Combinatorics of lattice QCD at strong coupling

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Lattice 2014
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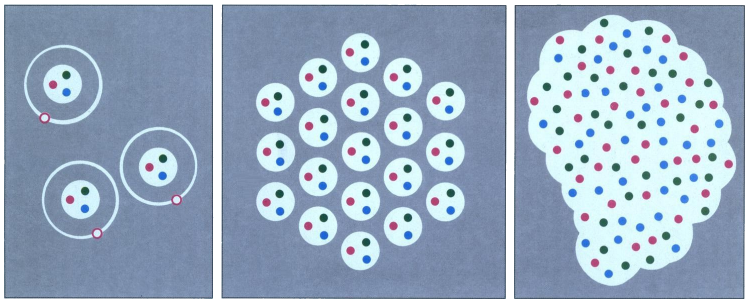
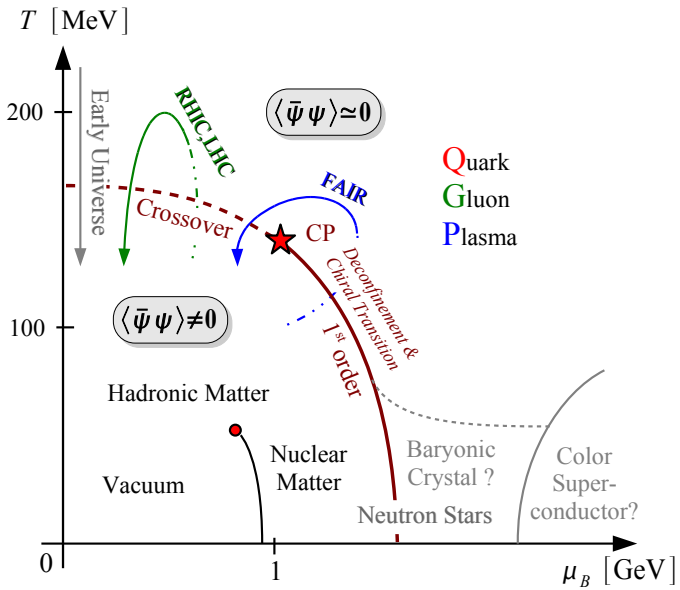


Outline

1. Motivation: lattice QCD at finite density
2. Integrals (Gauge, Grassmann) and Invariants
3. Static Limit of Strong Coupling LQCD (Wilson, staggered)
4. Expansions in κ, β

QCD phase diagram with lattice QCD

QCD (μ, T) **phase diagram**:



- rich phase structure conjectured: **chiral transition** and **nuclear transition**, more exotic phases at high density
- but: only little is known due to sign problem, available methods limited to $\mu/T \lesssim 1$ [de Forcrand PoS Lat2009]

Ways to study lattice QCD with Monte Carlo:

- integrate out Grassmann numbers first:
  $\rightarrow$ **fermion determinant** $\det M[U]$
  - pro: full gauge action;
  - con: severe sign problem at finite μ; expensive to go to chiral limit
- integrate out all gauge links first:
  $\rightarrow$ **Monomer-Dimer-System** (Staggered f.)
  [Rossi & Wolff Nucl. Phys. B285 (1984)]
  [Adams & Chandrasekharan Nucl. Phys. B662 (2003)]
  - pro: sign problem is very mild, chiral limit cheap
  - con: only valid at **strong coupling**;
  - but: gauge action can be expanded in β
  [de Forcrand *et. al.* [1406.4397] (2014)]
- integrate spatial gauge links first:
  $\rightarrow$ **3d effective theory** (Wilson f.) in terms of Polyakov loops [De Pietri *et al.* PRD 76 (2007)]
  [Fromm *et al.* PRL 110 (2013)]
  - pro: gauge corrections to high order
  - con: restricted to heavy quarks

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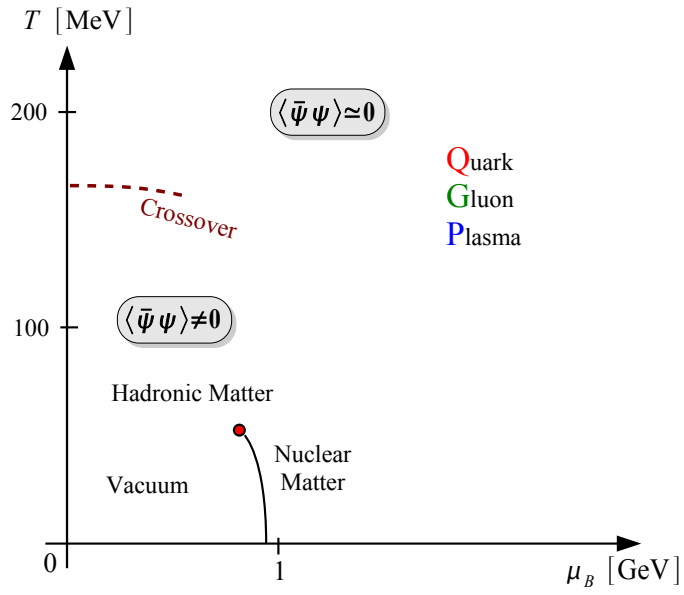

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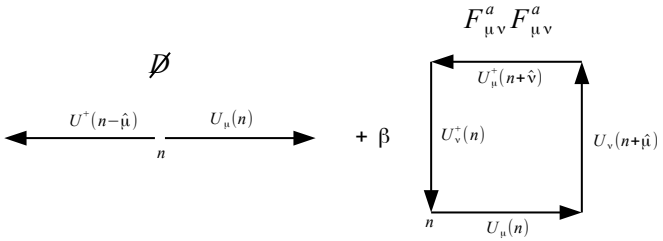
Strong Coupling Lattice QCD

Strong Coupling Limit: $\beta = \frac{2N_c}{g^2} \to 0$

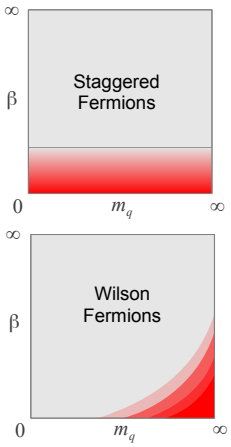
- allows to integrate out the gauge fields completely, as **link integration factorizes**
  $\to$ no fermion determinant
- drawback: lattice is maximally coarse



- Strong coupling LQCD shares important features with QCD:
  - exhibits **"confinement"**, only color singlet d.o.f. survive: **mesons** and **baryons**
  - and features a **nuclear (liqiud-gas) transition** at $a\mu_c$,
  $\to$ SC-LQCD is a great laboratory to study the full (μ, T) **phase diagram**

Staggered fermions:

- staggered fermions are spinless in SC-limit
- SC-partition function in **dimer representation** valid for **any quark mass**
- chiral symmetry: $U(1)_V \times U(1)_{55}$
- suitable to **study chiral dynamics!**



Wilson fermions:

- Wilson fermions have spin in SC-limit
- backtracking of fermions not allowed, $(\mathbb{1} - \gamma_\mu)(\mathbb{1} + \gamma_\mu) = 0$
  $\to$ dimer representation invovles spin, expansion in spatial hoppings needed;
  $\to$ restricted to **heavy quarks**
- no remnant chiral symmetry

$\to$ **is there a "physical" content they share at strong coupling or at $\mathcal{O}(\beta)$?**

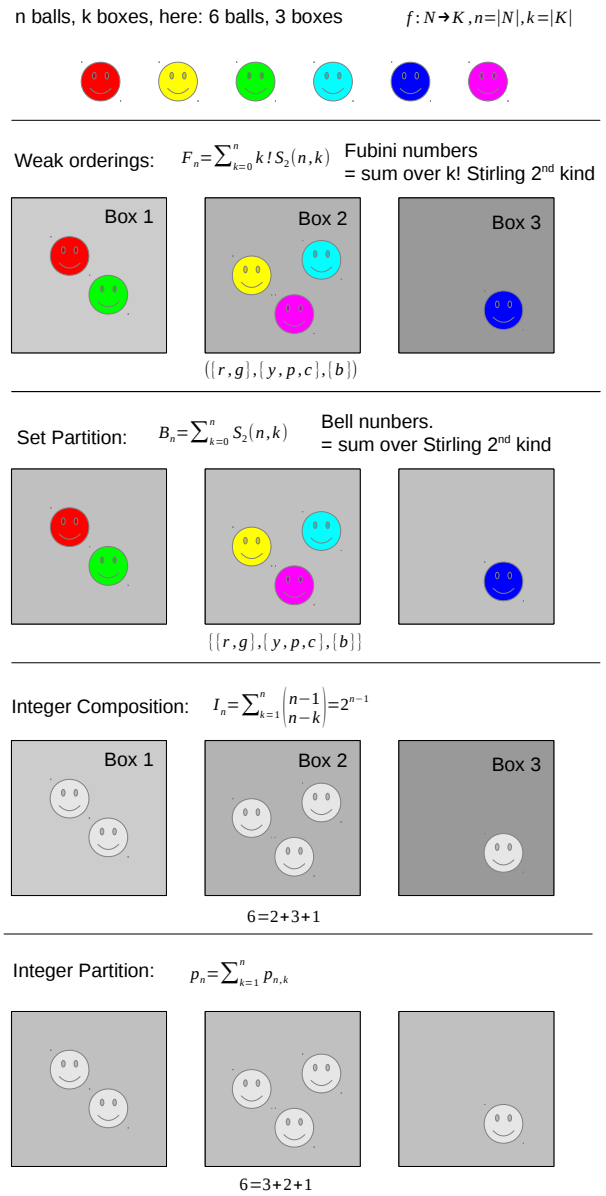
Combinatorical Paradigm: Balls into Boxes

How many ways are there to put n balls into k boxes?

- Various answers possible; many combinatorical questions reduce to this!
- Let $[n] = \{1, 2, \ldots n\}$, consider the maps from the set of balls $[n]$ into the set of boxes $[k]$.
- 12 kanonical answers (twelfefold way) related to **permutation symmetry:**

n balls	k boxes	$[n] \to [k]$ any placement	injective ($k \geq n$) (≤ 1 ball per box)	surjective ($k \leq n$) (≥ 1 ball per box)
dist.	dist.	k^n	$\frac{k!}{(k-n)!}$	$k! S_2(n, k)$
indist.	dist.	$\binom{k+n-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{k-1}$
dist.	indist.	$\sum_{j=0}^{k} S_2(n, j)$	$\begin{cases} 1 & k \geq n \\ 0 & \text{else} \end{cases}$	$S_2(n, k)$
indist.	indist.	$p_k(n+k)$	$\begin{cases} 1 & k \geq n \\ 0 & \text{else} \end{cases}$	$p_k(n)$

- here: combinatorical perspective on lattice QCD
- after link/grassmann integration: LQCD can be formulated in terms of **integer variables**



SU(N) group integrals: one-Link integrals (1)

Consider the **one-link integral** $z(x,\mu) = \int dU_\mu(x) e^{\mathrm{tr}_c\left[U_\mu(x)M^\dagger + U_\mu(x)^\dagger M\right]}$

[Creutz J.Math.Phys. 19 (1978), Eriksson *et al.* J.Math.Phys. 22 (1981)]

- for staggered fermions (cyclicity of trace in $S_F[\chi,\bar{\chi},U]$):

$$(M)_{ij} = \chi_i^f(x)\bar{\chi}_j^f(x+\hat{\mu}), \qquad (M^\dagger)_{ij} = -\chi_i^f(x+\hat{\mu})\bar{\chi}_j^f(x).$$

  with $i,j \in \{1,\dots N_\mathrm{c}\}$ and $f \in \{1,\dots N_\mathrm{f}\}$

- for Wilson fermions:

$$(M)_{ij} = \psi_i^{\beta,f}(x)(\mathbb{1}-\gamma_\mu)_{\alpha\beta}\bar{\psi}_j^{\alpha,f}(x+\hat{\mu}), \quad (M^\dagger)_{ij} = -\psi_i^{\beta,f}(x+\hat{\mu})(\mathbb{1}+\gamma_\mu)_{\alpha\beta}\bar{\psi}_j^{\alpha,f}(x)$$

  with α, β Dirac indices.

- link integral must be a gauge invariant:

$$z(x,\mu) = \sum_{k_1,\dots k_{N_\mathrm{c}+1}} \alpha_{k_1\dots k_{N_\mathrm{c}+1}} \det_c[M]^{k_1} \det_c\left[M^\dagger\right]^{k_2} \mathrm{tr}_c\left[MM^\dagger\right]^{k_3} \dots \mathrm{tr}_c\left[(MM^\dagger)^{N_\mathrm{c}-1}\right]^{k_{N_\mathrm{c}+1}}$$

- sum terminates due to Cayley Hamilton; example for staggered $N_\mathrm{f} = 2$:

$$(MM^\dagger)_{ij} = u_{x,i}((\bar{u}u)_{x+\hat{\mu},k}\bar{u}_{x,j} + (\bar{u}d)_{x+\hat{\mu},k}\bar{d}_{x,j}) + d_{x,i}((\bar{d}d)_{x+\hat{\mu},k}\bar{d}_{x,j} + (\bar{d}u)_{x+\hat{\mu},k}\bar{u}_{x,j})$$

SU(N) group integrals: one-Link integrals (2)

- For staggered fermions, $N_\mathrm{f} = 1$:

$$z(x,\mu) = \sum_{k=0}^{N_\mathrm{c}} \frac{(N_\mathrm{c}-k)!}{N_\mathrm{c}!k!}(M_x M_{x+\hat{\mu}})^k + (-1)^{N_\mathrm{c}} \bar{B}(x+\hat{\mu})B(x) + \bar{B}(x)B(x+\hat{\mu})$$

with $B(x) = \frac{1}{N_\mathrm{c}!}\epsilon_{i_1\ldots i_{N_\mathrm{c}}}\chi_{i_1}(x)\ldots\chi_{i_{N_\mathrm{c}}}(x)$

- determination of $\alpha_{k_1\ldots k_{N_\mathrm{c}+1}}$ via Grasmmann identities ($y = x + \hat{\mu}$):

$$e^{\bar{\chi}_y\chi_y} = \int d\chi_x d\bar{\chi}_x \int dU e^{\bar{\chi}_x\chi_x + \bar{\chi}_x U\chi_y - \bar{\chi}_y U^\dagger \chi_x} = \sum_{l=0}^{N_\mathrm{c}} \alpha_k \frac{N_\mathrm{c}!}{(N_\mathrm{c}-k)!}(\bar{\chi}_x\chi_x\bar{\chi}_y\chi_y)^k$$

- for Wilson fermions or staggered fermions with $N_\mathrm{f} > 1$:
  - meson hoppings ($M_x M_{x+\hat{\mu}}$) and baryon hoppings $\bar{B}(x)B(x+\hat{\mu})$ carry **spin/flavor**!
  - combinatorics more involved, but still **balls into boxes**

2 balls ($\bar{q}\,q$) into N_c boxes (mesons)

	$N_c(N_c-1)$					
M_1	1	1		2	2	
M_2	2		1	1		2
M_3		2	2		1	1

Reduced Haar Measure for SU(N_c)

- Let $P = \prod_{\mathcal{C}} U_\mu(x) = \mathrm{diag}(e^{i\phi_1}, \dots e^{i\phi_{N_c-1}}, e^{-i\sum_{k=1}^{N_c-1}\phi_k})$ be any closed loop of gauge links (e.g. Polyakov, Wilson loop)
- for SU(N_c) there are $N_c - 1$ gauge invariants, such as $L = \mathrm{tr}_c[P]$, L^*,...
- interested in integrals (mesonic: $\mathcal{O} = (LL^*)^n$, baryonic: $\mathcal{O} = (L^{N_c})^n$ for $n \in \mathbb{N}$):

$$\langle \mathcal{O}(L, L^* \dots)\rangle = \frac{1}{(2\pi)^{N_c-1}} \int d\phi_1 \dots d\phi_{N_c-1}\, V(L, L*, \dots)\mathcal{O}(L, L^*, \dots)$$

- $V(L, L^* \dots)$ obtained from **invariant Haar measure** $d\mu(\phi) = H(\phi)\prod_i d\phi_i$,

$$H(\phi) = \prod_{i>j} |e^{i\phi_i} - e^{i\phi_j}|^2 = \begin{vmatrix} N_c & \mathrm{tr}[P] & \dots & \mathrm{tr}[P^{N_c-1}] \\ \mathrm{tr}[P] & N_c & \dots & \mathrm{tr}[P^{N_c-2}] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{tr}[P^{N_c-1}] & \mathrm{tr}[P^{N_c-2}] & \dots & N_c \end{vmatrix}$$

- SU(2): $L^* = L$

$$\left\langle L^{2n}\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi 2\sin^2\phi(2\cos\phi)^{2n} = \frac{1}{2\pi}\int_{-2}^{2} dL\sqrt{4-L^2}L^{2n} = C_n = \frac{1}{n+1}\binom{2n}{n}$$

with C_n the **Catalan numbers**, based on 123-avoiding permutation patterns

Reduced Haar Measure for SU(N_c)

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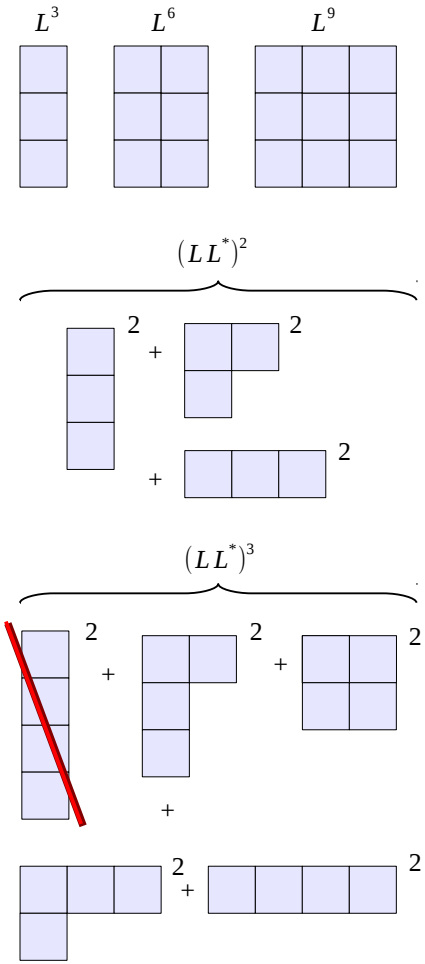
- SU(3): $\mathrm{tr}[P^2] = 9L^2 - 6L^{*2}$, hence:

$$\left\langle (LL^*)^n \right\rangle = \frac{1}{6(2\pi)^2} \int_{-\pi}^{\pi} d\phi_1 \int_{-\pi}^{\pi} d\phi_2 \sqrt{27 - 18LL^* - (LL^*)^2 + 8Re[L]^3}(LL^*)^n$$

Invariants (needed for $N_f > 1$ at strong coupling)

- for SU(N_c) (here: SU(3)): Generalizations of the Catalan numbers

	1 (mesonic)	L^3	L^6	L^9	L^{12}
1 (baryonic)	1	1	5	42	462
(LL^*)	1	3	21	210	2574
$(LL^*)^2$	2	11	98	1122	15015
$(LL^*)^3$	6	74	498	6336	91091
$(LL^*)^4$	23	225	2709	37466	...
$(LL^*)^5$	103	1173	15565	...	...
$(LL^*)^n$	1234-av. permutation				



- invariants of higher moments in L, L*: **restricted permutation patterns** which correpsond to dimensions of **standard young tableaux of bounded height**:

$$m_{N_c}(n) = \sum_{h(\lambda_n) \leq N_c} d^2_{\lambda_n} \leq n!, \quad b_{N_c}(n) = d_{n \times N_c}$$

$$mix_{N_c}(n_m, n_b) = \sum_{h(\lambda_{n_m}) \leq N_c} d_{n_b \times Nc, \lambda_{n_m}} \cdot d_{\lambda_{n_m}}$$

Generalized Lucas Polynomials for SU(N_c) / U(N_c)

Lucas n-step numbers: $F_k^{(n)} = \sum_{i=1}^{n} F_{k-i}^{(n)}$ (Fibonacci-like for $n = 2$)

- related to SU(3): the 3-step Lucas numbers

$$F_k^{(3)} = F_{k-1}^{(3)} + F_{k-2}^{(3)} + F_{k-3}^{(3)}, \qquad F_0^{(3)} = 3, \quad F_1^{(3)} = 1, \quad F_2^{(3)} = 3$$

  and correpsonding 3-step polynomials:

$$F_n^{(3)}(x,y,z) = \mathrm{tr}\left[\begin{pmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n\right], \quad \tilde{F}_n^{(3)}(x,y,z) = \mathrm{tr}\left[\begin{pmatrix} x & y & z \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}^n\right]$$

- it turns out: $\mathrm{tr}[P^n] = \tilde{F}_n(P)$ when identifiying
  $x = L = \mathrm{tr}[P]$, $y = L^* = \mathrm{tr}[P^\dagger]$, $z = D \equiv \det[P]$ the first orders are

$$\mathrm{tr}[P^0] = 3 \qquad \mathrm{tr}[P^1] = L$$

$$\mathrm{tr}[P^2] = L^2 - 2L^* \qquad \mathrm{tr}[P^3] = L^3 - 3LL^* + 3D$$

$$\mathrm{tr}[P^4] = L^4 - 4L^2L^* + 2L^{*2} + 4LD \qquad \mathrm{tr}[P^5] = L^5 - 5L^3L^* + 5LL^{*2} + 5L^2D - 5L^*D$$

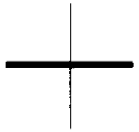
Grassmann Integrals

Grassmann integrals $\int d\xi\, \xi = 1$, $\int d\bar{\xi}\, \bar{\xi} = 1$, $\int d\xi d\bar{\xi}\, \bar{\xi}\xi = 1$ lead to **site weights**:

- staggered fermions: $\int \prod_c [d\chi_{c,x} d\bar{\chi}_{c,x}] e^{2am_q \bar{\chi}_{c,x}\chi_{c,x}} (\bar{\chi}_{c,x}\chi_{c,x})^{k_x} = \frac{N_c!}{n_x!}(2am_q)^{n_x}$
  with monomers $n_x = N_c - k$, determined by Grassmann constraint $k_x = \sum_{\pm\hat{\mu}} k_{\pm\hat{\mu}}(x)$,
  hence $n_x \in \{0, \dots N_c\}$, no monomers at baryonic sites.
- Wilson fermions: site weights (almost) cancel link weight $\sim \kappa^{-n_x/2}$,
  $n_x \in \{0, 2, \dots 4N_c\}$ Grassmann constraint leads to spin conservation:

$$\sum_{\hat{\mu}} (k_{+\hat{\mu}}(x) - k_{-\hat{\mu}}(x)) = 0$$

- Schwinger model (2-dim QED): Wilson fermions mapped on 7-vertex model
  [Salmhofer, Nucl. Phys. B362 (1991)]
- $N_c > 1$ too complicated to do by hand $\rightarrow$ automatize to obtain vertex model



$$= \int d\psi(x)\, d\bar{\psi}(x) \left(\bar{\psi}(x-e_1) T_1^{(-)} \psi(x) \bar{\psi}(x) T_1^{(+)} \psi(x-e_1) \right)$$
$$= \bar{\psi}_2(x-e_1)\psi_1(x-e_1)\psi_2(x+e_1)\bar{\psi}_1(x+e_1)$$

$$= \int d\psi\, d\bar{\psi}(x) \left(\bar{\psi}(x-e_1) T_1^{(-)} \psi(x) \bar{\psi}(x) T_1^{(+)} \psi(x-e_1) \right)$$
$$= \left(-\tfrac{1}{2}\right) \bar{\psi}_2(x-e_1)\psi_1(x-e_1)\chi_2(x+e_2)\bar{\chi}_1(x+e_2)$$

Strong Coupling Partition Function

Staggered Partition Function ($N_\mathrm{f} = 1$): all orders in hopping parameter

$$Z_{SC}(m_q, \mu, \gamma) = \sum_{\{k_b, n_x, \ell\}} \underbrace{\prod_{b=(x,\mu)} \frac{(N_\mathrm{c} - k_b)!}{N_\mathrm{c}! k_b!} \gamma^{2k_b \delta_{\mu 0}}}_{\text{meson hoppings } M_x M_y} \underbrace{\prod_x \frac{N_\mathrm{c}!}{n_x!} (2am_q)^{n_x}}_{\text{chiral condensate } M_x} \underbrace{\prod_\ell w(\ell, \mu)}_{\text{baryon hoppings } \bar{B}_x B_y}$$

$$k_b \in \{0, \dots N_\mathrm{c}\},\ n_x \in \{0, \dots N_\mathrm{c}\},\ \ell_b \in \{0, \pm 1\}$$

- Grassmann constraint: $n_x + \sum_{\hat{\mu} = \pm\hat{0}, \dots \pm\hat{d}} \left(k_{\hat{\mu}}(x) + \frac{N_\mathrm{c}}{2} |\ell_{\hat{\mu}}(x)| \right) = N_\mathrm{c}$
- weight $w(\ell, \mu)$ and sign $\sigma(\ell) = \pm 1$ for oriented loop ℓ depend on loop geometry

Wilson Partition Function ($N_\mathrm{f} = 1$, $N_\mathrm{c} = 1$): mapping on a **3-state vertex model**

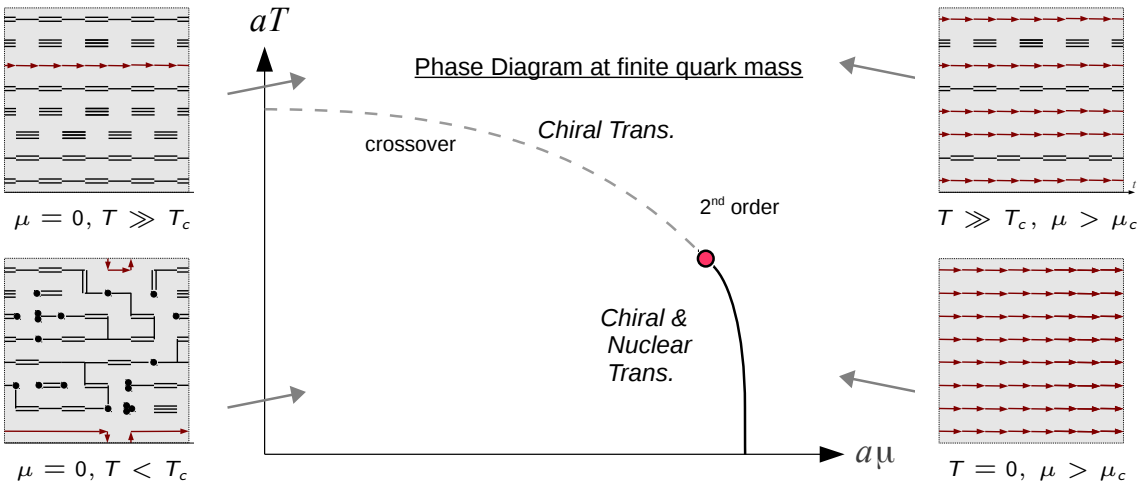
[c.f. Salmhofer, Nucl. Phys. B362 (1991), Scharnhorst, Nucl. Phys. B (1996)]

$$Z_{SC}(\kappa, \mu) = \sum_{\{k_b, n_x\}} \left(\sum_{s=0} N(s, \{k_b\}) 2^s \right)^2 \left(\frac{1}{2} \right)^{C(\{k_b\})} \prod_x \frac{1}{(2\kappa)^{n_x}} \qquad \begin{array}{l} k_b \in \{0, 1, 2\} \\ n_x \in \{0, 2, 4\} \end{array}$$

- $C(\{k_b\})$ counts how often a line bends, $N(s, \{k_b\})$ counts multiplicities of loops
- Grassmann constraint: $n_x + \sum_{\hat{\mu} = \pm 0, \dots, \pm d} k_{\hat{\mu}}(x) = 4$

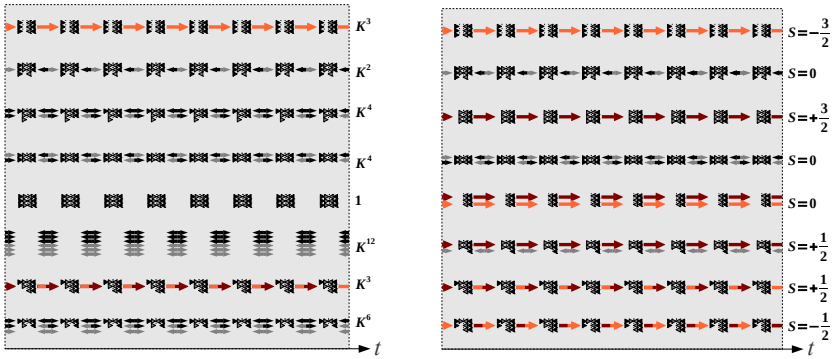
Static Limit (for $N_f = 1$): $Z_{SC} = \prod_{\vec{x}} Z_1(\vec{x})$

Staggered Fermions: $$Z_1(\mu, T) = N_c + 1 + \sum_{n=1}^{N_t N_c/2} \#(2am_q)^{2n} + 2\cosh(\mu_B/T)$$



- prefactors # related to *Fibonacci / Tribonacci / ...* suppressed at high T
- chiral restoration takes place when **spatial dimers become rare**
- nuclear and chiral transition **coincide**: $\langle \bar{\chi}\chi \rangle$ vanishes as baryonic crystal forms

Wilson fermions: $$Z_1(\mu, T) = \sum_{k=0}^{2N_c} T(k) K^{2k} + \sum_{k=0}^{N_c} P(k) K^{(2k+N_c)} 2\cosh(\mu_B/T) + K^{2N_c} 2\cosh(2\mu_B/T)$$



- quark mass dependence: $K = (2\kappa)^{N_\tau}$, $\kappa = \frac{1}{4+2am_q}$
- $T(k) = \binom{3+\min(k, 2N_c - k)}{3} = \sum_{q=0}^{k} D^{0q}$ (Triangle no.)

  $P(k) = (1+k)(1+N_c-k)$,
- same result as 3d-effective theory

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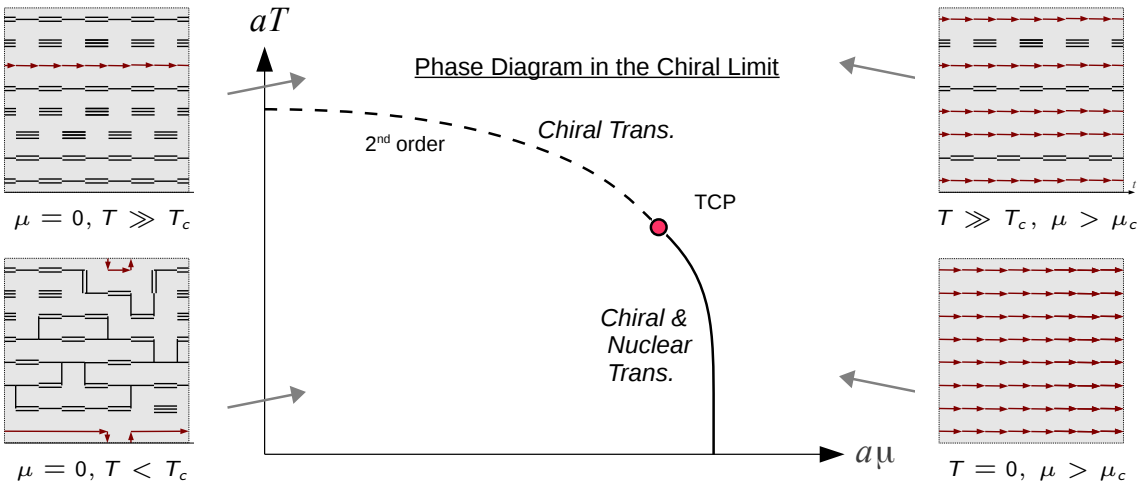

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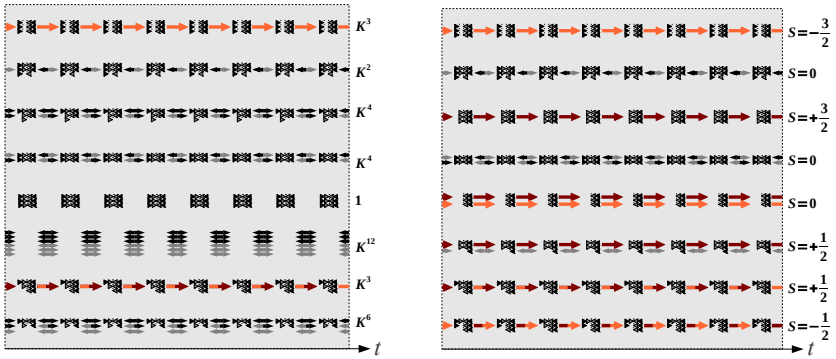
Wilson fermions: $$Z_1(\mu, T) = \sum_{k=0}^{2N_c} T(k) K^{2k} + \sum_{k=0}^{N_c} P(k) K^{(2k+N_c)} 2\cosh(\mu_B/T) + K^{2N_c} 2\cosh(2\mu_B/T)$$



- quark mass dependence: $K = (2\kappa)^{N_\tau}$, $\kappa = \frac{1}{4+2am_q}$
- $T(k) = \binom{3+\min(k, 2N_c - k)}{3} = \sum_{q=0}^{k} D^{0q}$ (Triangle no.)

  $P(k) = (1+k)(1+N_c-k)$,
- same result as 3d-effective theory

Static Limit (for $N_f > 1$)

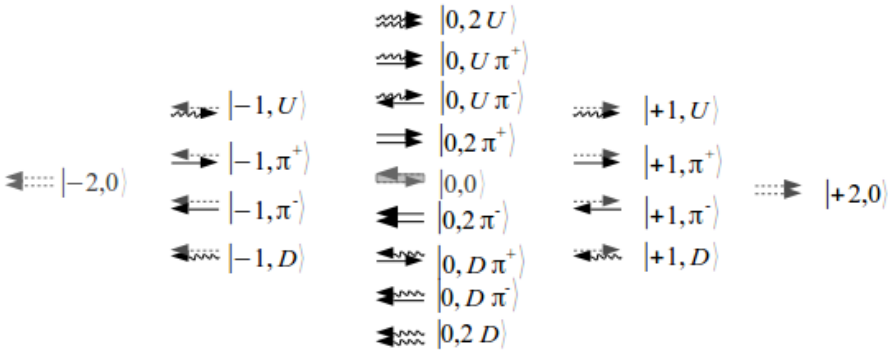
Staggered fermions (chiral limit): multiplicities due to **quantum numbers:**

$$|P_u, P_d, \ldots P_f, Q_{\pi^+}, Q_{K^+}, \ldots\rangle$$

- no. of mesonic static lines:

N_f	mesonic multiplicity
1	(N_c+1)
2	$(N_c+1)\frac{(2N_c+1)^2+1}{3}$ (octahedral no.)
3	$(N_c+1)\frac{11(N_c+1)^4+5(N_c+1)^2+4}{20}$
...	...
N_f	$\binom{2N_f}{N_c N_f}_{N_c} = \sum_{k=0}^{N_c N_f} \binom{N_f}{k}^2_{N_c}$

- no. of baryonic static lines: $\binom{N_c+N_f-1}{N_f-1}$



$N_f = 2, N_c = 2$: 19 states

[de Forcrand, U. hep-lat/1211.7322 (2012)]

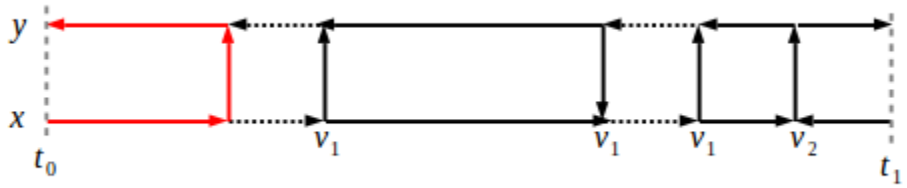
Wilson fermions: spin degeneracies according to HRG model, e.g. $N_c = 3$, $N_f = 2$:

	baryonic		mesonic
1	quenched limit	1	quenched limit
4+6+6+4=20	single baryon)	4	1 meson
1+4+10+20+10+4+1=50	two baryon	...	...
4+6+6=4=20	three baryon=hole	4	$4N_c N_f - 1$ mesons
1	4 baryons=saturation	1	$4N_c N_f$ mesons"chiral limit"

- results for mixed states and for all N_c, N_f determined

Transfer Matrix Approach to Hopping Parameter Expansion

- Hopping parameter expansion in κ
  $\rightarrow$ spatial meson/baryon hoppings



- The static lines are the in- and out states of the **transfer matrix:**

$$Z = \mathrm{Tr}[e^{\beta\mathcal{H}}], \quad \mathcal{H} = \frac{1}{2}\sum_{\langle x,y\rangle} J_x^+ J_y^-, \quad J^+ = \begin{pmatrix} 0 & & & \\ v_1 & 0 & & \\ & \ddots & \ddots & \\ & & v_{N_c} & 0 \end{pmatrix}, \quad v_k = \sqrt{\frac{k(1+N_c-k)}{N_c}}$$

- quantum numbers (spin, parity,flavor) are locally conserved
- **spin/parity/charge conservation:** transitions at spatial dimers, raising charges at one site, lowering at a neighboring site, e.g. staggered $N_f = 2$:

$$|\Delta S^z| = 1, \quad |\Delta Q_{\pi^0}| + |\Delta Q_{\pi^+}| = 1$$

$$\mathcal{H} = \frac{1}{2}\sum_{\langle x,y\rangle}\left(J^+_{U(x)}J^-_{U(y)} + J^+_{D(x)}J^-_{D(y)} + J^+_{\pi^+(x)}J^-_{\pi^+(y)} + J^+_{\pi^+(x)}J^-_{\pi^-(y)}\right)$$

[de Forcrand, U. [hep-lat/1211.7322] (2012)]

- works well for staggered fermions, should also work for Wilson fermions
  $\rightarrow$ **unified picture**, no need to compute expansion in κ, **sample it to all orders!**

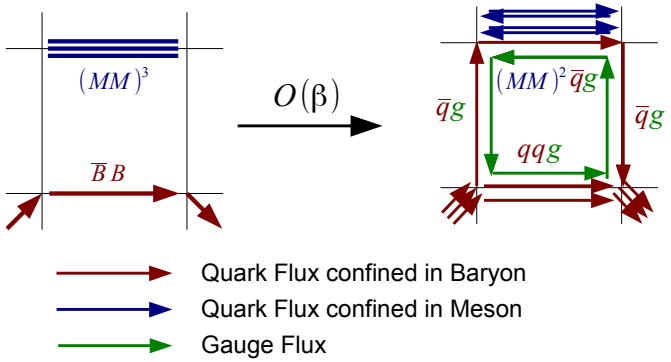
$\mathcal{O}(\beta)$ corrections for staggered fermions

QCD Partition function in terms of systematic expansion in β:

$$Z_{QCD} = \int d\chi d\bar{\chi} dU e^{S_G + S_F} = \int d\chi d\bar{\chi} Z_F \left\langle e^{S_G} \right\rangle_{Z_F}$$

$$\langle \mathcal{O} \rangle_{Z_F} = \frac{1}{Z_F} \int dU \mathcal{O} e^{-S_F}, \qquad Z_F = \int dU e^{-S_F} = \prod_{l=(x,\mu)} z(x,\mu)$$

- linearize gauge action:
  $\left\langle e^{S_G} \right\rangle_{Z_F} \simeq 1 + \frac{\beta}{2N_c} \sum_P \left\langle \mathrm{tr}[U_P + U_P^\dagger] \right\rangle_{Z_F}$
- evaluate plaquette expectation value before Grassmann integration:
  $\left\langle \mathrm{tr}[U_P + U_P^\dagger] \right\rangle_{Z_F} = \frac{1}{Z_F} \int dU \mathrm{tr}[U_P + U_P^\dagger] e^{S_F}$



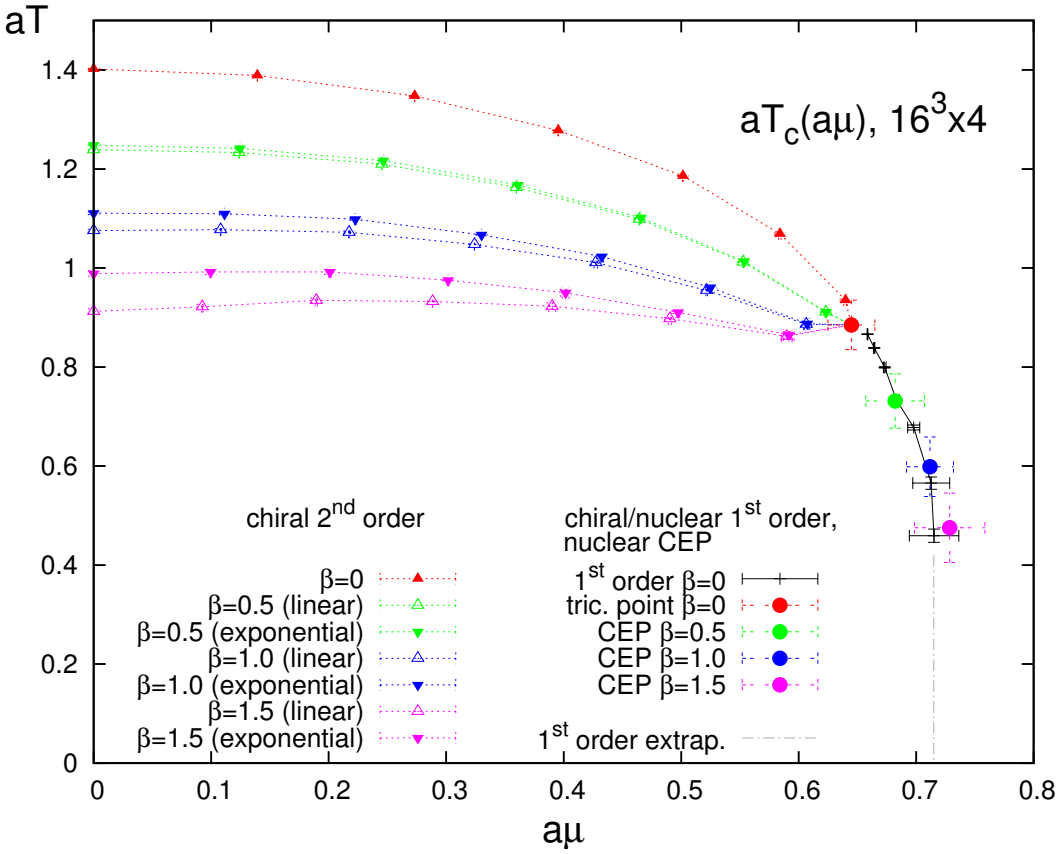
- one-Link integrals along excited plaquette: [Azakov & Aliev, Physica Scripta 38 (1988)] [Langelage, PhD thesis (2009)]

$$J_{ij} = \sum_{k=1}^{N_c} \underbrace{\frac{(N_c-k)!}{N_c!(k-1)!}(M_\chi M_\varphi)^{k-1}\bar{\chi}_j\varphi_i}_{mesons+\bar{q}g} - \underbrace{\frac{1}{N_c!(N_c-1)!}\epsilon_{ii_1i_2}\epsilon_{jj_1j_2}\bar{\varphi}_{i_1}\bar{\varphi}_{i_2}\chi_{j_1}\chi_{j_2}}_{qqg} - \underbrace{\frac{1}{N_c}\bar{B}_\varphi B_\chi \bar{\chi}_j\varphi_i}_{\substack{meson+qqg \\ baryon+\bar{q}g}}$$

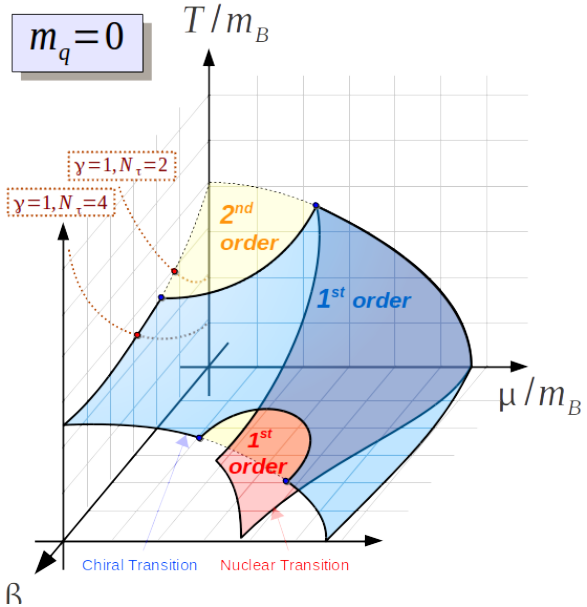
have combinatorical interpretation; color neutral, now not necessarily hadronic

SC-Phase diagram

- slope vanishes at the tricritial point and along the first order line
- $aT_c(\mu = 0)$ drops, whereas $a\mu_c(T = 0)$ does not (consistent with $\frac{T_c(\mu=0)}{3\mu_c(T=0)} = 0.82$ being too large compared to continuum ratio $\approx \frac{154\,\mathrm{MeV}}{0.93\,\mathrm{GeV}} = 0.165$)



open question: do the nuclear and chiral transition split?



[de Forcrand, Langelage, Philipsen, U. [1406.4397] (2014)]

Summary

- all spin and flavor **multiplicities** can be related to representations of SU(N_c) or the (restricted) permutation group.
- the weights of the paritition functions can be interpreted as symmetry factors related to **balls into boxes** problems
- the combinatorics of staggered fermions and Wilson fermions is very different at strong coupling.
- hopping parameter expansion can be combined with **transfer matrix approach**
  $\rightarrow$ Hamiltonian can be constructed
  $\rightarrow$ **quantum Monte Carlo** applicable
  (e.g. stochastic series expansion, continuous time Worm algorithm)
- was already successfullly applied to staggered fermions
- hope for Wilson fermions: simpler to approach light quarks with 4d dimer/flux approach compared to 3d Polyakov effective theory

Goal: compare staggered and Wilson fermions order by order in κ and β.

Backup Slide: Staggered Combinatorics

For $\mathrm{U}(N_c)$ the total number of possible meson line segments of length l, $\mathcal{N}_{\mathrm{tot}} = a_1(l)$, is given by the following recursive matrix relation

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N_c+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \ddots & & 0 \\ \vdots & & & \\ 1 & 0 & \dots & 0 \end{pmatrix}^{l} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad a_1(l) = \begin{cases} \mathrm{fib}(l) & N_c = 1 \\ \mathrm{fib}_2(l) & N_c = 2 \\ \mathrm{fib}_3(l) & N_c = 3 \end{cases} \tag{1}$$

$$\mathrm{fib}(l) = \mathrm{fib}(l-1) + \mathrm{fib}(l-2), \qquad \mathrm{fib}(0) = 1, \quad \mathrm{fib}(1) = 2 \tag{2}$$

$$\mathrm{fib}_2(l) = 2\mathrm{fib}_2(l-1) + \mathrm{fib}(l-2) - \mathrm{fib}(l-3), \qquad \mathrm{fib}_2(0) = 1, \quad \mathrm{fib}_2(1) = 3 \tag{3}$$

$$\mathrm{fib}_3(l) = 2\mathrm{fib}_2(l-1) + 3\mathrm{fib}(l-2) - \mathrm{fib}(l-3) - \mathrm{fib}(l-4), \qquad \mathrm{fib}_3(0) = 1, \quad \mathrm{fib}_3(1) = 4 \tag{4}$$

with $\mathrm{fib}(l)$ the Fibonacci series ($\mathrm{fib}(3) = 5$, $\mathrm{fib}(4) = 8$)

Backup Slide: SC + Plaquette Partition Function at $\mathcal{O}(\beta)$

partition function can be expanded up to $\mathcal{O}(1/g^{2N_c})$ as Grassmann integration terminates at this order:

$$Z = \int d\chi d\bar{\chi} Z_F \prod_P \left(1 + \frac{1}{g^2}\left(\prod_{l\in P} z_l\right)^{-1} \sum_{s=1}^{19} F_P^s + \dots\right)$$

- new set of **plaquette variables** $q_P \in \{0, \dots N_c\}$ and auxiliary variables

$$q_x = \sum_P^{x\in P} q_P \in \{0, \dots N_c\}, \quad q_b = \sum_P^{b\in P} q_P \in \{0, \dots N_c\}$$

- help to write down Z after Grassmann integration:

$$Z = \sum_{\{k,n,\ell,q\}} \prod_{b=(x,\mu)} w_b \prod_x w_x \prod_\ell w_\ell \prod_P w_P,$$

$$w_x = \frac{N_c!}{n_x!}(2am_q)^{n_x} v_i(x), \quad w_b = \frac{(N_c-k_b)!}{N_c!(k_b-q_b)!}, \quad w_P = g^{-2q_P}$$

$$n_x + \sum_{\hat{\mu}=\pm\hat{0},\dots\pm\hat{d}} \left(k_{\hat{\mu}}(x) + \frac{N_c}{2}|\ell_{\hat{\mu}}(x)|\right) = N_c + q_x$$