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Lattice 2014

Causal Space-Time on a Null Lattice with Hypercubic Coordination

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Outline

I. **Introduction: Two discretizations of causal manifolds**
   - Causal Dynamical Regge Triangulation (CDT) of Ambjørn and Loll
   - or on a topological lattice

II. **Topological Null Lattice**
   - Discretization of a causal 2,3,and 4 dimensional manifold

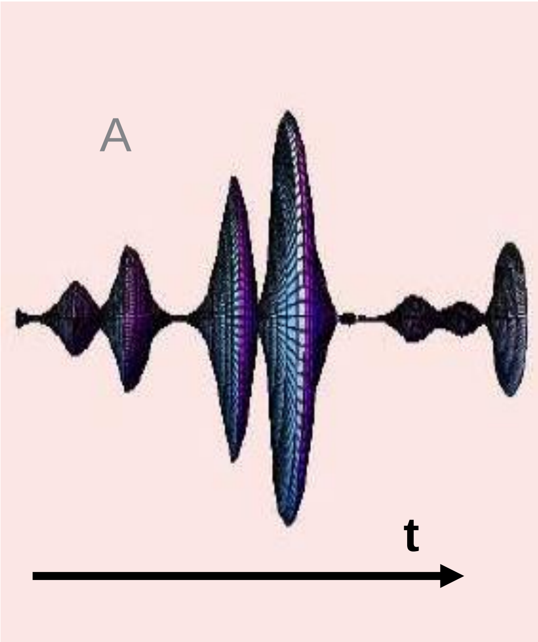
III. **The Manifold Constraint**
   - Construction of the Topological Lattice Theory (TLT)
     - for tetrads
     - for spinors

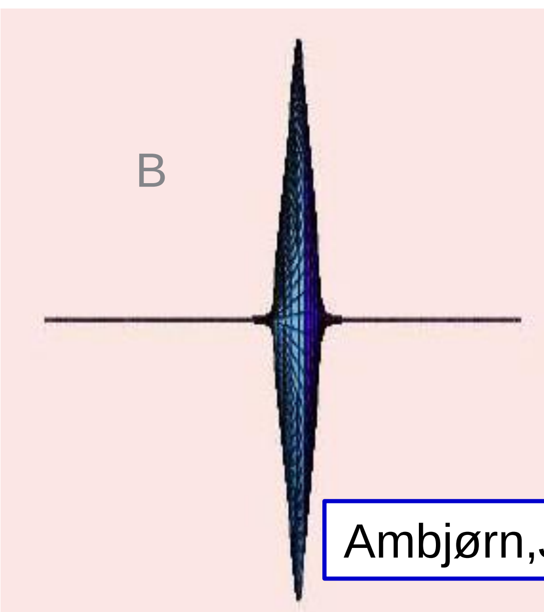
IV. **Outlook**
   - Invariant integration measures and observables
   - Regularization

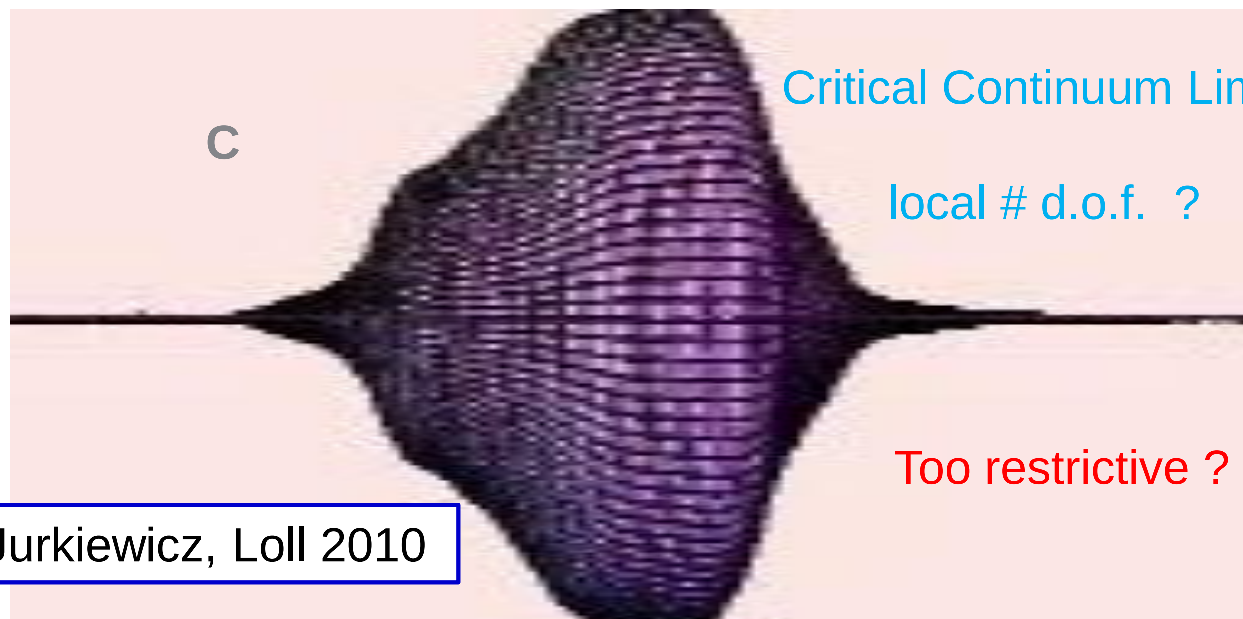
Causal Dynamical triangulation (CDT)

Foliated Gravity as QM of space histories (Ambjørn and Loll):

- Foliated space-time with spatial submanifolds at fixed temporal separation. Triangulation of spatial manifolds by simplexes of fixed size, but varying coordination number.
- Can be "rotated" to Euclidean space, but contrary to Regge QG simulates a subset of causal manifolds only.
- There exist 3 distinct phases: A, B and C that depend on the coupling constants (κ, λ).







Critical Continuum Limit ?

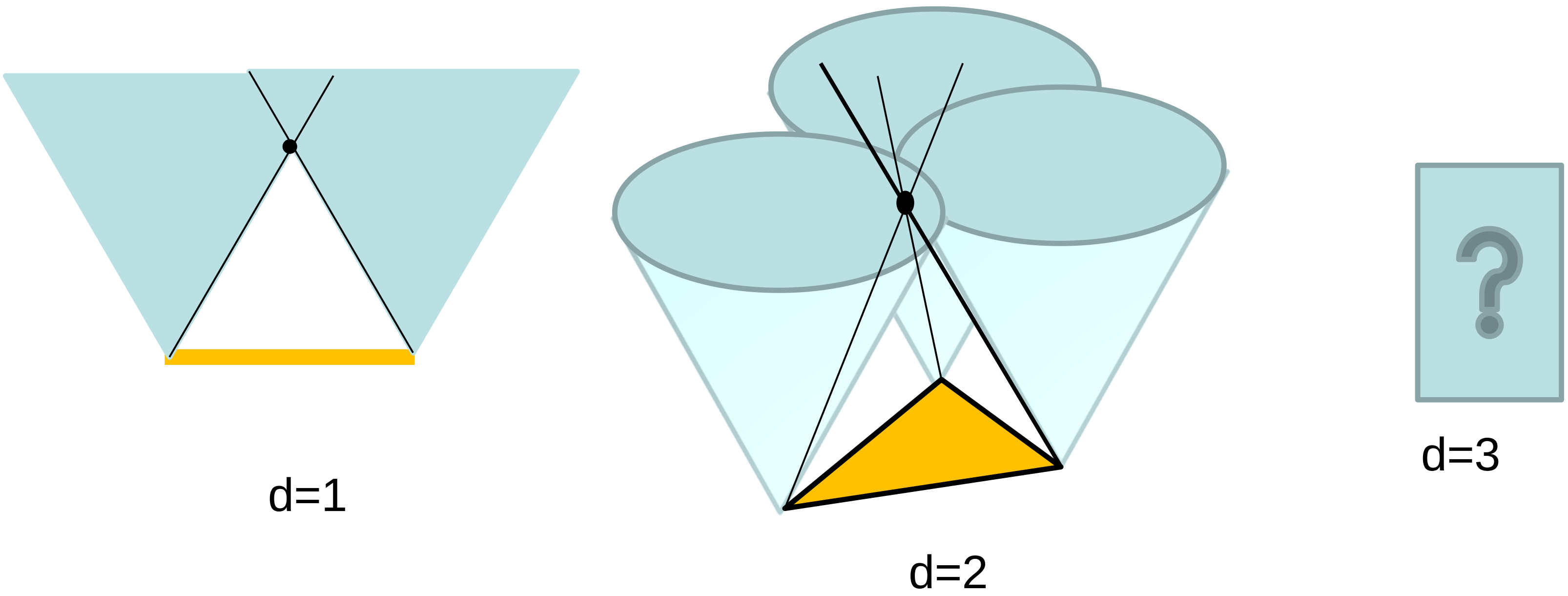
local # d.o.f. ?

Too restrictive ?

Ambjørn,Jurkiewicz, Loll 2010

Null Lattice Combinatorics

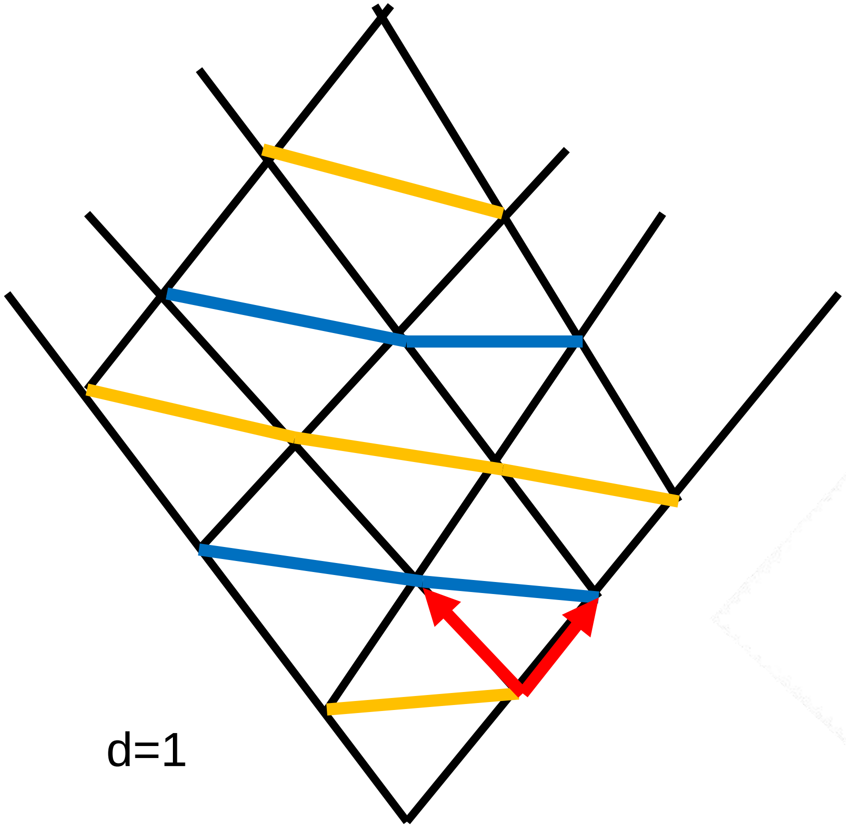
❑ Light-like signals naturally foliate a causal manifold:
The (future) light cones of a spatial line segment (in d=1), a spatial triangle (in d=2) and of a spatial tetrahedron (in d=3) intersect at a unique point:



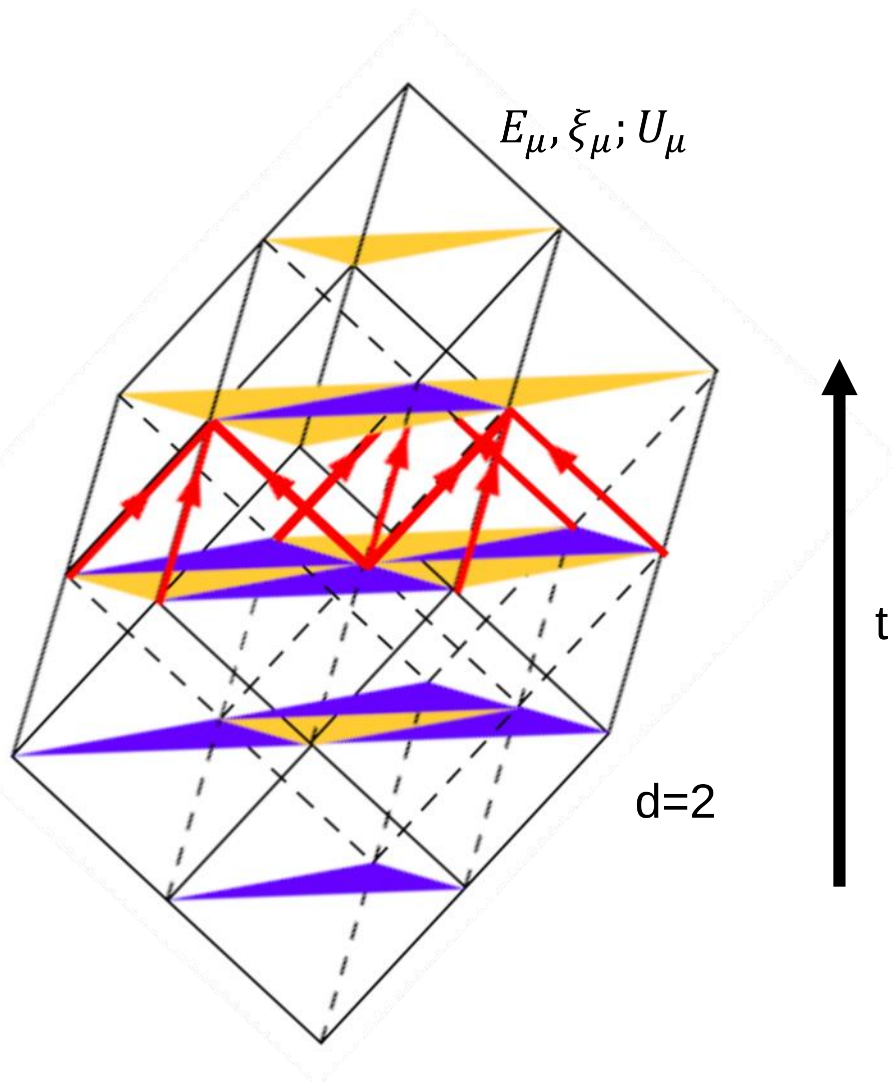
❑ Each spatial triangle (tetrahedron) maps to a point on a spatial (hyper-)surface with time-like separation
- if each vertex is common to 3 (4) otherwise disjoint triangles (tetrahedrons), the mapping between points of two timelike separated spatial surfaces is 1 to 1 !!!!
- the (spatial) coordination of a spatial vertex in d=1,2,3 dimensions thus is 2,6,12.

In 2 (3) spatial dimensions the (spatial) triangulation is hexagonal (tetrahedral) and fixed!
LENGTHS ARE VARIABLE: LATTICE IS TOPOLOGICALLY (HYPER)CUBIC ONLY

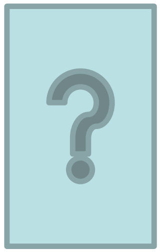
Null Lattices in 1+1 and 2+1 dimensions



d=1



d=2



d=3

Deformed: Actual geometry determined by "fields" $E_\mu(\boldsymbol{n}), \xi_\mu(\boldsymbol{n}); U_\mu(\boldsymbol{n})$ on lattice.

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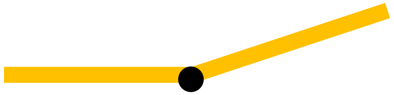
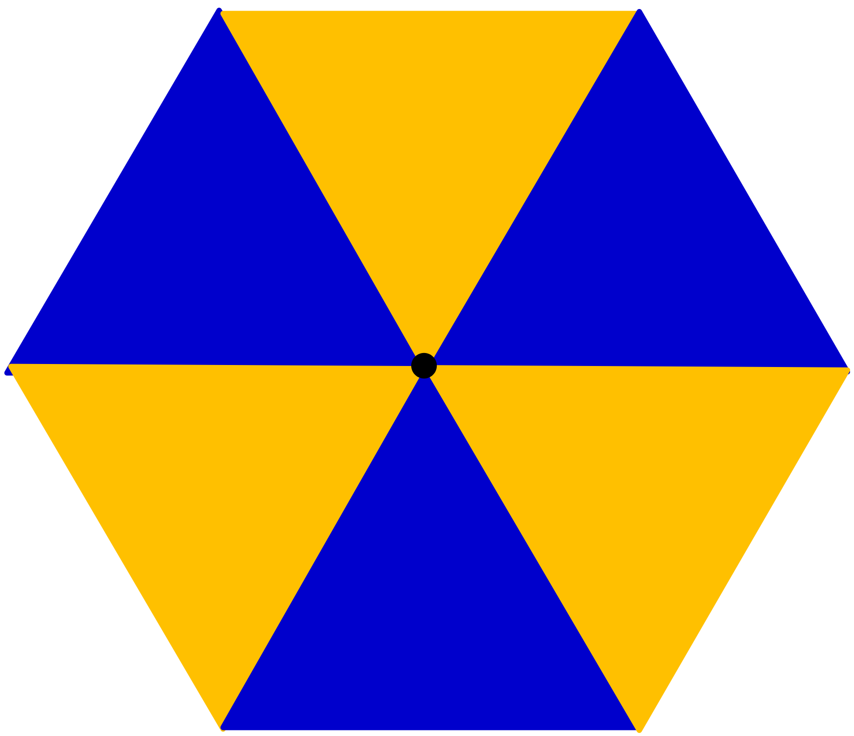
Spatial Triangulation in d=1,2,3

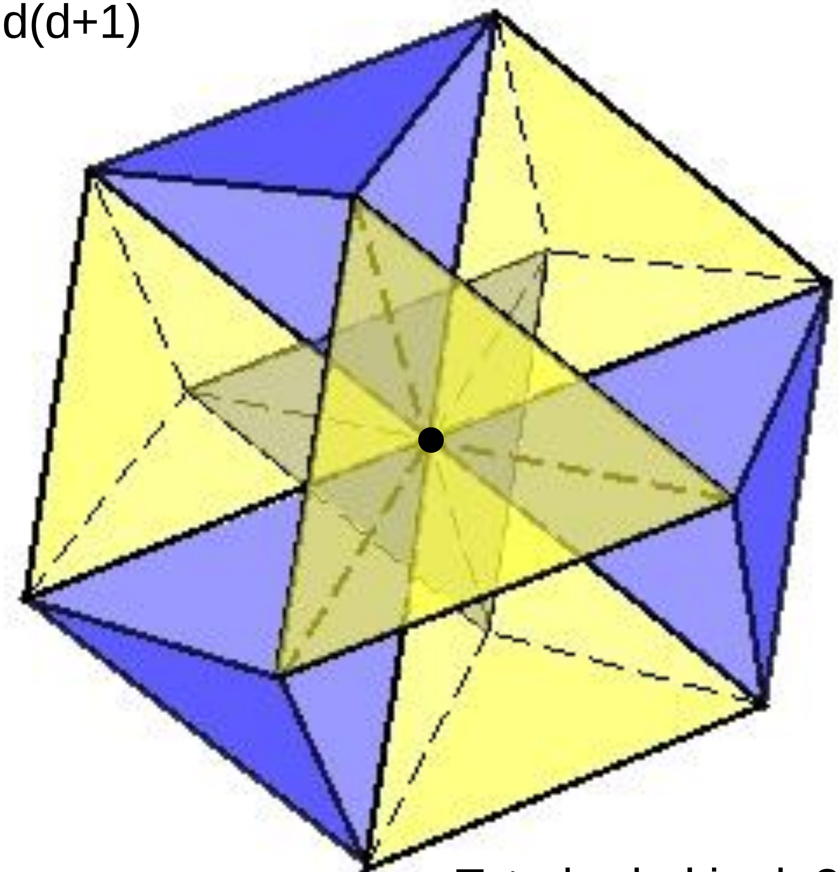
Coordination=d(d+1)



Linear in d=1
Coordination=2



Hexagonal in d=2
Coordination=6



Tetrahedral in d=3
Coordination=12

The tetrahedra in this triangulation of spatial hypersurfaces in general are neither equal nor regular !

Null Coframes

Co-frame, $e^\alpha = e^\alpha_\mu dx^\mu$ is a 1-form

Introducing a basis of anti-hermitian 2x2 matrices σ_α, this 1-form corresponds to an anti-hermitian matrix,

$$E^{A\dot{B}}_\mu(\boldsymbol{n}) = \frac{1}{l_P}\sigma^{A\dot{B}}_\alpha \int_{\boldsymbol{n}}^{\boldsymbol{n}+\mu} e^\alpha$$

where the line-integral is along the geodesic $[\boldsymbol{n}, \boldsymbol{n}+\mu]$. The lattice nodes have light-like or null separation

$$\Leftrightarrow \det E_\mu(\boldsymbol{n}) = 0 \Leftrightarrow E^{A\dot{B}}_\mu(\mathbf{n}) = i(\xi_\mu \otimes \xi^*_\mu)^{A\dot{B}} = i\xi^A_\mu(\mathbf{n})\xi^{*\dot{B}}_\mu(\mathbf{n})$$

Null links thus are described by complex bosonic spinors $\xi^A_\mu(\boldsymbol{n})$.

d.o.f. per site: spinors: 2x2x4-6[sl(2,C)]=10, 10-4 phases=6 metric d.o.f.

The Metric and a Skew Spinorial Form

Local (spatial) lengths on the null lattice are given by the scalar product :

$$\ell^2_{\mu\nu}(\mathbf{n}) = -2E_\mu(\mathbf{n})\cdot E_\nu(\mathbf{n}) = \mathrm{Tr}\varepsilon E^T_\mu(\mathbf{n})\varepsilon E_\nu(\mathbf{n}) = |f_{\mu\nu}(\mathbf{n})|^2 \geq 0$$

with the SL(2,C) invariant skew-symmetric tensor $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and the invariant skew product $f_{\mu\nu}(\mathbf{n}) = -f_{\nu\mu}(\mathbf{n}) = \xi^A_\mu(\mathbf{n})\varepsilon_{AB}\xi^B_\nu(\mathbf{n})$

The latter algebraically satisfies:

$$\boxed{\mathrm{Pf}(f(\mathbf{n})) = \frac{1}{8}\sum_{\mu\nu\rho\sigma}\varepsilon(\mu\nu\rho\sigma)f_{\mu\nu}(\mathbf{n})f_{\rho\sigma}(\mathbf{n}) = f_{12}(\mathbf{n})f_{34}(\mathbf{n}) + f_{13}(\mathbf{n})f_{42}(\mathbf{n}) + f_{14}(\mathbf{n})f_{23}(\mathbf{n}) = 0}$$

The $|\ell_{\mu\nu}|$ for $\mu \neq \nu$ are the 6 spatial lengths of the tetrahedron whose sides are $E_\mu(\boldsymbol{n}) - E_\nu(\boldsymbol{n})$. These vectors are all given in the same inertial system at $\boldsymbol{n}$.
The lengths are invariant under local SL(2,C) transformations $\xi^A_\mu(\boldsymbol{n}) \to g^A_B(\boldsymbol{n})\xi^B_\mu(\boldsymbol{n})$.

Since $Pf\left[f_{\mu\nu}(\boldsymbol{n})\right] = 0$, $\exists$ two vectors ξ^μ_A for which $\sum_\nu f_{\mu\nu}(\mathbf{n})\xi^\nu_A = 0, \text{ for } A = 1,2$

skew form: 2x6-2constraints=10,10-4 =6 only $|f_{\mu\nu}|$ related to metric

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The Manifold Condition

Any causal manifold can be discretized on a topological null-lattice with hypercubic coordination.

Example: Minkowski space-time with null coframes,

$$e_1^{\text{Mink}} = (-1,1,1,\sqrt{3}),\ e_2^{\text{Mink}} = (1,-1,1,\sqrt{3}),\ e_3^{\text{Mink}} = (1,1,-1,\sqrt{3}),\ e_4^{\text{Mink}} = (-1,-1,-1,\sqrt{3})$$

Independent of the node $\boldsymbol{n}$ (scaled and Lorentz-transformed frames here are equivalent). The Minkowski metric (distances) in these coordinates is:

$$-\ell_{\mu\nu}^2 = g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b = -4 \text{ for } \mu \neq \nu, \text{ vanishing for } \mu = \nu.$$

and $\det(e_\mu^a) = 16\sqrt{3} > 0$, preserving orientation.

BUT: NOT EVERY orientation-preserving configuration of null-frames on a lattice with hypercubic orientation represents a discretized manifold !!

Since the #d.o.f. is correct $\Leftrightarrow$ the additional conditions are topological and do not alter the #d.o.f.

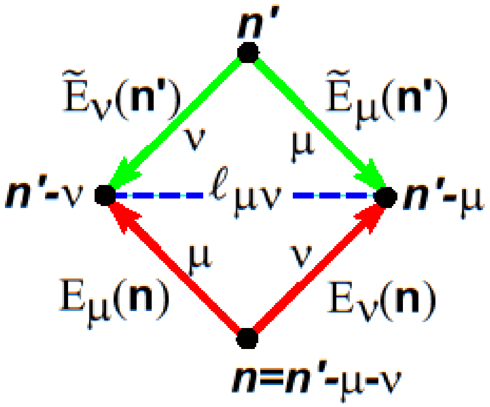
The Manifold Condition for Coframes

For a lattice configuration to be the triangulation of a manifold, distances between events common to two inertial systems ("overlapping atlases") must be the same:

$$-E_\mu(\boldsymbol{n})\cdot E_\nu(\boldsymbol{n})=\ell^2_{\mu\nu} = -\tilde{E}_\mu(\boldsymbol{n}')\cdot\tilde{E}_\nu(\boldsymbol{n}')$$

Forward null coframe
In inertial system at $\boldsymbol{n}$

Backward null coframe
In inertial system at $\boldsymbol{n}'$



For a fixed event $\boldsymbol{n}'$, the six spatial lengths $\mu \neq \nu$

$$0 < \ell^2_{\mu\nu}(\boldsymbol{n}') = -E_\mu(\boldsymbol{n}'-\mu-\nu)\cdot E_\nu(\boldsymbol{n}'-\mu-\nu)$$

are the sides of a spatial tetrahedron (the one formed by the 4 events $\boldsymbol{n}'-\mu, \mu = 1,\dots,4$ in the backward light cone of $\boldsymbol{n}'$). They can be assembled to a tetrahedron $\Leftrightarrow$ the lengths satisfy triangle inequalities. A configuration of (forward) null frames on this lattice is the discretization of a manifold if and only if:

$$\boxed{0 < \ell_{\mu\nu}(\mathbf{n}) < \ell_{\mu\rho}(\mathbf{n}) + \ell_{\rho\nu}(\mathbf{n}), \text{ for all } \mu,\nu,\rho \text{ at every site } \mathbf{n}}$$

The eBRST and TLT of the Manifold Condition

The manifold condition can be enforced by a local TLT with an equivariant BRST. The action of this TLT is:

$$
\begin{aligned}
S_{TLT} &= s\sum_{\mu,\nu,\mathbf{n}}[\bar{c}^{\mu\nu}(\mathbf{n})(E_\mu(\mathbf{n}-\mu-\nu)\cdot E_\nu(\mathbf{n}-\mu-\nu)-\tilde{E}_\mu(\mathbf{n})\cdot\tilde{E}_\nu(\mathbf{n}))+2b^{\mu\nu}(\mathbf{n})\tilde{E}_\mu(\mathbf{n})\cdot c_\nu(\mathbf{n})] \\
&= \sum_{\mu,\nu,\mathbf{n}} g^{\mu\nu}(\mathbf{n})[E_\mu(\mathbf{n}-\mu-\nu)\cdot E_\nu(\mathbf{n}-\mu-\nu)-\tilde{E}_\mu(\mathbf{n})\cdot\tilde{E}_\nu(\mathbf{n})]+ \\
&\qquad + 2[\bar{c}^{\mu\nu}(\mathbf{n})+\bar{\omega}^{\mu\nu}(\mathbf{n})]\tilde{E}_\mu(\mathbf{n})\cdot c_\nu(\mathbf{n})+2b^{\mu\nu}(\mathbf{n})[\tilde{E}_\mu(\mathbf{n})\cdot\phi(\mathbf{n})\cdot\tilde{E}_\nu(\mathbf{n})+c_\mu(\mathbf{n})\cdot c_\nu(\mathbf{n})]
\end{aligned}
$$

with real bosonic Lagrange multiplier fields $g^{\mu\nu}=g^{\nu\mu}$ enforcing 10 constraints on 16 (backward) tetrads $\tilde{E}_\mu^\alpha$, 16 anti-ghosts $\bar{c}^{\mu\nu}=\bar{c}^{\nu\mu}$ and $\bar{\omega}^{\mu\nu}=-\bar{\omega}^{\nu\mu}$, an equal number of 16 ghosts c_μ^α and 6 bosonic topological ghosts $\phi^{\alpha\beta}=-\phi^{\alpha\beta}$ (ghost#=2) and an equal number of topological antighosts $b^{\mu\nu}=-b^{\nu\mu}$ (ghost#=-2) and an additional bosonic $\widetilde{\mathrm{SL}}$(2,C) symmetry.

The TLT therefore has 10+16+6+6-6($\widetilde{\mathrm{SL}}$(2,C))=32 bosonic d.o.f. and 16+16=32 fermionic d.o.f. , or a total of 0 d.o.f. . The eBRST is :

$$
\begin{aligned}
s\tilde{E}_\mu^\alpha(\mathbf{n}) &= c_\mu^\alpha(\mathbf{n})+\omega^\alpha{}_\beta(\mathbf{n})\tilde{E}_\mu^\beta(\mathbf{n}) & s\bar{c}^{\mu\nu}(\mathbf{n}) &= g^{\mu\nu}(\mathbf{n})=g^{\nu\mu}(\mathbf{n})\\
sc_\mu^\alpha(\mathbf{n}) &= \phi^\alpha{}_\beta(\mathbf{n})\tilde{E}_\mu^\beta(\mathbf{n})+\omega^\alpha{}_\beta(\mathbf{n})c_\mu^\beta(\mathbf{n}) & sg^{\mu\nu}(\mathbf{n}) &= 0\\
s\phi^\alpha{}_\beta(\mathbf{n}) &= \omega^\alpha{}_\gamma(\mathbf{n})\phi^\gamma{}_\beta(\mathbf{n})-\phi^\alpha{}_\gamma(\mathbf{n})\omega^\gamma{}_\beta(\mathbf{n}) & sb^{\mu\nu}(\mathbf{n}) &= \bar{\omega}^{\mu\nu}(\mathbf{n})=-\bar{\omega}^{\nu\mu}(\mathbf{n})\\
s\omega^\alpha{}_\beta(\mathbf{n}) &= \omega^\alpha{}_\gamma(\mathbf{n})\omega^\gamma{}_\beta(\mathbf{n})-\phi^\alpha{}_\beta(\mathbf{n}) & s\bar{\omega}^{\mu\nu}(\mathbf{n}) &= 0
\end{aligned}
$$

The additional fields of the TLT can in fact be integrated out to give the triangle inequality constraints:

$$Z_{TLT}[E]\propto\prod_{\mathbf{n}}\prod_{\mu,\nu,\rho}\Theta[\ell_{\mu\rho}(\mathbf{n})+\ell_{\rho\nu}(\mathbf{n})-\ell_{\mu\nu}(\mathbf{n})]$$

The Manifold Condition for Spinors

The manifold condition for the spinors is: $\tilde{f}_{\mu\nu}(\boldsymbol{n}) \coloneqq \xi^A_\mu(\boldsymbol{n}-\mu-\nu)\varepsilon_{AB}\xi^B_\nu(\boldsymbol{n}-\mu-\nu)$

$|\tilde{f}_{\mu\nu}(\boldsymbol{n})| = |\tilde{\xi}^A_\mu(\boldsymbol{n})\varepsilon_{AB}\tilde{\xi}^B_\nu(\boldsymbol{n})|$ or $\boxed{\tilde{\xi}^A_\mu(\boldsymbol{n})\varepsilon_{AB}\tilde{\xi}^B_\nu(\boldsymbol{n}) = e^{i\,\varphi_{\mu\nu}(\boldsymbol{n})}\ \tilde{f}_{\mu\nu}(\boldsymbol{n})}$

where the 6 real phases $\varphi_{\mu\nu}(\boldsymbol{n}) = \varphi_{\nu\mu}(\boldsymbol{n})$ for $\mu \neq \nu$ are not all independent, because $Pf\left[e^{i\,\varphi_{\mu\nu}(\boldsymbol{n})}\ \tilde{f}_{\mu\nu}(\boldsymbol{n})\right] = \boldsymbol{0}$.

Proposition:
If $Pf\left[e^{i\,\varphi_{\mu\nu}(\boldsymbol{n})}\ \tilde{f}_{\mu\nu}(\boldsymbol{n})\right] = \boldsymbol{0} \Leftrightarrow$ $\exists$ a U^4(1) transformation $\xi^A_\mu(\boldsymbol{n}) \to e^{-i\psi_\mu(\boldsymbol{n})}\xi^A_\mu(\boldsymbol{n})$
such that $Pf\left[\ \widetilde{f'}_{\mu\nu}(\boldsymbol{n})\right] = \boldsymbol{0}$.

Proof: Choose $\psi_1(\boldsymbol{n}) = -\psi_2(\boldsymbol{n})$; $\psi_3(\boldsymbol{n}) = -\psi_4(\boldsymbol{n})$ and denote $\psi_1(\boldsymbol{n}) \pm \psi_3(\boldsymbol{n}) = \psi_\pm(\boldsymbol{n})$
The 2 relations decouple and can be solved uniquely (modulo b.c.):

$$\psi_+(\boldsymbol{n}-\boldsymbol{2}-\boldsymbol{4}) - \psi_+(\boldsymbol{n}-\boldsymbol{1}-\boldsymbol{3}) = \varphi_{12}(\boldsymbol{n}) + \varphi_{34}(\boldsymbol{n}) - \varphi_{13}(\boldsymbol{n}) - \varphi_{42}(\boldsymbol{n})$$
$$\psi_-(\boldsymbol{n}-\boldsymbol{2}-\boldsymbol{3}) - \psi_-(\boldsymbol{n}-\boldsymbol{1}-\boldsymbol{4}) = \varphi_{12}(\boldsymbol{n}) + \varphi_{34}(\boldsymbol{n}) - \varphi_{14}(\boldsymbol{n}) - \varphi_{23}(\boldsymbol{n})$$

manifold condition $\Leftrightarrow$ $\exists$ $U^4(1)$ transformation $\ni$ $Pf\left(\tilde{f}_{\mu\nu}(\boldsymbol{n})\right) = 0$

The Manifold TLT for Spinors

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The TLT of the manifold condition is obtained by using it to fix (part of) the $U^4(1)$.

BRST:

$$s\,\xi_1^A(\boldsymbol{n}) = \frac{-i}{2}\psi_1(\boldsymbol{n})\xi_1^A(\boldsymbol{n}) \qquad s\,\xi_1^{*A}(\boldsymbol{n}) = \frac{i}{2}\psi_1(\boldsymbol{n})\xi_1^{*A}(\boldsymbol{n})$$

$$s\,\xi_2^A(\boldsymbol{n}) = \frac{i}{2}\psi_1(\boldsymbol{n})\xi_2^A(\boldsymbol{n}) \qquad s\,\xi_2^{*A}(\boldsymbol{n}) = \frac{-i}{2}\psi_1(\boldsymbol{n})\xi_2^{*A}(\boldsymbol{n}) \quad ;\, s^2 = 0$$

$$s\,\xi_3^A(\boldsymbol{n}) = \frac{-i}{2}\psi_3(\boldsymbol{n})\xi_3^A(\boldsymbol{n}) \qquad s\,\xi_3^{*A}(\boldsymbol{n}) = \frac{i}{2}\psi_3(\boldsymbol{n})\xi_3^{*A}(\boldsymbol{n})$$

$$s\,\xi_4^A(\boldsymbol{n}) = \frac{i}{2}\psi_3(\boldsymbol{n})\xi_4^A(\boldsymbol{n}) \qquad s\,\xi_4^{*A}(\boldsymbol{n}) = \frac{-i}{2}\psi_3(\boldsymbol{n})\xi_4^{*A}(\boldsymbol{n})$$

$$s\,\bar\psi_+(\boldsymbol{n}) = b_+(\boldsymbol{n}) \qquad s\,\bar\psi_-(\boldsymbol{n}) = b_-(\boldsymbol{n})$$

$$\tilde f_{\mu\nu}(\boldsymbol{n}) \coloneqq \xi_\mu^A(\boldsymbol{n}-\mu-\nu)\varepsilon_{AB}\xi_\nu^B(\boldsymbol{n}-\mu-\nu)$$

$$\mathrm{Pf}\left(\tilde f(\boldsymbol{n})\right) = \tilde f_{12}(\boldsymbol{n})\tilde f_{34}(\boldsymbol{n}) + \tilde f_{13}(\boldsymbol{n})\tilde f_{42}(\boldsymbol{n}) + \tilde f_{14}(\boldsymbol{n})\tilde f_{32}(\boldsymbol{n})$$

$$\psi_+(\boldsymbol{n}) = \psi_1\,(\boldsymbol{n}-\boldsymbol{4}-\boldsymbol{2}) + \psi_3\,(\boldsymbol{n}-\boldsymbol{4}-\boldsymbol{2}) - \psi_1(\boldsymbol{n}-\boldsymbol{1}-\boldsymbol{3}) - \psi_3\,(\boldsymbol{n}-\boldsymbol{1}-\boldsymbol{3})$$

$$\psi_-(\boldsymbol{n}) = \psi_1\,(\boldsymbol{n}-\boldsymbol{3}-\boldsymbol{2}) - \psi_3\,(\boldsymbol{n}-\boldsymbol{3}-\boldsymbol{2}) - \psi_1(\boldsymbol{n}-\boldsymbol{1}-\boldsymbol{4}) + \psi_3\,(\boldsymbol{n}-\boldsymbol{1}-\boldsymbol{4})$$

$$S_{\mathrm{TLT}} = s\sum_{\boldsymbol{n}}(\bar\psi_+(\boldsymbol{n}) + i\bar\psi_-(\boldsymbol{n}))\mathrm{Pf}\left(\tilde f(\boldsymbol{n})\right) + (\bar\psi_+(\boldsymbol{n}) - i\bar\psi_-(\boldsymbol{n}))\mathrm{Pf}\left(\tilde f^*(\boldsymbol{n})\right) + \alpha\, b_+(\boldsymbol{n})\bar\psi_-(\boldsymbol{n})$$

$$= \sum_{\boldsymbol{n}} b_+(\boldsymbol{n})(\,\mathrm{Pf}\left(\tilde f(\boldsymbol{n})\right) + \mathrm{Pf}\left(\tilde f^*(\boldsymbol{n})\right)) + i b_-(\boldsymbol{n})\left(\mathrm{Pf}\left(\tilde f(\boldsymbol{n})\right) - \mathrm{Pf}\left(\tilde f^*(\boldsymbol{n})\right)\right) + \alpha\, b_+(\boldsymbol{n}) b_-(\boldsymbol{n})$$

$$+2\left[\bar\psi_+(\boldsymbol{n})\,\mathrm{Im}\left(\tilde f_{13}(\boldsymbol{n})\tilde f_{42}(\boldsymbol{n})\right) + \bar\psi_-(\boldsymbol{n})\mathrm{Re}\left(\tilde f_{14}(\boldsymbol{n})\tilde f_{32}(\boldsymbol{n})\right)\right]\psi_+(\boldsymbol{n})$$

$$+2\left[\bar\psi_+(\boldsymbol{n})\,\mathrm{Im}\left(\tilde f_{14}(\boldsymbol{n})\tilde f_{32}(\boldsymbol{n})\right) + \bar\psi_-(\boldsymbol{n})\mathrm{Re}\left(\tilde f_{13}(\boldsymbol{n})\tilde f_{42}(\boldsymbol{n})\right)\right]\psi_-(\boldsymbol{n})$$

The TLT measure

One can integrate out the real bosonic fields b_+, b_- as well as the fermionic ghosts $\bar{\psi}_+,\ \bar{\psi}_-,\ \psi_+$ and ψ_- to arrive at the TLT contribution to the measure:

$$d\mu_{\mathrm{TLT}} = \prod_{\boldsymbol{n}} \tilde{V}(\boldsymbol{n})\ \exp\left[-\frac{i}{\alpha}\left|\mathrm{Pf}\left(\tilde{f}(\boldsymbol{n})\right)\right|^2\right] \prod_{\mu=1}^{4} d\psi_\mu$$

with $$\mathrm{Pf}\left(\tilde{f}(\boldsymbol{n})\right) = \tilde{f}_{12}(\boldsymbol{n})\tilde{f}_{34}(\boldsymbol{n}) + \tilde{f}_{13}(\boldsymbol{n})\tilde{f}_{42}(\boldsymbol{n}) + \tilde{f}_{14}(\boldsymbol{n})\tilde{f}_{32}(\boldsymbol{n})$$

and $$\tilde{V}(\boldsymbol{n}) = \frac{-i}{48}\sum\nolimits_{\mu\nu\rho\sigma} \epsilon(\mu\nu\rho\sigma)\tilde{f}_{\mu\nu}(\boldsymbol{n})\tilde{f}^*_{\nu\rho}(\boldsymbol{n})\tilde{f}_{\rho\sigma}(\boldsymbol{n})\ \tilde{f}^*_{\sigma\mu}(\boldsymbol{n})$$

where $$\tilde{f}_{\mu\nu}(\boldsymbol{n}) \coloneqq \xi^A_\mu(\boldsymbol{n}-\mu-\nu)\varepsilon_{AB}\xi^B_\nu(\boldsymbol{n}-\mu-\nu)$$

Note: Observables of the lattice theory do NOT DEPEND on the parameter α , but lattice configurations generally are triangulations of causal manifolds for $\alpha \longrightarrow 0$ only!

The dependence on a particular "direction" used in the construction of the TLT has disappeared (as it must).

Conclusion and Outlook

Summary:

- Causal description of discretized space-time on a null lattice with fixed hypercubic coordination.

- Local eBRST actions that encode the topological manifold condition.

Outlook:

- Integration Measures and Regularization
- Invariants and Observables
- eBRST localization to compact SU(2) structure group of spatial rotations.

Hope You Enjoyed the Talk. Thank You.

Invariant Integral Measures & Regularization

The integration measures for the spinors and SL(2,C) transport matrices are dictated by SL(2,C) invariance: $E_\mu(\mathbf{n}) \longrightarrow g(\mathbf{n})E_\mu(\mathbf{n})g^\dagger(\mathbf{n})$ with $g(\mathbf{n}) \in SL(2,\mathbb{C})$

or $\xi^A_\mu(\mathbf{n}) \longrightarrow g^A{}_B(\mathbf{n})\xi^B_\mu(\mathbf{n})$, and $\xi^c_{\mu A}(\mathbf{n}) \longrightarrow \xi^c_{\mu B}(\mathbf{n})g^{-1B}{}_A(\mathbf{n})$

Parameterizing: $\xi_\mu(\mathbf{n}) = \sqrt{\tau_\mu(\mathbf{n})}e^{i\psi_\mu(\mathbf{n})/2}\begin{pmatrix} e^{-i\varphi_\mu(\mathbf{n})/2}\cos(\theta_\mu(\mathbf{n})/2) \\ e^{i\varphi_\mu(\mathbf{n})/2}\sin(\theta_\mu(\mathbf{n})/2)\end{pmatrix}$

$$E_\mu(\mathbf{n}) = i\xi_\mu(\mathbf{n}) \otimes \bar{\xi}_\mu(\mathbf{n}) = i\tau_\mu(\mathbf{n})P_\mu(\mathbf{n}) = i\tau_\mu(\mathbf{n})\,\frac{1}{2}(\mathbb{1} - i\vec{\sigma}\hat{r}_\mu(\mathbf{n}))$$

The SL(2,C) invariant measure for the tetrads is:

$$d\mu[E_\mu(\mathbf{n})] \propto d^4E_\mu\ \delta^+(E_\mu \cdot E_\mu) \propto \tau_\mu d^+\tau_\mu d\Omega(\hat{r}_\mu)$$

The invariant measure of SL(2,C) matrices: $U^A_{\mu B}(\mathbf{n}) \to g^A{}_C(\mathbf{n})U^C_{\mu D}(\mathbf{n})g^{-1D}{}_B(\mathbf{n}+\mu)$

is the (non-compact) SL(2,C) Haar measure: $d\mu\big[U_\mu(\boldsymbol{n})\big] = d\mu_{SL(2,C)}$

So far NO UV-Regularization: the discretization can locally be arbitrary fine.

Coordinate invariant local regularization: $V(\boldsymbol{n}) \geq \varepsilon \to 0^+$

Invariants and Observables

Basic SL(2,C) invariants

Closed loops: $C^{(r)}(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_r) := \mathrm{Tr}U[\mathbf{n}_1, \mathbf{n}_2]U[\mathbf{n}_2, \mathbf{n}_3]\ldots U[\mathbf{n}_r, \mathbf{n}_1]$,

Open strings: $O^{(r)}_{\mu\nu}(\mathbf{n}_0, \mathbf{n}_1, \ldots, \mathbf{n}_r) := \xi^c_\mu(\mathbf{n}_0)U[\mathbf{n}_0, \mathbf{n}_1]U[\mathbf{n}_1, \mathbf{n}_2]\ldots U[\mathbf{n}_{r-1}, \mathbf{n}_r]\xi_\nu(\mathbf{n}_r)$

SL(2,C) parallel transport

Most local examples:

$$f_{\mu\nu}(\mathbf{n}) := \xi^c_{\mu A}(\mathbf{n})\xi^A_\nu(\mathbf{n}) = \xi^A_\mu(\mathbf{n})\varepsilon_{AB}\xi^B_\nu(\mathbf{n}) =: O^{(0)}_{\mu\nu}(\mathbf{n}) ,$$

$$\psi_{\mu\nu\rho}(\mathbf{n}) := \xi^c_{\mu A}(\mathbf{n})U^A_{\rho B}(\mathbf{n})\xi^B_\nu(\mathbf{n}+\rho) =: O^{(1)}_{\mu\nu}(\mathbf{n}, \mathbf{n}+\rho) ,$$

$$\chi_{\mu\nu\rho\sigma}(\mathbf{n}) := \xi^c_{\mu A}(\mathbf{n})U^A_{\rho B}(\mathbf{n})U^B_{\sigma C}(\mathbf{n}+\rho)\xi^C_\nu(\mathbf{n}+\rho+\sigma) =: O^{(2)}_{\mu\nu}(\mathbf{n}, \mathbf{n}+\rho, \mathbf{n}+\rho+\sigma)$$

Observables are real scalars of SL(2,C) invariant densities. Some are:

4-volume

$$\mathcal{V}_{\mu\nu\rho\sigma}(\mathbf{n}) := -i f^*_{\mu\nu}(\mathbf{n}) f_{\nu\rho}(\mathbf{n}) f^*_{\rho\sigma}(\mathbf{n}) f_{\sigma\mu}(\mathbf{n}) = 2\varepsilon(\mu\nu\rho\sigma)\det[E^\alpha_\tau]$$

Hilbert-Palatini term

$$\mathcal{P}_{\mu\nu\rho\sigma}(\mathbf{n}) := i\chi_{\mu\nu\rho\sigma}(\mathbf{n})\chi^*_{\mu\nu\sigma\rho}(\mathbf{n}) = i\mathrm{Tr}\varepsilon E^T_\mu(\mathbf{n})\varepsilon U_\rho(\mathbf{n})U_\sigma(\mathbf{n}+\rho)E_\nu(\mathbf{n}+\rho+\sigma)U^\dagger_\rho(\mathbf{n}+\sigma)U^\dagger_\sigma(\mathbf{n})$$

Partial Localization

No matter what the invariant action, we must factor the infinite volume of the non-compact SL(2,C) structure group!

We localize to the compact SU(2) subgroup of spatial rotations by requiring:

$$\boxed{\tau_1(\mathbf{n}) = \tau_2(\mathbf{n}) = \tau_3(\mathbf{n}) = \tau_4(\mathbf{n}) =: \tau(\mathbf{n})}$$

Physically we are choosing a local inertial system at **n** where the 4 non-collinear events on the forward light cone are simultaneous. This local inertial system always exists and is unique up to spatial rotations.

The partial gauge fixing is local and ghost-free. It is unique and the Jacobian can be evaluated explicitly. The residual SU(2) structure group is compact !

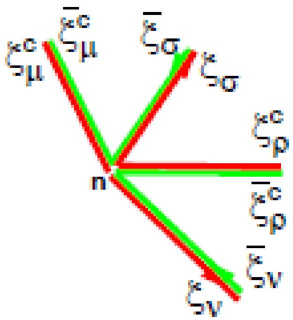
The partially gauge-fixed lattice integration measure in polar parametrization becomes:

$$d\mu_{LAT} = \prod_{\mathbf{n}} \tau^7(\mathbf{n}) \left|\hat{V}(\mathbf{n})\right| d^+\tau(\mathbf{n}) \prod_{\mu} d\Omega(\hat{r}_\mu(\mathbf{n}))\, dU_\mu(\boldsymbol{n})$$

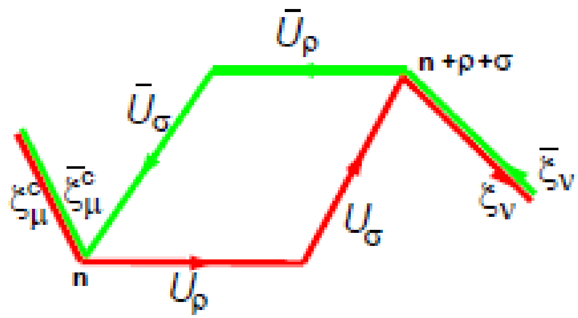
where $\hat{V}(\mathbf{n}) = \frac{1}{16} \det \begin{pmatrix} \hat{r}_1 - \hat{r}_4 \\ \hat{r}_2 - \hat{r}_4 \\ \hat{r}_3 - \hat{r}_4 \end{pmatrix}_{\mathbf{n}}$ is the 3-volume of the (spatial) tetrahedron and is

related to the 4-volume by: $V(\mathbf{n}) = \frac{1}{48} \sum_{\mu\nu\rho\sigma} \varepsilon(\mu\nu\rho\sigma) \mathcal{V}_{\mu\nu\rho\sigma}(\mathbf{n}) = \det E^\alpha_\mu(\mathbf{n}) \xrightarrow[p.g.f.]{} \tau^4(\boldsymbol{n})\hat{V}(\boldsymbol{n})$

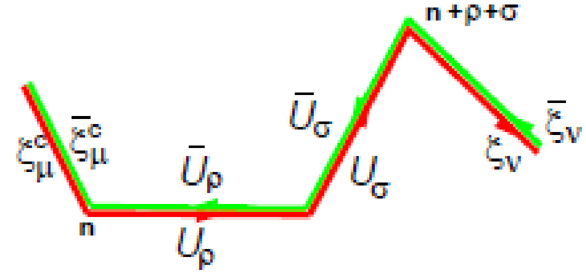
Lattice Actions



volume



Hilbert-Palatini



Holst

$$S^L_{\mathrm{HP}} = \sum_{\mu\nu\rho\sigma} \varepsilon(\mu\nu\rho\sigma) \sum_{\mathbf{n}\in\Lambda} [\mathcal{P}_{\mu\nu\rho\sigma}(\mathbf{n}) + \frac{\lambda}{12}\mathcal{V}_{\mu\nu\rho\sigma}(\mathbf{n})]$$

Letting $\xi \to \lambda^{-1/4}\xi$, the Hilbert-Palatini action is proportional to

$$\beta = 12/\lambda = 12/\Lambda\ell_P^2 = 3c^3/(4Gh\Lambda) \approx 2\ x\ 10^{120}$$

Is the only dimensionless coupling of the lattice model.
Without cosmological constant - no critical limit!
In units $c = \hbar = \ell_P = 1$, critical and thermodynamic limits coincide if the average 4-volume of the universe is fixed (λ is the Lagrange multiplier).

RUTGERS Hilbert-Palatini GR with Null Co-Frames

$$S_{\mathrm{HP}} = \frac{1}{l_P^2} \int_M e^\alpha \wedge e^\beta \wedge [\frac{\Lambda}{6} e^\gamma \wedge e^\delta - R^{\gamma\delta}(\omega)] \varepsilon_{\alpha\beta\gamma\delta}$$

Co-frame, $e^\alpha = e^\alpha_\mu dx^\mu$ is a 1-form

so(3,1) curvature $R^{\alpha\beta}(\omega) = d\omega^{\alpha\beta} + \omega^\alpha{}_\gamma \wedge \omega^{\gamma\beta}$ is a 2-form

constructed from an SL(2,C) connection 1-form $\omega^{\alpha\beta} = -\omega^{\beta\alpha}$

Introducing a basis of anti-hermitian 2x2 matrices σ_α, the co-frame 1-form corresponds to an anti-hermitian matrix,

$$E_\mu^{A\dot{B}}(\boldsymbol{n}) = \frac{1}{l_P} \sigma_\alpha^{A\dot{B}} \int_{\boldsymbol{n}}^{\boldsymbol{n}+\mu} e^\alpha$$

line-integral on geodesic $[\boldsymbol{n}, \boldsymbol{n}+\mu]$. The lattice nodes have light-like separation

$$\Leftrightarrow \det E_\mu(\boldsymbol{n}) = 0 \Leftrightarrow E_\mu^{A\dot{B}}(\mathbf{n}) = i(\xi_\mu \otimes \xi_\mu^*)^{A\dot{B}} = i\xi_\mu^A(\mathbf{n})\xi_\mu^{*\dot{B}}(\mathbf{n})$$

Null links thus are described by complex bosonic spinors $\xi_\mu^A(\boldsymbol{n})$.

Hilbert-Palatini GR with Null Co-Frames

First order formulation for a scalar:

$$S_\phi = \mathrm{Tr}\int_M e^\alpha \wedge e^\beta \wedge e^\gamma \wedge [B^\delta d\phi - \tfrac{1}{2}B^\sigma B_\sigma e^\delta]\varepsilon_{\alpha\beta\gamma\delta}$$

and a spinor: $S_\psi = \int_M e^\alpha \wedge e^\beta \wedge e^\gamma \wedge \bar\psi\sigma^\delta \mathcal{D}\psi\varepsilon_{\alpha\beta\gamma\delta} = \int_M d^4x\, \det(e^\gamma_\mu)\, \bar\psi\sigma^\alpha e^\mu_\alpha \mathcal{D}_\mu\psi$

Co-frame, $e^\alpha = e^\alpha_\mu dx^\mu$ is a 1-form

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Introducing a basis of anti-hermitian 2x2 matrices σ_α, the co-frame 1-form corresponds to an anti-hermitian matrix,

$$E_\mu^{A\dot B}(\boldsymbol{n}) = \frac{1}{\ell_P}\sigma_\alpha^{A\dot B}\int_{\boldsymbol{n}}^{\boldsymbol{n}+\mu} e^\alpha$$

on the link $[\boldsymbol{n}, \boldsymbol{n}+\mu]$. The separation between these nodes is light-like or null

$$\Leftrightarrow \det E_\mu(\boldsymbol{n}) = 0 \Leftrightarrow E_\mu^{A\dot B}(\mathbf{n}) = i(\xi_\mu \otimes \xi_\mu^*)^{A\dot B} = i\xi_\mu^A(\mathbf{n})\xi_\mu^{*\dot B}(\mathbf{n})$$

Null links are described by bosonic spinors $\xi_\mu^A(\boldsymbol{n})$.