Renormalization of Flavor Singlet and Nonsinglet Fermion Bilinear Operators


Sphinx [Cyprus, 5th century B.C.]
Metropolitan Museum of Art, NY

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Overview

- Renormalization of fermion bilinears $\mathcal{O} = \bar{\psi} \Gamma \psi$ on the lattice, where $\Gamma = \hat{1}, \gamma_5, \gamma_\mu, \gamma_5 \gamma_\mu, \gamma_5 \sigma_{\mu \nu}$

- We consider both flavor singlet and nonsinglet operators ($\bar{\psi} \lambda^a \Gamma \psi$)

- Action: Symanzik gluons, Wilson/clover fermions, stout links

  [Includes twisted mass actions, SLiNC action]

- Calculation of the fermion self energy, up to two loops: $\Sigma^L_\psi(q, a_L)$

- Calculation of the 2-pt Green’s functions, up to two loops: $\Sigma^L_\Gamma(q, a_L)$

- Calculation of the lattice two-loop renormalization functions:

  $Z^L,Y^L, Z^L,Y^\Gamma (Y : RI' and \overline{MS} schemes)$

  - Results computed in an arbitrary covariant gauge
  - Generalization to fermionic fields in an arbitrary representation

- By-product: Quark mass multiplicative renormalization, $Z^L,m_{RI'}$

- Prerequisites: Calculation of 1-loop renormalization functions:

  [For Gluons $A$, Ghosts $c$, Gauge parameter $\alpha$, Coupling constant $g$]

  $Z^L,A_{RI'}$, $Z^L,c_{RI'}$, $Z^L,\alpha_{RI'}$, $Z^L,g_{RI'}$
Motivation


- Non-perturbative estimates: preferred (but feasible? precise? often...). In all cases, a comparison with perturbation theory is desirable.


- $Z_m$ : Essential ingredient in computation of quark masses.

- Prototype for the renormalization of operators such as:
  - $\bar{\psi} \Gamma D^\mu \psi$ (appearing in Hadron Structure Functions)
  - $(\bar{s} \Gamma_1 d)(\bar{s} \Gamma_2 d)$ (appearing in $\Delta S = 2$ transitions, etc.)

- Generalize to other actions (Symanzik gluons, Stout links).

- Singlet vs. nonsinglet renormalization: Appears first at 2 loops. Non-perturbative determination is difficult. [QCDSF, talk by H. Perlt] Relevant for studies of $\eta – \eta'$ mesons, etc.
Lattice Action: Symanzik Gluons, Clover Fermions

- Complete discretized action: \( S = S_G + S_F^W + S_{SW} \), where:

\[
S_F^W = -\frac{a^4}{2} \sum_{n, \mu} \left[ \frac{1}{a} \overline{\Psi}(\vec{n}) \left( (r - \gamma_\mu) \tilde{U}_{\vec{n}, \vec{n}+\hat{\mu}} \Psi(\vec{n} + \hat{\mu}) \\
+ (r + \gamma_\mu) \tilde{U}^\dagger_{\vec{n}-\hat{\mu}, \vec{n}} \Psi(\vec{n} - \hat{\mu}) - \left( 2r + \frac{Ma}{2} \right) \Psi(\vec{n}) \right) \right]
\]

\( \tilde{U}_{\vec{n}, \vec{n}+\hat{\mu}} \): Stout links

\[
S_{SW} = -a^5 \sum_n c_{SW} \overline{\Psi}(\vec{n}) \frac{1}{4} \sigma_{\mu\nu} \hat{G}_{\mu\nu}(\vec{n}) \Psi(\vec{n})
\]

and: \( \hat{G}_{\mu\nu}(\vec{n}) = \frac{1}{8a^2} \left[ Q_{\mu\nu}(\vec{n}) - Q_{\nu\mu}(\vec{n}) \right], \)

\[
Q_{\mu\nu} = U_{\vec{n}, \vec{n}+\hat{\mu}} U_{\vec{n}+\hat{\mu}, \vec{n}+\hat{\nu}} U_{\vec{n}+\hat{\nu}, \vec{n}} + U_{\vec{n}, \vec{n}+\hat{\nu}} U_{\vec{n}+\hat{\nu}, \vec{n}+\hat{\mu}} U_{\vec{n}+\hat{\mu}, \vec{n}} + U_{\vec{n}, \vec{n}+\hat{\mu}} U_{\vec{n}+\hat{\mu}, \vec{n}+\hat{\nu}} U_{\vec{n}+\hat{\nu}, \vec{n}} + U_{\vec{n}, \vec{n}-\hat{\mu}} U_{\vec{n}+\hat{\mu}, \vec{n}-\hat{\nu}} U_{\vec{n}-\hat{\nu}, \vec{n}} + U_{\vec{n}, \vec{n}-\hat{\nu}} U_{\vec{n}+\hat{\nu}, \vec{n}} + U_{\vec{n}+\hat{\mu}, \vec{n}+\hat{\nu}} U_{\vec{n}+\hat{\nu}, \vec{n}+\hat{\mu}} \]

\( c_{SW} \): Free parameter
\[ \tilde{U}_{\vec{n}, \vec{n} + \hat{\mu}} = e^{iQ_{\hat{\mu}}(\vec{n})} U_{\vec{n}, \vec{n} + \hat{\mu}} \]

\[ Q_{\hat{\mu}}(\vec{n}) = \frac{\omega}{2i} \left[ V_{\hat{\mu}}(\vec{n}) U_{\vec{n}, \vec{n} + \hat{\mu}}^\dagger - U_{\vec{n}, \vec{n} + \hat{\mu}} V_{\hat{\mu}}(\vec{n})^\dagger \right. \]

\[ \left. - \frac{1}{N_c} \text{Tr} \left( V_{\hat{\mu}}(\vec{n}) U_{\vec{n}, \vec{n} + \hat{\mu}}^\dagger - U_{\vec{n}, \vec{n} + \hat{\mu}} V_{\hat{\mu}}(\vec{n})^\dagger \right) \right] \]

\[ V_{\hat{\mu}}(\vec{n}) = \sum_{\hat{\nu} \neq \hat{\mu}} (U_{\vec{n}, \vec{n} + \hat{\nu}} U_{\vec{n} + \hat{\nu}, \vec{n} + \hat{\mu} + \hat{\nu}} U_{\vec{n} + \hat{\mu} + \hat{\nu}, \vec{n} + \hat{\mu}} + U_{\vec{n}, \vec{n} - \hat{\nu}} U_{\vec{n} - \hat{\nu}, \vec{n} + \hat{\mu} - \hat{\nu}} U_{\vec{n} + \hat{\mu} - \hat{\nu}, \vec{n} + \hat{\mu}}) \]

\[ \omega : \text{Free parameter} \]
**Gluon Action**

\[
S_G = \frac{2}{g^2} \left[ c_0 \sum_{\text{Plaq}} \text{Re Tr} (1 - U_{\text{Plaq}}) + c_1 \sum_{\text{Rect}} \text{Re Tr} (1 - U_{\text{Rect}}) + c_2 \sum_{\text{Chair}} \text{Re Tr} (1 - U_{\text{Chair}}) + c_3 \sum_{\text{Parall}} \text{Re Tr} (1 - U_{\text{Parall}}) \right]
\]

where: \( c_0 + 8c_1 + 16c_2 + 8c_3 = 1 \)

<table>
<thead>
<tr>
<th>Action</th>
<th>( c_0 )</th>
<th>( c_1 )</th>
<th>( c_2 )</th>
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Common sets of values for Symanzik coefficients
Preliminaries

- Denoting all bare quantities in the Lagrangian with the subscript “o”, the corresponding renormalized quantities read:

\[ A_\mu^o = \sqrt{Z_A} A_\mu, \quad c_\alpha^o = \sqrt{Z_c} c_\alpha, \quad \psi^o = \sqrt{Z_\psi} \psi \]

\[ g^o = \mu^e Z_g g, \quad \alpha^o = Z_\alpha^{-1} Z_A \alpha \]

where \( \mu \) is the mass scale introduced to ensure the coupling constant has the correct dimensionality in \( d \) dimensions and \( d = 4 - 2\epsilon \).

- The \(RI'\) renormalization scheme is defined by imposing renormalization conditions on matrix elements at a scale \( \bar{\mu} \), where (just as in \( MS\)):

\[ \bar{\mu} = \mu (4\pi/e^{\gamma_E})^{1/2} \]

- Mass-independent renormalization scheme \( \implies \) All renormalization functions are calculable at vanishing renormalized mass:

\[ m^o \to m_{cr} = m_1 g^2_\circ + \mathcal{O}(g^4_\circ) \]

- Free parameters:
  - Coupling constant \( g_0 \), number of flavors \( N_f \), number of colors \( N_c \), clover coefficient \( c_{SW} \), stout coefficient \( \omega \), gauge parameter \( \alpha \).

- Numerical results for specific values of Symanzik parameters \( c_i \).
Definition of 2-point Green’s Functions

• Fermion and ghost self energy in Euclidean space:
  \[ \Sigma_\psi(q, a_L) = iq + m_\circ + \mathcal{O}(g_\circ^2) \quad , \quad \Sigma_c(q, a_L) = q^2 + \mathcal{O}(g_\circ^2) \]

• Gluon propagator with radiative corrections:
  \[ G_{\mu\nu}^L(q, a_L) = \frac{1}{q^2} \left[ \frac{\delta_{\mu\nu} - q_\mu q_\nu / q^2}{\Pi_T(a_L q)} + \alpha_\circ \frac{q_\mu q_\nu / q^2}{\Pi_L(a_L q)} \right] \]
  \[ (\Pi_{T,L}(a_L q) = 1 + \mathcal{O}(g_\circ^2)) \]

• \( \mathcal{O}_\Gamma \), 2-point amputated Green’s functions:
  \[ \Sigma_S(q a_L) = \hat{1} \Sigma_S^{(1)}(q a_L) \]
  \[ \Sigma_P(q a_L) = \gamma_5 \Sigma_P^{(1)}(q a_L) \]
  \[ \Sigma_V(q a_L) = \gamma_\mu \Sigma_V^{(1)}(q a_L) + \frac{q_\mu q / q^2}{q^2} \Sigma_V^{(2)}(q a_L) \]
  \[ \Sigma_{AV}(q a_L) = \gamma_5 \gamma_\mu \Sigma_{AV}^{(1)}(q a_L) + \gamma_5 \frac{q_\mu q / q^2}{q^2} \Sigma_{AV}^{(2)}(q a_L) \]
  \[ \Sigma_T(q a_L) = \gamma_5 \sigma_{\mu\nu} \Sigma_T^{(1)}(q a_L) + \gamma_5 \frac{q_\mu q_\nu - q_\nu q_\mu}{q^2} \Sigma_T^{(2)}(q a_L) \]

\[ \left( \Sigma_T^{(1)}(q a_L) = 1 + \mathcal{O}(g_\circ^2) , \quad \Sigma_T^{(2)}(q a_L) = \mathcal{O}(g_\circ^2) \right) \]
The RI’ Renormalization Scheme

- Renormalization conditions for the fermion and ghost self-energy:
  \[
  \lim_{a_L \to 0} \left[ Z_{\psi}^{L,RI'}(a_L \bar{\mu}) \frac{\text{tr} \left( \Sigma_\psi(q, a_L) \psi \right)}{(4i q^2)} \right]_{q^2 = \bar{\mu}^2} = 1
  \]
  \[
  \lim_{a_L \to 0} \left[ Z_{c}^{L,RI'}(a_L \bar{\mu}) \frac{\Sigma_c(q, a_L)}{q^2} \right]_{q^2 = \bar{\mu}^2} = 1
  \]

- Renormalization conditions for \( Z_A \) and \( Z_\alpha \):
  \[
  \lim_{a_L \to 0} \left[ Z_{A}^{L,RI'}(a_L \bar{\mu}) \Pi_T(a_L q) \right]_{q^2 = \bar{\mu}^2} = \lim_{a_L \to 0} \left[ Z_{\alpha}^{L,RI'}(a_L \bar{\mu}) \Pi_L(a_L q) \right]_{q^2 = \bar{\mu}^2} = 1
  \]

- Renormalization conditions for \( Z_g \), on the lattice:
  \[
  \lim_{a_L \to 0} \left[ Z_{\psi}^{L,RI'} \left( Z_{A}^{L,RI'} \right)^{1/2} Z_{g}^{L,RI'} G_{A\bar{\psi}\psi}(q, a_L) \right]_{q^2 = \bar{\mu}^2} = G_{A\bar{\psi}\psi}^{\text{finite}}
  \]
  or, equivalently:
  \[
  \lim_{a_L \to 0} \left[ Z_{c}^{L,RI'} \left( Z_{A}^{L,RI'} \right)^{1/2} Z_{g}^{L,RI'} G_{A\bar{c}c}(q, a_L) \right]_{q^2 = \bar{\mu}^2} = G_{A\bar{c}c}^{\text{finite}}
  \]

where the gluon-fermion-antifermion (gluon-ghost-antighost) 1PI vertex function \( G_{A\bar{\psi}\psi}^{\text{finite}} \) (\( G_{A\bar{c}c}^{\text{finite}} \)) is required to be the same as the one stemming from the continuum (dimensional regularization, \( \overline{\text{MS}} \)).
The RI’ Renormalization Scheme

- We define renormalized operators by:
  \[ \mathcal{O}_{\Gamma}^{RI'} = Z_{\Gamma}^{L,RI'}(a_L \bar{\mu}) \mathcal{O}_\Gamma \quad (\mathcal{O}_\Gamma = \bar{\psi} \Gamma \psi) \]

- Renormalization conditions for \( \mathcal{O}_\Gamma \):
  \[ \lim_{a_L \to 0} \left[ Z_{\psi}^{L,RI'} Z_{\Gamma}^{L,RI'} \Sigma^{(1)}_{\Gamma} (qa_L) \right]_{q^2=\bar{\mu}^2} = 1 \]
  \( \Sigma^{(2)}_{\Gamma} \) is not included.

- Alternative scheme, more appropriate for nonperturbative renormalization:
  \[ \lim_{a_L \to 0} \left[ Z_{\psi}^{L,RI'} Z_{\Gamma}^{L,RI'} \frac{\text{tr}(\Gamma \Sigma_{\Gamma}(qa_L))}{\text{tr}(\Gamma \Gamma)} \right]_{q^2=\bar{\mu}^2} = 1 \]

- Difference between two schemes (for V, A, T):
  - Finite
  - Deducible from lower loop calculation (plus continuum results).
Conversion to the $\overline{MS}$ Scheme

- Conversion of the coupling constant and of the gauge parameter to $\overline{MS}$:
  \[ g_{RI'} = g_{\overline{MS}} \quad \text{and} \quad \alpha_{RI'} = \left( \frac{Z_{\overline{MS}}^L/Z_{RI'}^L}{Z_{\overline{MS}}^A / Z_{RI'}^A} \right) \alpha_{\overline{MS}} \]

- Ratios of $Z$'s must be regularization independent. Having computed the renormalization functions in $RI'$, we can construct their $\overline{MS}$ counterparts using conversion factors ($DR \equiv$ Dimensional Regularization):
  \[ C_i(a, \alpha) \equiv \frac{Z_{L,RI'}^i}{Z_{L,\overline{MS}}^i} = \frac{Z_{i}^{DR,RI'}}{Z_{i}^{DR,\overline{MS}}}, \quad \text{where } i : A, c, \psi, S, V, T. \]

- In the case of the scalar, vector and tensor operators, the renormalization functions, $Z_{\Gamma,\overline{MS}}^L$ can be obtained by:
  \[ Z_{\Gamma,\overline{MS}}^L = Z_{\Gamma,RI'}^L / C_{\Gamma}(g, \alpha) \]

- For pseudoscalar and axial vector: In order to satisfy Ward identities, extra finite factors $Z_5^P(g)$ and $Z_5^{AV}(g)$, calculable in DR, are required:
  \[ O_P = Z_5^P(g) Z_{P}^{DR,\overline{MS}} O_P^0, \quad O_{AV}^{\overline{MS}} = Z_5^{AV}(g) Z_{AV}^{DR,\overline{MS}} O_{AV}^0 \]
  \[ Z_{P}^{\overline{MS}} = Z_{P}^{L,RI'} / (C_S Z_5^P), \quad Z_{AV}^{\overline{MS}} = Z_{AV}^{L,RI'} / (C_V Z_5^{AV}) \]

- $Z_5$ differ for flavor singlet and nonsinglet.
Conversion Factors

\[ C_A(g, \alpha) = 1 + \frac{g^2}{36(16\pi^2)} \left[ (9\alpha^2 + 18\alpha + 97) N_c - 40N_f \right] \]

\[ C_c(g, \alpha) = 1 + \frac{g^2}{16\pi^2} N_c \]

\[ C_\psi(g, \alpha) = 1 - \frac{g^2}{16\pi^2} c_F \alpha + \frac{g^4}{8(16\pi^2)^2} c_F \left[ (8\alpha^2 + 5) c_F + 14 N_f \right. \\
\left. - (9\alpha^2 - 24\zeta(3) \alpha + 52\alpha - 24\zeta(3) + 82) N_c \right] \]

\[ C_S(g, \alpha) = 1 + \frac{g^2}{16\pi^2} c_F (\alpha + 4) + \frac{g^4}{24(16\pi^2)^2} c_F \left[ (24\alpha^2 + 96\alpha - 288\zeta(3) + 57) c_F \right. \\
\left. + 166 N_f - (18\alpha^2 + 84\alpha - 432\zeta(3) + 1285) N_c \right] \]

\[ Z_5^P(g) = 1 - \frac{g^2}{16\pi^2} (8 c_F) + \frac{g^4}{(16\pi^2)^2} \left( \frac{2}{9} c_F N_c + \frac{4}{9} c_F N_f \right) \]

\[ (c_F = (N_c^2 - 1)/(2 N_c) \quad \zeta(x): \text{Riemann's zeta function}) \]
Conversion Factors

\[ C_V(g, \alpha) = 1 + \mathcal{O}(g^8) \]

\[ C_T(g, \alpha) = 1 + \frac{g^2}{16\pi^2} c_F \alpha + \frac{g^4}{216 (16\pi^2)^2} c_F \left[ (216\alpha^2 + 4320\zeta(3) - 4815) c_F 
- 626 N_f + (162\alpha^2 + 756\alpha - 3024\zeta(3) + 5987) N_c \right] \]

\[ Z_{AV,s}^5(g) = 1 - \frac{g^2}{16\pi^2} (4 c_F) + \frac{g^4}{(16\pi^2)^2} \left( 22 \frac{c_F^2}{9} c_F N_c + 31 \frac{1}{18} c_F N_f \right) \]

\[ Z_{AV,ns}^5(g) = 1 - \frac{g^2}{16\pi^2} (4 c_F) + \frac{g^4}{(16\pi^2)^2} \left( 22 \frac{c_F^2}{9} c_F N_c + \frac{2}{9} c_F N_f \right) \]

[S. A. Larin, J. A. Gracey]
Feynman Diagrams

- 2-loop diagrams contributing to $\mathcal{Z}_{\psi}^{L,\text{RI}'}$: [$Z_A$, $Z_\alpha$, $Z_c$, $Z_g$ needed to 1 loop]


- Several diagrams, evaluated individually, are IR divergent.

- Previously known with Wilson gluons / clover fermions. Differences?
  - No new $\ln^2(\bar{\mu}a_L)$ terms
  - Gluon propagator numerically inverted (more efficient)
  - Plethora of terms in vertices, due to stout links
  - e.g.: First diagram is no longer back-of-the-envelope: $>10,000$ terms
Feynman Diagrams

- Two-loop diagrams contributing to $Z_{\Gamma}^{L,R,I'}$ (flavor nonsinglet):

- Wavy (solid, dotted) lines: gluons (fermions, ghosts).
- Solid box: vertex from the measure part of the action.
- Solid circle: fermion mass counterterm.
- Crosses denote the Dirac matrices: $\hat{1}, \gamma_5, \gamma_\mu, \gamma_5 \gamma_\mu, \gamma_5 \sigma_\mu \nu$. 
Results

- One-loop results: Known for most actions we have considered.

- Two-loop results: Computation still in progress.

- Focus on the difference $Z^\text{singlet}_\Gamma - Z^\text{non-singlet}_\Gamma$:
  
  A two-loop effect, computation just completed.

- Two-loop diagrams contributing to $Z^\text{singlet}_\Gamma$: 
  
  

$$\mathcal{O}_{\text{singlet}} \equiv \sum_f \bar{\psi}_f \Gamma \psi_f$$

- $Z^\text{singlet}_\Gamma - Z^\text{non-singlet}_\Gamma = \mathcal{O}(g^4_0) \implies$ All other factors ($Z_\psi$, ...) are set to 1.

- Similarly, conversion factors can be set to 1
  
  $\implies$ Same difference in all renormalization schemes.

- The contribution of these diagrams to $Z_P$, $Z_V$, $Z_T$ vanishes identically
  
  $\implies$ Only $Z_S$ and $Z_{AV}$ are affected.
Results for the Scalar Operator

\[ Z_S^{\text{singlet}}(\bar{\mu}a_L) - Z_S^{\text{non-singlet}}(\bar{\mu}a_L) \]
\[ = -\frac{g_0^4}{(4\pi)^4} c_F N_f \left[ (s_{00} + s_{01} c_{SW} + s_{02} c_{SW}^2 + s_{03} c_{SW}^3 + s_{04} c_{SW}^4) \right. \]
\[ + (s_{10} + s_{11} c_{SW} + s_{12} c_{SW}^2 + s_{13} c_{SW}^3) \omega \]
\[ + (s_{20} + s_{21} c_{SW} + s_{22} c_{SW}^2) \omega^2 \]
\[ + (s_{30} + s_{31} c_{SW}) \omega^3 \]
\[ + s_{40} \omega^4 \left] + O\left( g_0^6 \right) \right. \]
\[ [c_F = (N_c^2 - 1)/(2 N_c)] \]

- Numerical constants \( s_{i,j} \): Computed for various sets of values of the Symanzik coefficients.

- Computation done in general gauge. Result gauge independent. [As it should in \( \overline{MS} \) and, to 2 loops, in all schemes.]

- For the scalar operator: The result is actually scale independent.
### Results for the Scalar Operator

<table>
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<tr>
<th>Wilson</th>
<th>TL Symanzik</th>
<th>Iwasaki</th>
<th>DBW2</th>
</tr>
</thead>
<tbody>
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<td>$s_{00}$</td>
<td>107.76(2)</td>
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- Errors stem from numerical integration over loop momenta
Results for the Axial Vector Operator

\[ Z_{AV}^{\text{singlet}}(\bar{\mu}a_L) - Z_{AV}^{\text{non-singlet}}(\bar{\mu}a_L) \]

\[ = -\frac{g_0^4}{(4\pi)^4} c_F N_f \left[ (a_{00} + a_{01} c_{SW} + a_{02} c_{SW}^2 + a_{03} c_{SW}^3 + a_{04} c_{SW}^4) \right. \]

\[ + (a_{10} + a_{11} c_{SW} + a_{12} c_{SW}^2 + a_{13} c_{SW}^3) \omega \]

\[ + (a_{20} + a_{21} c_{SW} + a_{22} c_{SW}^2) \omega^2 \]

\[ + (a_{30} + a_{31} c_{SW}) \omega^3 \]

\[ + a_{40} \omega^4 + 6 \ln(\bar{\mu}^2 a_L^2) \right] + \mathcal{O}(g_0^6) \]

- Numerical constants \( a_{i,j} \): Depend on Symanzik coefficients.
- Again: Computation done in general gauge, result gauge independent.
- For the axial vector operator: Scale dependence (related to axial anomaly).
- Presence of \( \gamma_5 q^\mu \bar{q}/q^2 \) term in Green’s function
  \[ \Rightarrow \text{In alternative } RI’ \text{ scheme, add } (g_0^4/(4\pi)^4) c_F N_f \text{ to the above result.} \]
# Results for the Axial Vector Operator

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<thead>
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<tr>
<td>$a_{00}$</td>
<td>2.051(2)</td>
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</tr>
<tr>
<td>$a_{03}$</td>
<td>2.1103(3)</td>
<td>1.7260(1)</td>
<td>1.1251(2)</td>
<td>0.42834(6)</td>
</tr>
<tr>
<td>$a_{04}$</td>
<td>0.0434(2)</td>
<td>0.01636(1)</td>
<td>-0.01074(5)</td>
<td>-0.0171(2)</td>
</tr>
<tr>
<td>$a_{10}$</td>
<td>43.75(1)</td>
<td>36.66(1)</td>
<td>25.827(9)</td>
<td>12.576(8)</td>
</tr>
<tr>
<td>$a_{11}$</td>
<td>76.993(3)</td>
<td>57.190(3)</td>
<td>31.768(2)</td>
<td>9.550(2)</td>
</tr>
<tr>
<td>$a_{12}$</td>
<td>44.260(4)</td>
<td>29.363(2)</td>
<td>12.962(1)</td>
<td>2.3370(6)</td>
</tr>
<tr>
<td>$a_{13}$</td>
<td>-4.4660(6)</td>
<td>-3.3740(5)</td>
<td>-1.8710(2)</td>
<td>-0.50863(4)</td>
</tr>
<tr>
<td>$a_{20}$</td>
<td>-126.45(1)</td>
<td>-92.853(7)</td>
<td>-50.378(1)</td>
<td>-14.391(1)</td>
</tr>
<tr>
<td>$a_{21}$</td>
<td>-259.59(3)</td>
<td>-175.65(2)</td>
<td>-81.45(1)</td>
<td>-17.159(5)</td>
</tr>
<tr>
<td>$a_{22}$</td>
<td>-107.48(1)</td>
<td>-67.737(8)</td>
<td>-27.500(3)</td>
<td>-4.5270(5)</td>
</tr>
<tr>
<td>$a_{30}$</td>
<td>295.76(3)</td>
<td>198.78(2)</td>
<td>90.96(1)</td>
<td>18.6645(6)</td>
</tr>
<tr>
<td>$a_{31}$</td>
<td>400.05(5)</td>
<td>253.87(3)</td>
<td>104.74(1)</td>
<td>17.954(2)</td>
</tr>
<tr>
<td>$a_{40}$</td>
<td>-348.41(4)</td>
<td>-220.12(3)</td>
<td>-90.11(1)</td>
<td>-15.236(2)</td>
</tr>
</tbody>
</table>
Plots

\[ Z_{\Gamma}^{\text{singlet}}(\bar{\mu}a_L) - Z_{\Gamma}^{\text{non-singlet}}(\bar{\mu}a_L) = -\frac{g_0^4}{(4\pi)^4} c_F N_f \cdot \zeta \]

- \( \zeta \) depends on:
  - \( c_{SW} \)
  - \( \omega \)
  - Symanzik coefficients
  - \( \bar{\mu}a_L \) (axial vector case)

- Plot \( \zeta \) at \( \bar{\mu}a_L = 1 \), vs. \( c_{SW} \) or vs. \( \omega \),
  for specific values of Symanzik coefficients

- Plots for Scalar and Axial Vector
Plots (Scalar)

- $\zeta$ vs. $c_{SW} \ [\omega = 0]$

\[ Z_{\Gamma}^{\text{singlet}}(\bar{\mu}a_L) - Z_{\Gamma}^{\text{non-singlet}}(\bar{\mu}a_L) = -\frac{g_0^4}{(4\pi)^4} c_F N_f \cdot \zeta \]
Plots (Scalar)

• $\zeta$ vs. $c_{SW}$ [$\omega = 0.1$]

\[ Z_{\Gamma}^{\text{singlet}}(\bar{\mu}a_L) - Z_{\Gamma}^{\text{non-singlet}}(\bar{\mu}a_L) = -\frac{g_0^4}{(4\pi)^4} c_F N_f \cdot \zeta \]
- $\zeta$ vs. $c_{SW}$ [$\omega = 0.1$] Zoom-in

$Z_{\Gamma}^{\text{singlet}}(\bar{\mu}a_L) - Z_{\Gamma}^{\text{non-singlet}}(\bar{\mu}a_L) = -\frac{g_0^4}{(4\pi)^4} c_F N_f \cdot \zeta$
Plots (Scalar)

- $\zeta$ vs. $\omega$ [$c_{SW} = 0$]

$Z_{\Gamma}^{\text{singlet}}(\bar{\mu} a_L) - Z_{\Gamma}^{\text{non-singlet}}(\bar{\mu} a_L) = -\frac{g_0^4}{(4\pi)^4} c_F N_f \cdot \zeta$
Plots (Scalar)

• $\zeta$ vs. $\omega$ [$c_{SW} = 1.0$]

$$Z_{\Gamma}^{\text{singlet}}(\bar{\mu}a_L) - Z_{\Gamma}^{\text{non-singlet}}(\bar{\mu}a_L) = -\frac{g_0^4}{(4\pi)^4} c_F N_f \cdot \zeta$$
Plots (Scalar)

- $\zeta$ vs. $\omega$ [$c_{SW} = 2.65$]

\[
Z^{\text{singlet}}(\bar{\mu}a_L) - Z^{\text{non-singlet}}(\bar{\mu}a_L) = -\frac{g_0^4}{(4\pi)^4} c_F N_f \cdot \zeta
\]
Plots (Axial Vector)

- $\zeta$ vs. $c_{SW}$ [$\omega = 0$]

\[
Z^{\text{singlet}}_{\Gamma}(\bar{\mu}a_L) - Z^{\text{non-singlet}}_{\Gamma}(\bar{\mu}a_L) = -\frac{g_0^4}{(4\pi)^4}c_F N_f \cdot \zeta
\]
Plots (Axial Vector)

- $\zeta$ vs. $c_{SW}$ [$\omega = 0.1$]

- $Z_{\Gamma}^{\text{singlet}}(\bar{\mu}a_L) - Z_{\Gamma}^{\text{non-singlet}}(\bar{\mu}a_L) = -\frac{g_0^4}{(4\pi)^4}c_F N_f \cdot \zeta$
Plots (Axial Vector)

- $\zeta$ vs. $\omega$ [$c_{SW} = 0$]

\[ Z_{\Gamma}^{\text{singlet}}(\bar{\mu}a_L) - Z_{\Gamma}^{\text{non-singlet}}(\bar{\mu}a_L) = -\frac{g_0^4}{(4\pi)^4} c_F N_f \cdot \zeta \]
Plots (Axial Vector)

- $\zeta$ vs. $\omega$ [$c_{SW} = 1.0$]

$$Z^{\text{singlet}}_\Gamma (\bar{\mu} a_L) - Z^{\text{non-singlet}}_\Gamma (\bar{\mu} a_L) = -\frac{g_0^4}{(4\pi)^4} c_F N_f \cdot \zeta$$
Plots (Axial Vector)

- $\zeta$ vs. $\omega$ \([c_{SW} = 2.65]\)

\[
Z^{\text{singlet}}_{\Gamma}(\bar{\mu}a_L) - Z^{\text{non-singlet}}_{\Gamma}(\bar{\mu}a_L) = -\frac{g_0^4}{(4\pi)^4} c_F N_f \cdot \zeta
\]
Plots with ETMC action (Scalar)

- Iwasaki gluon action, $N_f = 4 (2+1+1)$, $\beta = 2N_c/g^2 = 1.95$
• Iwasaki gluon action, $N_f = 4 (2+1+1)$, $\beta = 2N_c/g^2 = 1.95$

\[
\begin{align*}
Z^\text{sing.} - Z^\text{non-sing.} \\
\omega
\end{align*}
\]

- Blue line: $c_{SW} = 0$
- Pink dashed line: $c_{SW} = 1$
- Green line: $c_{SW} = 2.65$
• Iwasaki gluon action, $N_f = 4 (2+1+1)$, $\beta = 2N_c/g^2 = 1.95$
Iwasaki gluon action, $N_f = 4 (2+1+1)$, $\beta = 2N_c/g^2 = 1.95$
Plots with SLiNC action (Scalar)

- Tree-level Symanzik gluon action, $N_f = 3$, $\beta = 2N_c \cdot c_0 / g^2 = 5.5$
- Tree-level Symanzik gluon action, \(N_f = 3\), \(\beta = 2N_c \cdot c_0 / g^2 = 5.5\)
Plots with SLiNC action (Axial Vector)

- Tree-level Symanzik gluon action, $N_f = 3$, $\beta = 2N_c \cdot c_0 / g^2 = 5.5$
Plots with SLiNC action (Axial Vector)

- Tree-level Symanzik gluon action, \( N_f = 3, \beta = 2N_c \cdot c_0/g^2 = 5.5 \)

\[
Z_{AV}^{\text{sing.}} - Z_{AV}^{\text{non-sing.}}
\]

- \( c_{SW} = 0 \)
- \( c_{SW} = 1 \)
- \( c_{SW} = 2.65 \)
Future Extensions

- Compute $Z^\text{singlet}_\Gamma(\vec{\mu}a_L)$ and $Z^\text{non-singlet}_\Gamma(\vec{\mu}a_L)$ individually
  - Compare with non-perturbative results for $Z^\text{non-singlet}_\Gamma(\vec{\mu}a_L)$

- Extend to different actions, e.g. with more steps of stout smearing
  - Additional contributions: more convergent $\Rightarrow$ More automatic treatment
  - Far more complicated vertices ($\sim 10^6$ terms for two smearing steps)

- Computation for several variants of staggered fermion actions

- Study of other operators:
  - Extended versions of $\bar{\psi}\Gamma\psi$:
    - Lots of additional terms, but superficially convergent $\Rightarrow$ Easy!?
  - Other twist-2 operators used in GPDs ($\bar{\psi}\gamma^\mu D^\nu\psi$, etc.)
  - 4-fermion operators (e.g. in $\Delta S = 2$ transitions: $(\bar{s}\Gamma d)(\bar{s}\Gamma' d)$, etc.)
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THANK YOU