Quark number density at imaginary chemical potential and its extrapolation to large real chemical potential by the effective model

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Lattice 2014
June 23rd, 2014
New York, USA
1. The lattice results of the quark number density at imaginary chemical potential
2. The extrapolation of lattice data from imaginary to real chemical potential
3. The determination of the strength of the vector interaction in the NJL-type effective models from the lattice results
4. The model prediction for the QCD phase diagram and the mass-radius relation of neutron stars
Introduction

- Lattice QCD - At real chemical potential ($\mu$): sign problem

- The vector interaction
  - In mean field approx.,
    \[ G_V(\bar{q}\gamma_{\mu}q)^2 \rightarrow G_V\langle n \rangle(\bar{q}\gamma_0q) \]
    \[ G_V : \text{the vector coupling} \]

✓ We calculate the quark number density by lattice QCD at finite imaginary $\mu$.
✓ We determine the strength of the vector interaction from the lattice results in order to construct a reliable effective model.
✓ We predict the QCD phase diagram at high densities and the mass-radius relation of neutron stars.
We calculate the quark number density by **lattice QCD** at finite imaginary $\mu$.

We determine the strength of the vector interaction from the lattice results in order to construct a **reliable effective model**.

We predict the QCD phase diagram at high densities and the mass-radius relation of neutron stars.
In the imaginary $\mu (\mu_1)$ region, QCD has two properties.

- Roberge-Weiss (RW) periodicity

$$Z \left( \frac{\mu_1}{T} \right) = Z \left( \frac{\mu_1}{T} + \frac{2\pi}{3} \right)$$

- RW transition

Lattice setup

Lattice action: renormalization group improved Iwasaki gauge action and clover improved Wilson fermion action ($N_f=2$)

Lattice size: $N_x \times N_y \times N_z \times N_t = 8^2 \times 16 \times 4$

$T_c \simeq 171\text{MeV}$ at $\mu = 0$


$m_{PS}/m_V = 0.80$ line of constant physics


Parameters and temperatures

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\beta$</th>
<th>$T/T_c$</th>
<th>$\mu_I/T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.141139</td>
<td>1.80</td>
<td>0.93</td>
<td>$\pi/3$</td>
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<tr>
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<td>1.85</td>
<td>0.99</td>
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<tr>
<td>0.136931</td>
<td>2.00</td>
<td>1.35</td>
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Chemical potential

$\mu_I/T = 0 \sim \pi/3$
Quark number density at imaginary chemical potential

The quark number density is an odd function of $\mu$.

Confinement region: $n_q = \sum_{n} F_0^{(n)} \sin \left( 3n \frac{\mu I}{T} \right)$

Deconfinement region: $n_q = \sum_{n} p_0^{(2n+1)} \left( \frac{\mu I}{T} \right)^{2n+1}$

All temperatures

Below $T_c$
Extrapolation from imaginary to real chemical potential

At imaginary chemical potential

\[ T < T_c \]
\[
\begin{align*}
{f_F}^1 \left( \frac{\mu_1}{T} \right) &= F_o^{(1)} \sinh \left( 3 \frac{\mu_1}{T} \right) \\
{f_F}^2 \left( \frac{\mu_1}{T} \right) &= F_o^{(1)} \sinh \left( 3 \frac{\mu_1}{T} \right) + F_o^{(2)} \sinh \left( 6 \frac{\mu_1}{T} \right)
\end{align*}
\]

\[ T > T_{RW} \]
\[
\begin{align*}
{f_p}^3 \left( \frac{\mu_1}{T} \right) &= p_o^{(1)} \left( \frac{\mu_1}{T} \right) + p_o^{(3)} \left( \frac{\mu_1}{T} \right)^3 \\
{f_p}^5 \left( \frac{\mu_1}{T} \right) &= p_o^{(1)} \left( \frac{\mu_1}{T} \right) + p_o^{(3)} \left( \frac{\mu_1}{T} \right)^3 + p_o^{(5)} \left( \frac{\mu_1}{T} \right)^5
\end{align*}
\]

At real chemical potential

\[ T < T_c \]
\[
\begin{align*}
{f_F}^1 \left( \frac{\mu}{T} \right) &= F_o^{(1)} \sin \left( 3 \frac{\mu}{T} \right) \\
{f_F}^2 \left( \frac{\mu}{T} \right) &= F_o^{(1)} \sin \left( 3 \frac{\mu}{T} \right) + F_o^{(2)} \sin \left( 6 \frac{\mu}{T} \right)
\end{align*}
\]

\[ T > T_{RW} \]
\[
\begin{align*}
{f_p}^3 \left( \frac{\mu}{T} \right) &= p_o^{(1)} \left( \frac{\mu}{T} \right) - p_o^{(3)} \left( \frac{\mu}{T} \right)^3 \\
{f_p}^5 \left( \frac{\mu}{T} \right) &= p_o^{(1)} \left( \frac{\mu}{T} \right) - p_o^{(3)} \left( \frac{\mu}{T} \right)^3 + p_o^{(5)} \left( \frac{\mu}{T} \right)^5
\end{align*}
\]
Extrapolation from imaginary to real chemical potential

At imaginary chemical potential

\[
T < T_c
\]

\[
f_F^1 \left( \frac{\mu_I}{T} \right) = F_o^{(1)} \sin \left( \frac{3 \mu_I}{T} \right)
\]

\[
f_F^2 \left( \frac{\mu_I}{T} \right) = F_o^{(1)} \sin \left( \frac{3 \mu_I}{T} \right) + F_o^{(2)} \sin \left( \frac{6 \mu_I}{T} \right)
\]

\[
T > T_{RW}
\]

\[
f_p^3 \left( \frac{\mu_I}{T} \right) = p_o^{(1)} \left( \frac{\mu_I}{T} \right) + p_o^{(3)} \left( \frac{\mu_I}{T} \right)^3
\]

\[
f_p^5 \left( \frac{\mu_I}{T} \right) = p_o^{(1)} \left( \frac{\mu_I}{T} \right) + p_o^{(3)} \left( \frac{\mu_I}{T} \right)^3 + p_o^{(5)} \left( \frac{\mu_I}{T} \right)^5
\]
Extrapolation from imaginary to real chemical potential

At real chemical potential

\[ T < T_c \]

\[
\begin{align*}
    f_F^1 \left( \frac{\mu}{T} \right) &= F_o^{(1)} \sinh \left( \frac{3\mu}{T} \right) \\
    f_F^2 \left( \frac{\mu}{T} \right) &= F_o^{(1)} \sinh \left( \frac{3\mu}{T} \right) + F_o^{(2)} \sinh \left( \frac{6\mu}{T} \right)
\end{align*}
\]

\[
\frac{F_o^{(2)} \sinh(6\mu/T)}{F_o^{(1)} \sinh(3\mu/T)} \sim 0.1
\]

\[ T > T_{RW} \]

\[
\begin{align*}
    f_p^3 \left( \frac{\mu}{T} \right) &= p_o^{(1)} \left( \frac{\mu}{T} \right) - p_o^{(3)} \left( \frac{\mu}{T} \right)^3 \\
    f_p^5 \left( \frac{\mu}{T} \right) &= p_o^{(1)} \left( \frac{\mu}{T} \right) - p_o^{(3)} \left( \frac{\mu}{T} \right)^3 + p_o^{(5)} \left( \frac{\mu}{T} \right)^5
\end{align*}
\]

\[
\frac{p_o^{(5)} \left( \frac{\mu}{T} \right)^5}{p_o^{(3)} \left( \frac{\mu}{T} \right)^3} \sim 0.1
\]
Extrapolation from imaginary to real chemical potential

At real chemical potential

\[
T < T_c \begin{cases} 
  f_1^F \left( \frac{\mu}{T} \right) = F_0^{(1)} \sinh \left( \frac{3\mu}{T} \right) \\
  f_2^F \left( \frac{\mu}{T} \right) = F_0^{(1)} \sinh \left( \frac{3\mu}{T} \right) + F_0^{(2)} \sinh \left( \frac{6\mu}{T} \right)
\end{cases}
\]

\[
\frac{F_0^{(2)} \sinh(6\mu/T)}{F_0^{(1)} \sinh(3\mu/T)} \sim 0.1
\]

\[
T/T_c=0.99 : \quad \mu/T \lesssim 0.445
\]

\[
T > T_{RW} \begin{cases} 
  f_3^F \left( \frac{\mu}{T} \right) = p_0^{(1)} \left( \frac{\mu}{T} \right) - p_0^{(3)} \left( \frac{\mu}{T} \right)^3 \\
  f_5^F \left( \frac{\mu}{T} \right) = p_0^{(1)} \left( \frac{\mu}{T} \right) - p_0^{(3)} \left( \frac{\mu}{T} \right)^3 + p_0^{(5)} \left( \frac{\mu}{T} \right)^5
\end{cases}
\]

\[
\frac{p_0^{(5)} \left( \frac{\mu}{T} \right)^5}{p_0^{(3)} \left( \frac{\mu}{T} \right)^3} \sim 0.1
\]

\[
T/T_c=1.20 : \quad \mu/T \lesssim 0.723
\]
Extrapolation from imaginary to real chemical potential

The upper limit of the reliable region

The reliable region is expanded as temperature increases.

A guideline
Two-phase model

- We consider the two-phase model in order to treat the quark-hadron phase transition.

**Hadron phase**: Walecka model

\[
L_{\text{QHD}} = \bar{\psi} (i\gamma^\mu \partial_\mu - m_N - g_N \varphi - g_\omega \omega^\mu \gamma_\mu) \psi \\
+ \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{4} \Omega^{\mu\nu} \Omega_{\mu\nu} - U_{\text{QHD}} (\varphi, \omega_\mu)
\]

\[
\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu
\]

\[
U_{\text{QHD}} = \frac{1}{2} m_\sigma^2 \varphi^2 + \frac{1}{3} g_2 \varphi^3 + \frac{1}{4} g_3 \varphi^4 - \frac{1}{2} m_\omega^2 \omega^\mu \omega_\mu
\]


**Quark phase**: The entanglement-PNJL model (EPNJL model)

\[
L_{\text{EPNJL}} = \bar{q} (\gamma^\nu D_\nu + \hat{m}_0 - \gamma_4 \hat{\mu}) q + G_S(\Phi) [(\bar{q}q)^2 + (\bar{q}i\gamma_5 q)^2] \\
- G_V(\Phi) (\bar{q}\gamma_\mu q)^2 + U(\Phi[A], \Phi^*[A], T)
\]

\[
G_S(\Phi) = G_S[1 - \alpha_1 \Phi \Phi^* - \alpha_2 (\Phi^3 + \Phi^{*3})], \quad G_V(\Phi) = \alpha_3 G_S(\Phi)
\]

Determination of the strength of the vector interaction

- The normalized quark number density \( n_q / n_{SB} \)

<table>
<thead>
<tr>
<th>( T/T_c )</th>
<th>( n_q / n_{SB} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.20</td>
<td>0.5303</td>
</tr>
<tr>
<td>1.35</td>
<td>0.5587</td>
</tr>
</tbody>
</table>

The values of \( n_q / n_{SB} \) in the limit of \( \mu = 0 \).

The EPNJL model with \( G_V \) has good agreement with the lattice results, when \( \alpha_3 = G_V(\Phi) / G_S(\Phi) = 0.33 \).

Model prediction

- The QCD phase diagram

\[ T \text{ [GeV]} \]

\[ \mu_B \text{ [GeV]} \]

\( w/ \ G_V \)

\( w/o \ G_V \)

\( \mu_B \) : Baryon chemical potential, \( \mu_B = 3\mu \)

The critical chemical potential at \( T=0 \) is \( \mu_B^{(c)} \sim 1.6[\text{GeV}] \).

- Mass-radius relation of neutron stars

\[ M/M_{\odot} \]

\[ R[\text{km}] \]

Demorest: w/o vector, w/ vector

The present model prediction is consistent with the two solar mass observations of neutron stars.

Summary

• We calculate the quark number density at imaginary chemical potential.
• We study the extrapolation of the quark number density from imaginary to real chemical potential. The upper limit of reliable extrapolation region becomes large as temperature increases from $T_c$.
• We have determined the strength of the vector interaction in the EPNJL model from $n_q/n_{SB}$ calculated by lattice QCD simulations.
• The model prediction is consistent with the two solar mass observations of neutron stars. In this model, the QCD phase transition takes place in the inner core of neutron stars if the mass is larger than two solar masses.
• The critical chemical potential at $T=0$ is $\mu_B^{(c)} \sim 1.6\,[\text{GeV}]$. 