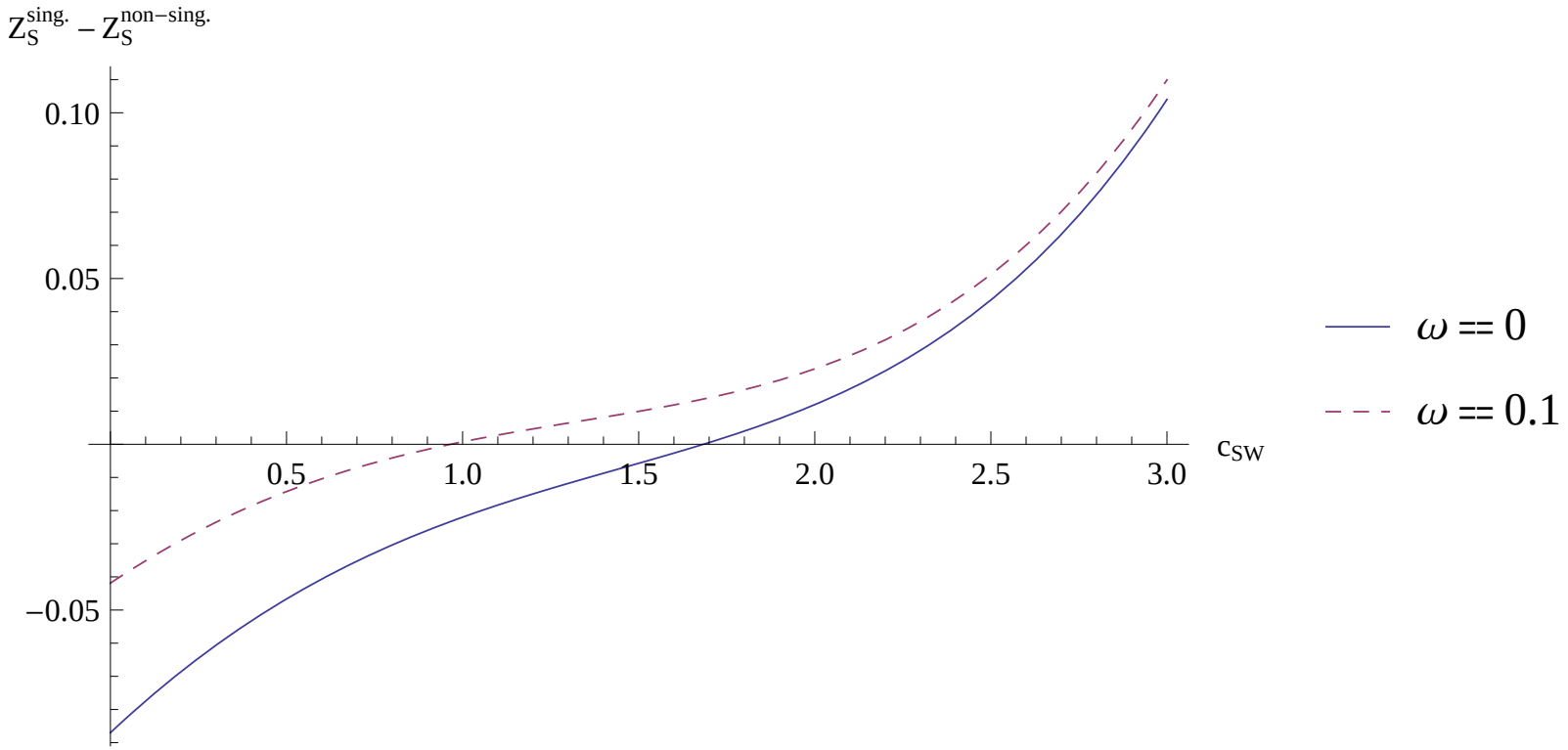
Plots with ETMC action (Scalar)

- Iwasaki gluon action, N_f = 4 (2+1+1), $\beta = 2N_c/g^2 = 1.95$



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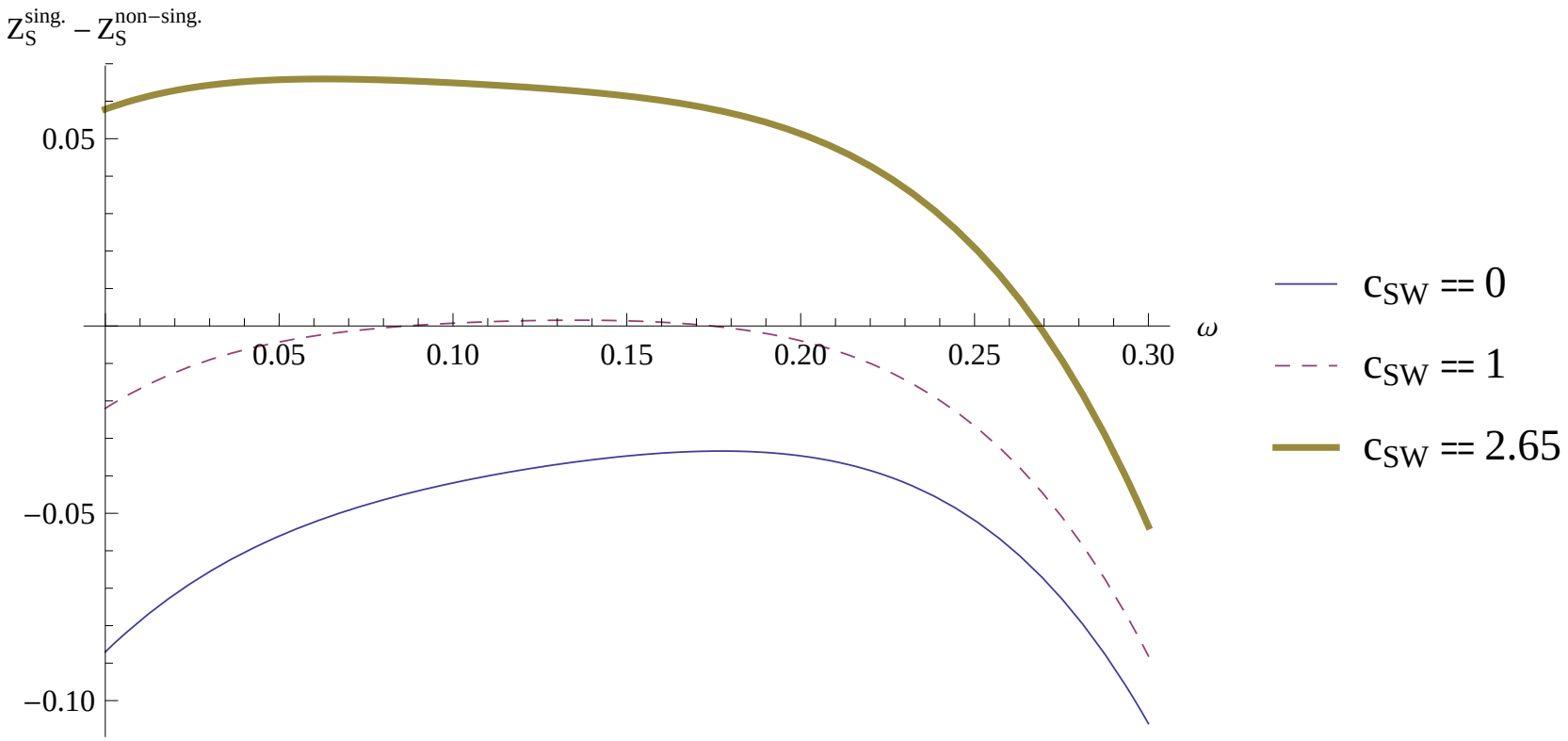

Plots with ETMC action (Axial Vector)

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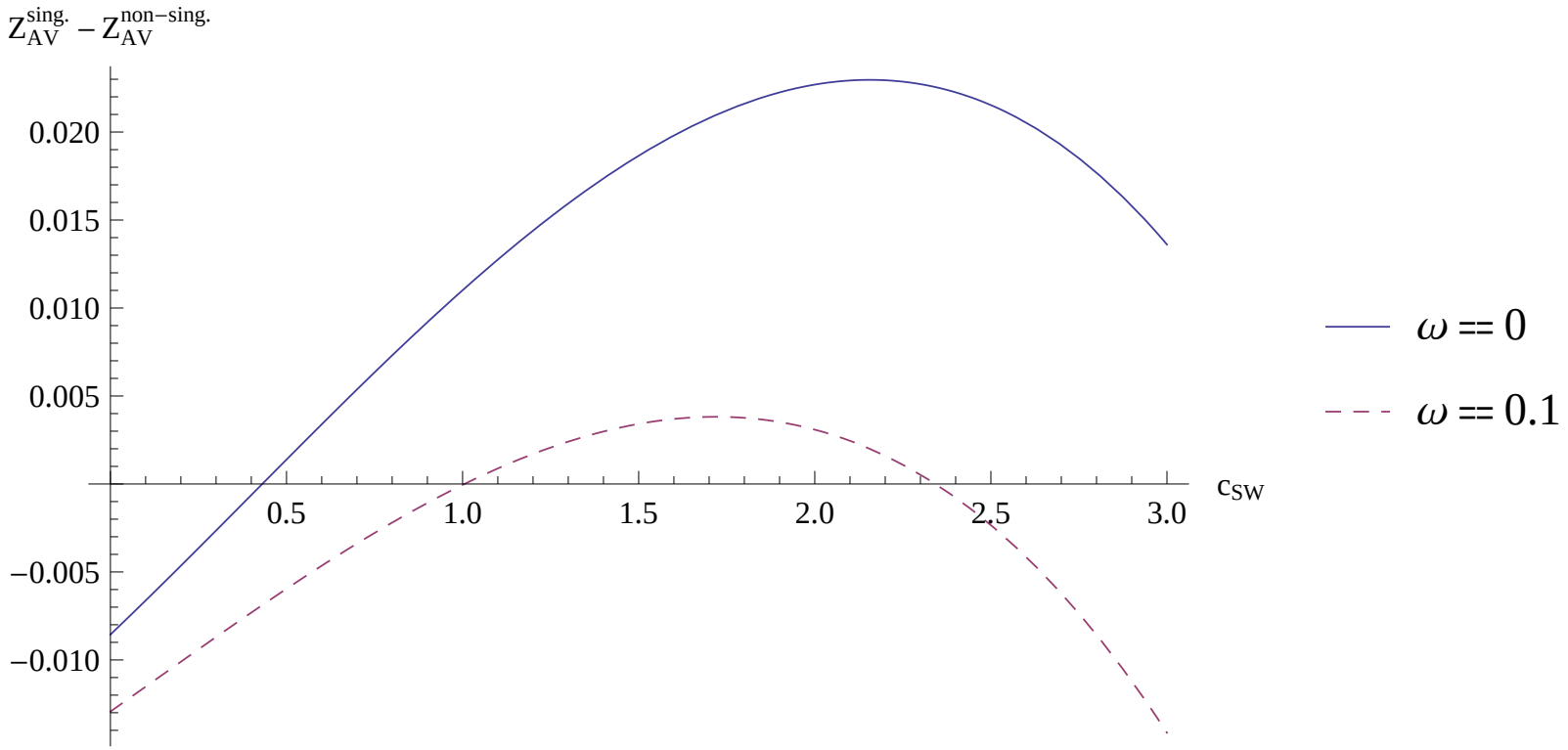

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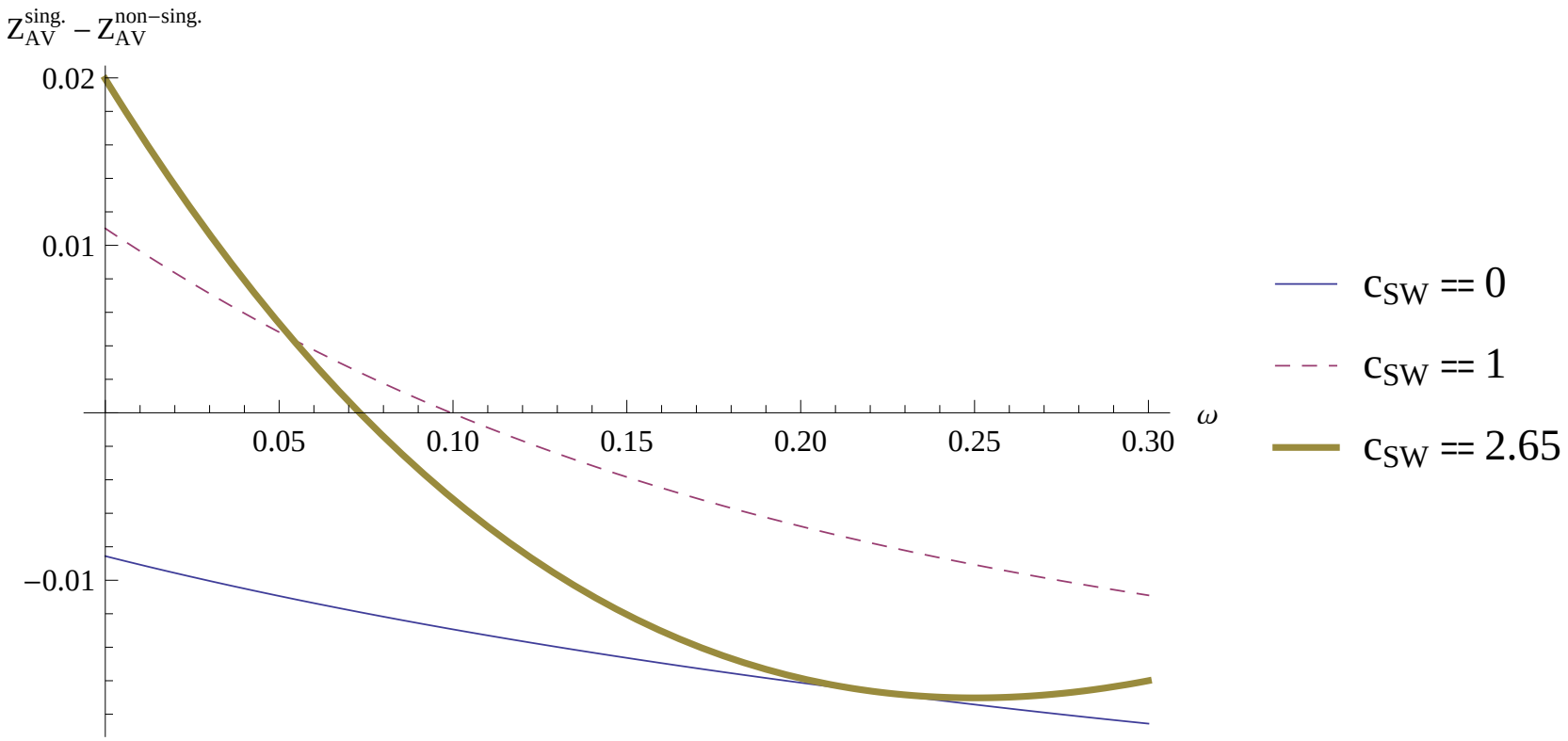
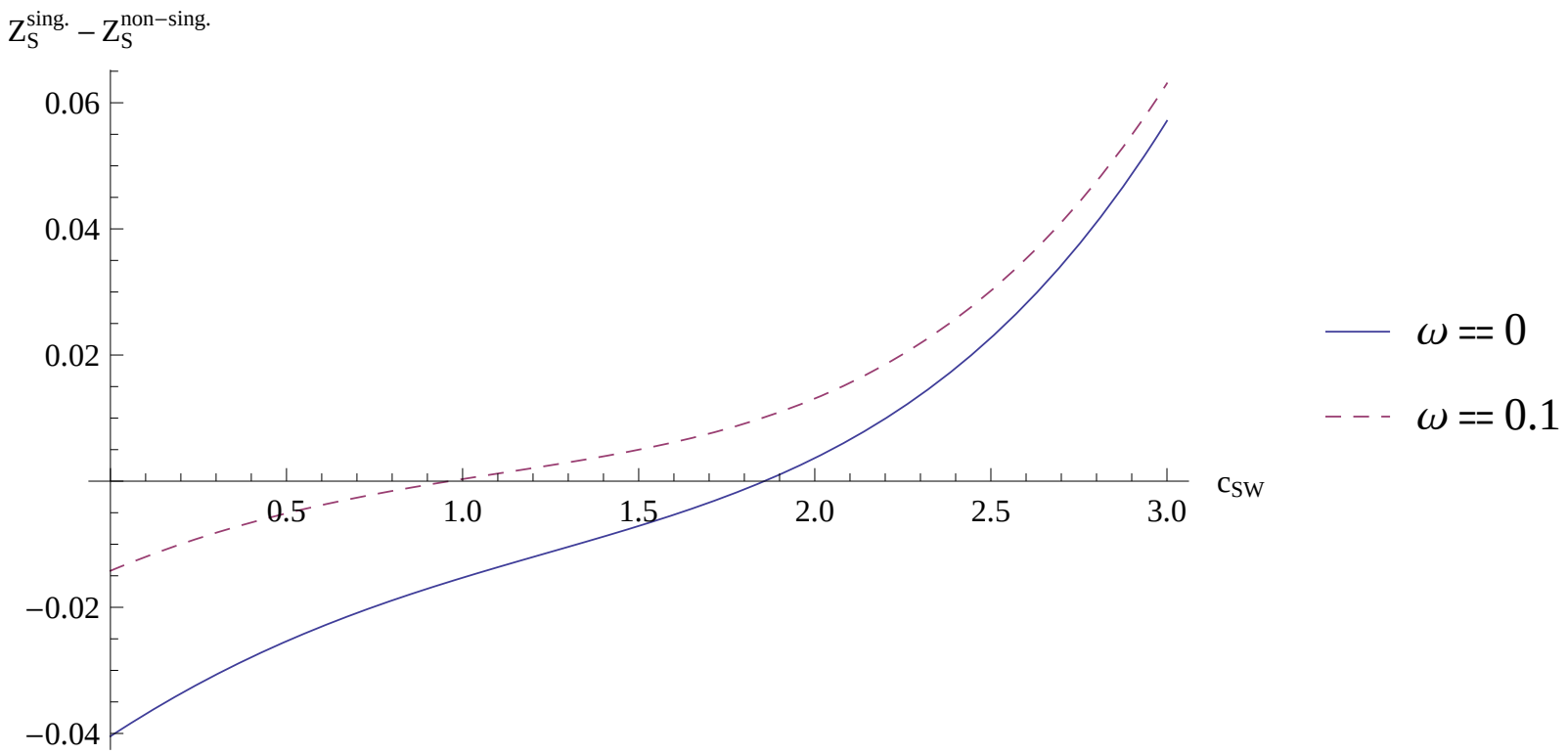


Plots with SLiNC action (Scalar)

- Tree-level Symanzik gluon action, N_f = 3, $\beta = 2N_c \cdot c_0/g^2 = 5.5$



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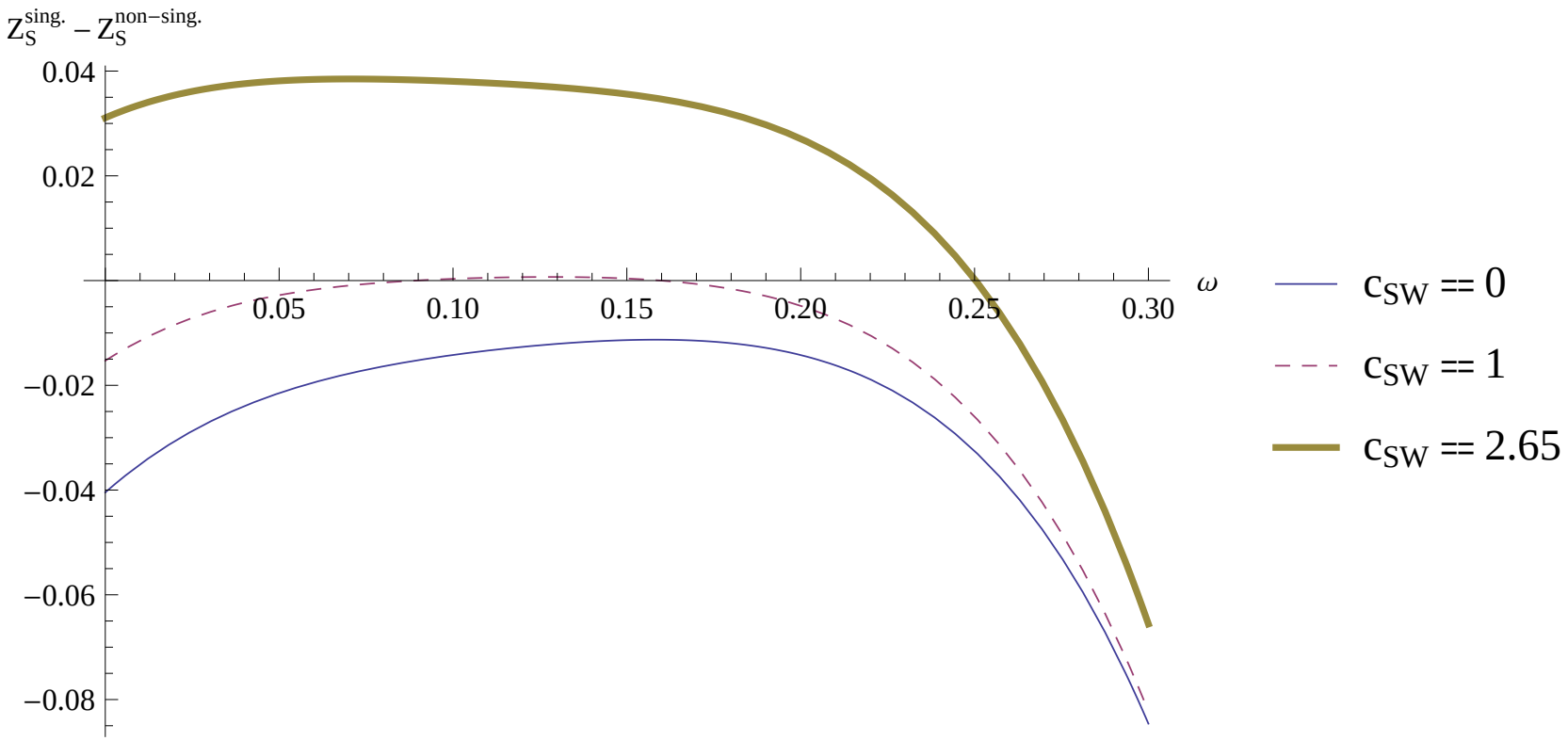

Plots with SLiNC action (Axial Vector)

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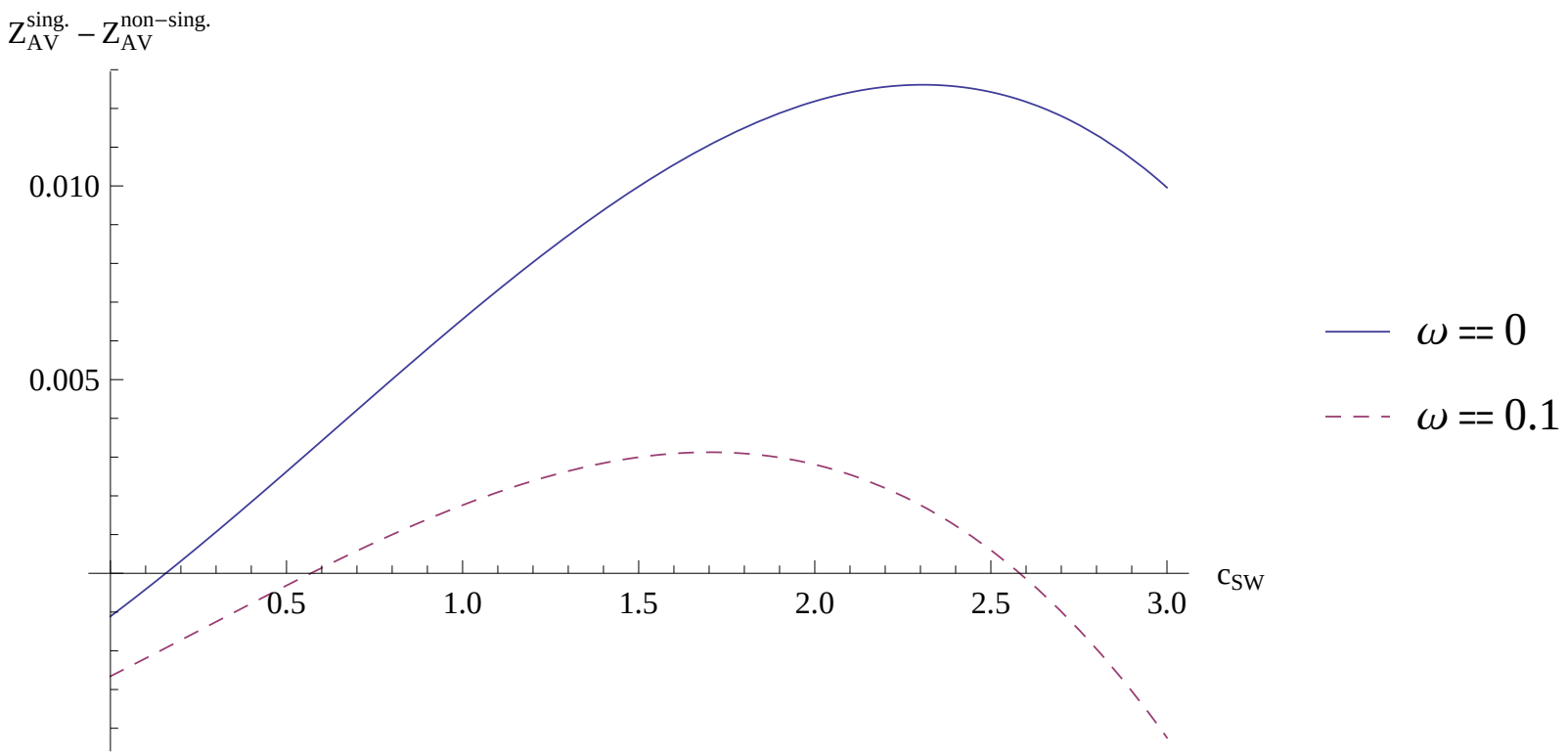

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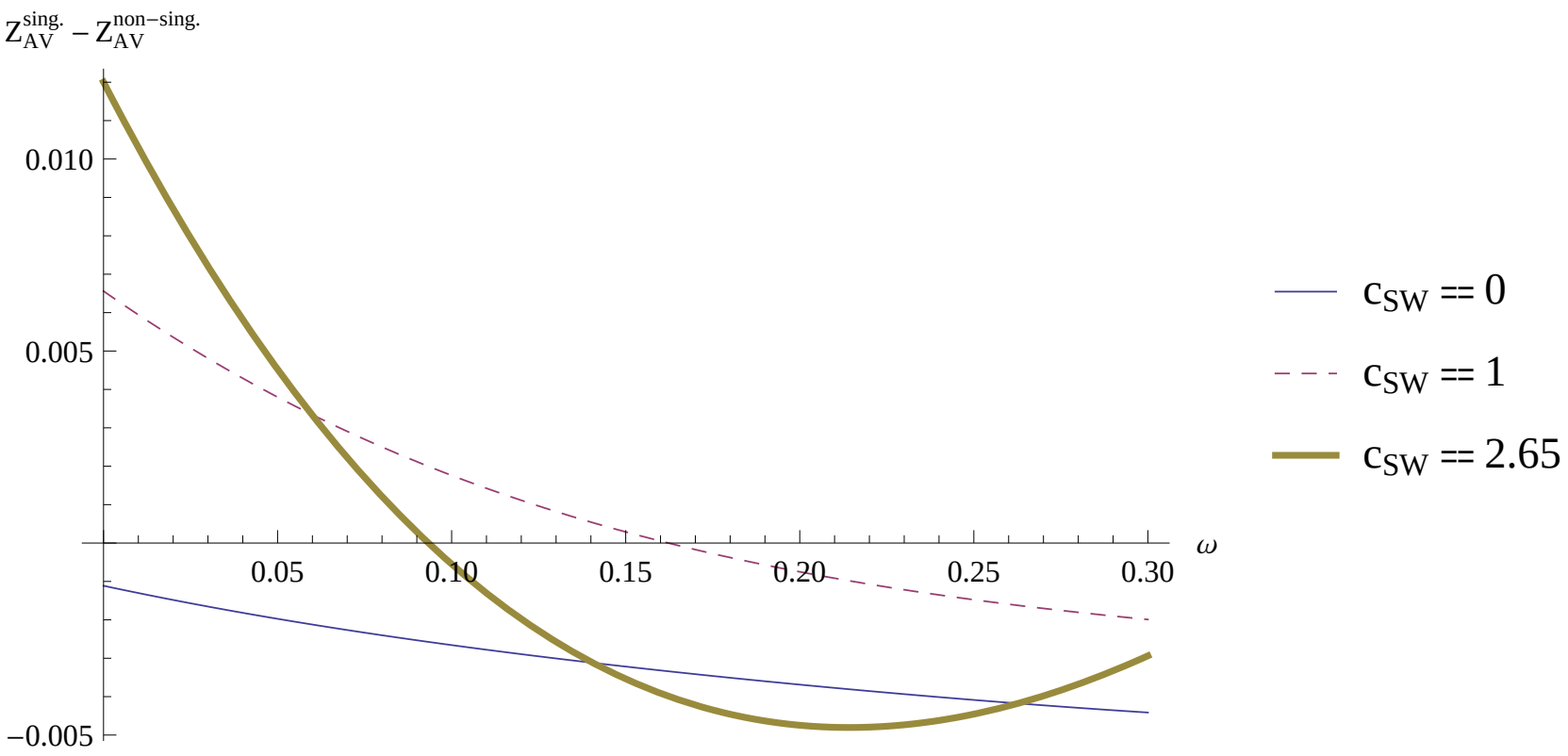

Future Extensions

- Compute $Z_\Gamma^{\rm singlet}(\bar{\mu} a_{\rm L})$ and $Z_\Gamma^{\rm non-singlet}(\bar{\mu} a_{\rm L})$ individually
  - Compare with non-perturbative results for $Z_\Gamma^{\rm non-singlet}(\bar{\mu} a_{\rm L})$
- Extend to different actions, e.g. with more steps of stout smearing
  - Additional contributions: more convergent $\Rightarrow$ More automatic treatment
  - Far more complicated vertices ($\sim 10^6$ terms for two smearing steps)
- Computation for several variants of staggered fermion actions
- Study of other operators:
  - Extended versions of $\bar{\psi}\Gamma\psi$:
    Lots of additional terms, but superficially convergent $\Rightarrow$ Easy!?
  - Other twist-2 operators used in GPDs ($\bar{\psi}\gamma^\mu D^\nu \psi$, etc.)
  - 4-fermion operators (e.g. in $\Delta S = 2$ transitions: $(\bar{s}\,\Gamma d)(\bar{s}\,\Gamma' d)$, etc.)

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THANK YOU