Lattice N=4 SYM

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The lattice SUSY problem

\[ \{ Q_\alpha, \bar{Q}_{\dot{\alpha}} \} = 2\sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu \]

- \( P_\mu \) generator of infinitesimal translations.
- Broken on lattice.
- Only discrete subgroup preserved.
- with lattice spacing \( a \)

\[ x \rightarrow T_\mu x = x + a\hat{\mu} \]
Failure of Leibnitz rule

- So why not just change the algebra to
  \[ \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2i\sigma^{\mu}_{\alpha\dot{\alpha}} \frac{1}{a} (T_\mu - 1) \equiv 2i\sigma^{\mu}_{\alpha\dot{\alpha}} \nabla_\mu \]

- Notice that
  \[ \nabla_\mu \phi(x) = \frac{1}{a} [\phi(x + a\hat{\mu}) - \phi(x)] = \partial_\mu \phi(x) + \frac{a}{2} \partial^2_\mu \phi(x) + O(a^2) \]

- Problem: essential difference between these (\( \partial_\mu \) and \( \nabla_\mu \)) at finite lattice spacing.

- The Leibnitz rule.
  \[ \nabla_\mu [\phi(x)\chi(x)] = \frac{1}{a} [\phi(x + a\hat{\mu})\chi(x + a\hat{\mu}) - \phi(x)\chi(x)] \]
  \[ = \nabla_\mu \phi(x)\chi(x) + \phi(x)\nabla_\mu \chi(x) + a\nabla_\mu \phi(x)\nabla_\mu \chi(x) \]

Dondi & Nicolai, 1977
Problems for interacting theory

- SUSY algebra on elementary fields
- Not on polynomials of fields
• Result: $O(a)$ artifact:

$$
\delta_\epsilon S = i[\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}, S] = ia(\epsilon^\alpha X_\alpha - \bar{\epsilon}_{\dot{\alpha}} \bar{X}_{\dot{\alpha}})
$$

• But we send $a \to 0$ at the end of our calculations, so who cares?
Counterterm/renormalization in N=1 SYM

- Failure of Leibnitz rule $\Rightarrow O(a)$ term

\[ \langle \partial_\mu S_\mu(x)O(y) \rangle = m_0 \langle \chi(x)O(y) \rangle + a \langle O_{11/2}(x)O(y) \rangle + \text{contact terms} \]

\[ O_{11/2}^R = Z_{11/2}O_{11/2} + \frac{1}{a} (Z_S - 1) \partial_\mu S_\mu + \frac{1}{a} Z_T \partial_\mu T_\mu + \frac{1}{a^2} Z \chi \chi + \sum_j Z_{11/2}^{(j)} O_{11/2}^{(j)R} \]

\[ S_\mu = -\sigma_{\rho\nu} \gamma_\mu \text{Tr}(F_{\rho\nu} \lambda), \quad T_\mu = 2\gamma_\nu \text{Tr}(F_{\mu\nu} \lambda), \quad \chi = \sigma_{\mu\nu} \text{Tr}(F_{\mu\nu} \lambda) \]

\[ \langle \partial_\mu S_\mu^R(x)O^R(y) \rangle = (m_0 - \frac{Z_\chi}{a}) \langle \chi(x)O^R(y) \rangle + a \cdot \text{finite + contact terms} \]

\[ S_\mu^R = Z_S S_\mu + Z_T T_\mu \]

- $Z_T/Z_S$ must be determined to test for SUSY
- Fine-tuning just $m_0$ agrees w/ expectations

Farchioni et al. 2001
Wilson fermion N=4 SYM

• How bad is an entirely conventional approach, say using Wilson fermions?

• We cannot impose SU(4) R symmetry because it is chiral, and Wilson fermions violate chiral symmetry.

• However, we can impose SO(4) flavor symmetry with the fermions in 4’s (vector) and the scalars in 6 (antisymmetric tensor).

• To determine the number of fine-tunings, we write down the most general renormalizable action consistent with these constraints.
\[ S = \int d^4x \ Tr\{-\frac{1}{2g_r^2}F_{\mu\nu}F_{\mu\nu} + \frac{i}{g_r^2}\bar{\lambda}_i\sigma^\mu D_\mu \lambda_i + \frac{1}{g_r^2}D_\mu \phi_mD_\mu \phi_m + m_\phi^2\phi_m\phi_m \\
+ m_\lambda(\lambda_i\lambda_i + \bar{\lambda}_i\bar{\lambda}_i) + \kappa_1\phi_m\phi_m\phi_n\phi_n + \kappa_2\phi_m\phi_n\phi_m\phi_n + y_1(\lambda_i[\phi_{ij}, \lambda_j] + \bar{\lambda}_i[\phi_{ij}, \bar{\lambda}_j]) \\
+ y_2\epsilon_{ijkl}(\lambda_i[\phi_{jk}, \lambda_l] + \bar{\lambda}_i[\phi_{jk}, \bar{\lambda}_l])\} \\
+ \int d^4x \ \{\kappa_3(Tr\phi_m\phi_m)^2 + \kappa_4Tr\phi_m\phi_nTr\phi_m\phi_n\} \]

• We achieved the first three coefficients by rescaling the fermion and scalar.
• We are left with 8 parameters to fine-tune: hopeless.
• Goal of modern formulations: reduce the number of fine-tunings.
• Method: lattice symmetries that restrict the long distance effective action.
Twisted N=4

- We form the twisted rotation group from an SO(4) subgroup of the flavor (R symmetry) group SU(4):

  \[ SO(4)' = \text{diag}[SO(4)_E \times SO(4)_R] \]

  \[ \lambda^I_{\alpha} \rightarrow \Psi_{\alpha\beta} \]

- Then it is natural to expand on the five gamma matrices (a=1,…,5):

  \[ \Psi = \frac{1}{2} \eta + \psi_a \gamma_a + \frac{i}{2} \chi_{ab} [\gamma_a, \gamma_b] \]
Given the 5d language of the fermions, it is also natural to package up the bosons in a 5d way:

\[ \mathcal{A}_a = A_a + iB_a, \quad \overline{\mathcal{A}}_a = A_a - iB_a \]
Q invariant action

\[ S = \frac{1}{2g^2} (Q \Lambda + S_{\text{closed}}) \]

\[ \Lambda = \int d^4 x \, \text{Tr}(\chi_{mn} F_{mn} + \eta [\overline{D}_m, D_m] - \frac{1}{2} \eta d) \]

\[ S_{\text{closed}} = -\frac{1}{4} \int d^4 x \, \text{Tr} \epsilon_{mnrpq} \chi_{pq} \overline{D}_r \chi_{mn} \]

\[ QA_m = \psi_m, \quad Q\psi_m = 0, \quad Q\overline{A}_m = 0 \]

\[ Q\chi_{mn} = -\overline{F}_{mn}, \quad Q\eta = d, \quad Qd = 0 \]
Lattice discretization

- In the lattice theory we switch to link variables for the gauge fields

\[ U_a(x) = e^{A_a(x)}, \quad \overline{U}_a(x) = U_a^\dagger(x) = e^{-A_a(x)} \]

- Physically, \( U_a(x) \) is a link that goes from

\[ x \to x + a e_a \]

and \( \overline{U}_a(x) \) is a link between the same pair of sites but going in the opposite direction.

Catterall, 0712.2532
• The five $e_\alpha$ are basis vectors of the $A_4^*$ lattice.

\begin{align*}
e_1 &= \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right) \\
e_2 &= \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right) \\
e_3 &= \left( 0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}} \right) \\
e_4 &= \left( 0, 0, -\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right) \\
e_5 &= \left( 0, 0, 0, -\frac{4}{\sqrt{20}} \right)
\end{align*}
These vectors are also used to construct an important orthogonal matrix

\[ O_{a\mu} = e_\mu^a, \quad O_{a5} = \frac{1}{\sqrt{5}}, \quad a = 1, \ldots, 5, \quad \mu = 0, \ldots, 3 \]

The bosonic fields of the usual formulation of N=4 SYM are obtained as

\[ \mathcal{V}_\mu = A_\mu + i\phi_{\mu+1} = O_{a\mu}A_a, \quad \phi_5 + i\phi_6 = O_{a5}A_a \]

Unsal, hep-th/0603046
Under gauge transformations, the link variables transform in the usual way:

\[
\mathcal{U}_a(x) \rightarrow g(x)\mathcal{U}_a(x)g^\dagger(x + e_a)
\]

\[
\overline{\mathcal{U}}_a(x) \rightarrow g(x + e_a)\overline{\mathcal{U}}_a(x)g^\dagger(x)
\]

The transformations of all of our other fields are dictated by this index related prescription

\[
\eta(x) \rightarrow g(x)\eta(x)g^\dagger(x), \quad \psi_a(x) \rightarrow g(x)\psi_a(x)g^\dagger(x + e_a)
\]

\[
\chi_{ab}(x) \rightarrow g(x + e_a + e_b)\chi_{ab}(x)g^\dagger(x)
\]
• Next we have to figure out how to discretize the covariant derivatives.

• For the field strength, the following has the right continuum limit:

\[ F_{ab}(x) = D_a^{(+)} U_b(x) = U_a(x) U_b(x + e_a) - U_b(x) U_a(x + e_b) \]

• We also introduce derivatives for term 2 of the action:

\[ [\bar{D}_a, D_a] \to \bar{D}_a^{(-)} U_a(x) = U_a(x) \bar{U}_a(x) - \bar{U}_a(x - e_a) U_a(x - e_a) \]

• It is easy to see this has the right continuum limit.
The lattice version of $Q$ transformations is a fairly straightforward transcription from the continuum:

\[ Q U_a = \psi_a, \quad Q \psi_a = 0, \quad Q \overline{U}_a = 0 \]

\[ Q \chi_{ab}(x) = \overline{F}_{ab}(x) \equiv \overline{U}_b(x + e_a) U_a(x) - \overline{U}_a(x + e_b) U_b(x) \]

\[ Q \eta = d, \quad Q d = 0 \]
• Then the Q exact action requires the replacements in the “gauge fermion”

\[ \chi_{ab} F_{ab} : \quad F_{ab}(x) = D_a^{(+)} U_b(x) = U_a(x) U_b(x + e_a) - U_b(x) U_a(x + e_b) \]

\[ \eta[D_a, D_a] \rightarrow \eta D_a^{(-)} U_a(x) = \eta(x) [U_a(x) U_a(x) - U_a(x - e_a) U_a(x - e_a)] \]

\[ S_{Q-\text{exact}} = \sum_x Q \text{Tr} \{ \chi_{ab} F_{ab} + \eta D_a^{(-)} U_a - \frac{1}{2} \eta d \} \]

• The action has a shift invariance (using equations of motion):

\[ \eta \rightarrow \eta + \epsilon \mathbf{1} \]
The Q-closed term is a little more work

$$\chi_{de} \overline{D}_c^{(-)} \chi_{ab} = \chi_{de}(\cdots) [\chi_{ab}(x) \overline{U}_c(\cdots) - \overline{U}_c(\cdots) \chi_{ab}(x - e_c)]$$

$$= \chi_{de}(x - e_d - e_e - e_c) [\chi_{ab}(x) \overline{U}_c(x - e_c) - \overline{U}_c(x - e_c + e_a + e_b) \chi_{ab}(x - e_c)]$$
• For the closure of this term, an important property is the lattice Bianchi identity

\[ \epsilon_{abcde} \overline{D}_c^{(-)} \overline{F}_{ab} = 0 \]
• If we have a renormalization scheme that preserves the lattice structure (including the symmetries), then we can enumerate the terms in the most general long distance effective action.

• There is only one Q-closed operator allowed by the lattice symmetries and it is already present.

\[ S_{Q\text{-closed}} = -\frac{\alpha^4}{4} \sum_x \epsilon_{abcde} \text{Tr}(\chi_{de} \overline{D}_c^{(-)} \chi_{ab}) \]
Q-exact terms must be fermionic, so they take the general form

\[ Q \text{Tr}[\Psi f(U, \bar{U})] \]

Taking into account the lattice gauge invariance and \( S_5 \) symmetry, we have (up to irrelevant operators)

\[ Q \text{Tr}(\chi_{ab} U_a U_b) - Q \text{Tr}(\chi_{ab} U_b U_a) = Q \text{Tr}(\chi_{ab} D_a^{(+) U_b}) \]
With $\eta$ we have lots of operators but shift invariance reduces to a few combinations

$$Q\text{Tr}[\eta(x)\overline{U}_a(x-e_a)U_a(x-e_a)], \quad Q\text{Tr}(\eta d)$$

$$Q\text{Tr}[\eta(x)U_a(x)\overline{U}_a(x)], \quad Q\text{Tr}\eta, \quad Q\{\text{Tr}\eta\text{Tr}(U_a\overline{U}_a)\}$$

$$\downarrow$$

$$Q\text{Tr}[\eta\overline{D}_a(-)U_a], \quad Q\text{Tr}(\eta d)$$

$$Q\text{Tr}(\eta U_a\overline{U}_a) - \frac{1}{N} Q\{\text{Tr}\eta\text{Tr}(U_a\overline{U}_a)\}$$
Thus the renormalizable long distance theory is

\[ S = \sum_x Q \text{Tr}\{\alpha_1 \chi_{ab} D_a^{(+)} U_b + \alpha_2 \eta \overline{D}_a^{(-)} U_a - \frac{\alpha_3}{2} \eta d}\]

\[ + \sum_x \beta_1 Q\{\text{Tr}(\eta U_a \overline{U}_a) - \frac{1}{N} \text{Tr}\eta \text{Tr}(U_a \overline{U}_a)\} \]

\[ - \frac{\alpha_4}{4} \sum_x \epsilon_{abcd} \text{Tr}(\chi_{de} \overline{D}_c^{(-)} \chi_{ab}) \]

Seems to be 4 fine-tunings. This is far fewer than a naive approach would yield.
• Act with $Q$ and then rescale the fermions and auxiliary field:
  \[ \eta \rightarrow \lambda_\eta \eta, \quad \chi_{ab} \rightarrow \lambda_\chi \chi_{ab}, \quad \psi_a \rightarrow \lambda_\psi \psi_a, \quad d \rightarrow \lambda_d d \]

• The action becomes

\[
\text{Tr}\{-\alpha_1 \mathcal{F}_{ab} \mathcal{F}_{ab} - \alpha_1 \lambda_\chi \lambda_\psi \chi_{ab} \mathcal{D}^{(+)}_{[a} \psi_{b]} + \alpha_2 \lambda_d d \mathcal{D}^{(-)}_a \mathcal{U}_a - \alpha_2 \lambda_\eta \lambda_\psi \eta \mathcal{D}^{(-)}_a \psi_a \\
- \frac{\alpha_3}{2} \lambda_d^2 d^2 - \frac{\alpha_4}{4} \lambda_\chi^2 \epsilon_{abcde} \chi_{de} \mathcal{D}^{(-)}_c \chi_{ab} \} + \beta \{ \lambda_d \text{Tr}(d \mathcal{U}_a \bar{\mathcal{U}}_a) - \lambda_\eta \lambda_\psi \text{Tr}(\eta \psi_a \bar{\mathcal{U}}_a) \\
- \frac{1}{N} \lambda_d \text{Tr}d \text{Tr}(\mathcal{U}_a \bar{\mathcal{U}}_a) + \frac{1}{N} \lambda_\eta \lambda_\psi \text{Tr} \eta \text{Tr}(\psi_a \bar{\mathcal{U}}_a) \}\]
• Use freedom to set

\[ \alpha_1 \lambda \chi \lambda \psi = \alpha_1, \quad \alpha_2 \lambda_d = \alpha_1, \quad \alpha_2 \lambda \eta \lambda \psi = \alpha_1, \quad \alpha_4 \lambda^2 \chi = \alpha_1 \]

• Solution:

\[ \lambda_\eta = \sqrt{\frac{\alpha_1^3}{\alpha_4 \alpha^2}}, \quad \lambda \chi = \frac{1}{\lambda \psi} = \sqrt{\frac{\alpha_1}{\alpha_4}}, \quad \lambda_d = \frac{\alpha_1}{\alpha_2} \]

• Define

\[ \alpha'_3 = \alpha_3 \left( \frac{\alpha_1}{\alpha_2} \right)^2, \quad \beta' = \beta \frac{\alpha_1}{\alpha_2} \]
• Action is now

\[
\text{Tr}\left\{ -\alpha_1 \bar{F}_{ab} F_{ab} - \alpha_1 \chi_{ab} D_{[a}^{(+)} \psi_{b]} + \alpha_1 d \bar{D}_{a}^{(-)} U_a - \alpha_1 \eta D_{a}^{(-)} \psi_a \\
- \frac{\alpha_3'}{2} d^2 - \frac{\alpha_1}{4} \epsilon_{abcd} \chi_{de} \bar{D}_{c}^{(-)} \chi_{ab} \right\} + \beta' \left\{ \text{Tr}(d U_a \bar{U}_a) - \text{Tr}(\eta \psi_a U_a) \right\} \\
- \frac{1}{N} \text{TrdTr}(U_a \bar{U}_a) + \frac{1}{N} \text{Tr}\eta \text{Tr}(\psi_a \bar{U}_a) \}
\]

• Only 2 fine-tunings:

\[\alpha'_3 \to \alpha_1, \quad \beta' \to 0\]

Cf. clover fermions, also 2 fine-tunings.
One less, one left

- Actually, we showed in our previous work that the moduli space is not lifted at any order of lattice perturbation theory.
- Here it is crucial that the partition function is a topological quantity, so that the one-loop result holds to all orders.
- But the $\beta$ term would lift the moduli space, so it is actually forbidden.
- Thus we are left with a single fine-tuning.
The other 15 SUSYs

- The supercharge also has the KD structure

\[ Q = Q + Q_a \gamma_a + \frac{i}{2} Q_{ab} [\gamma_a, \gamma_b] \]

- We can work out the other 15 SUSYs using discrete R invariances of the action (on-shell). For a fixed and b,c,etc. not equal to a,

\[ R_a : \]

\[ \eta \rightarrow 2\psi_a, \quad \psi_a \rightarrow \frac{1}{2} \eta, \quad \psi_b \rightarrow -\chi_{ab} \]

\[ \chi_{ab} \rightarrow -\psi_b, \quad \chi_{bc} \rightarrow \frac{1}{2} \epsilon_{bcagh} \chi_{gh} \]

\[ \mathcal{D}_a \rightarrow \mathcal{D}_a, \quad \overline{\mathcal{D}}_a \rightarrow \overline{\mathcal{D}}_a, \quad \mathcal{D}_b \rightarrow \overline{\mathcal{D}}_b, \quad \overline{\mathcal{D}}_b \rightarrow \mathcal{D}_b \]
This leads to the five SUSYs

\[ Q_a A_b = \frac{1}{2} \delta_{ab} \eta, \quad Q_a \bar{A}_b = -\chi_{ab}, \quad Q_a \psi_b = \frac{1}{2} \delta_{ab} d_a + (1 - \delta_{ab}) [\bar{D}_a, D_b] \]

\[ Q_a \chi_{bc} = -\frac{1}{2} \epsilon_{abcde} [D_d, D_e], \quad Q_a \eta = 0, \quad Q_a d_a = 0 \]

\[ d_a = [\bar{D}_a, D_a] - \sum_{m \neq a} [\bar{D}_m, D_m] \]
• Then there are 10 other discrete R symmetries:

\[ R_{ab} : \]

\[
\eta \to 2\chi_{ab}, \quad \psi_a \to \psi_b, \quad \psi_b \to -\psi_a, \quad \psi_c \to \frac{1}{2} \epsilon_{cabgh} \chi_{gh}
\]

\[
\chi_{ab} \to -\frac{1}{2} \eta, \quad \chi_{ac} \to \chi_{bc}, \quad \chi_{bc} \to -\chi_{ac}, \quad \chi_{gh} \to -\epsilon_{ghabc} \psi_c
\]

\[
\mathcal{D}_{a,b} \to \overline{\mathcal{D}}_{a,b}, \quad \overline{\mathcal{D}}_{a,b} \to \mathcal{D}_{a,b}, \quad \mathcal{D}_c \to \mathcal{D}_c, \quad \overline{\mathcal{D}}_c \to \overline{\mathcal{D}}_c
\]
Then one gets 10 more supercharges by applying these to $Q$:

\[ Q_{ab} A_c = \frac{1}{2} \epsilon_{abcgh} \chi_{gh}, \quad Q_{ab} \bar{A}_c = \delta_{ac} \psi_b - \delta_{bc} \psi_a, \quad Q_{ab} \psi_c = \epsilon_{abcgh} F_{gh} \]

\[ Q_{ab} \chi_{cd} = \frac{1}{4} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) d_{ab} + \delta_{ac} [\mathcal{D}_b, \bar{\mathcal{D}}_d] - \delta_{bc} [\mathcal{D}_a, \bar{\mathcal{D}}_d] \]

\[ Q_{ab} \eta = 2 \mathcal{F}_{ab}, \quad Q_{ab} d_{ab} = 0 \]

\[ d_{ab} = - [\bar{\mathcal{D}}_a, \mathcal{D}_a] - [\bar{\mathcal{D}}_a, \mathcal{D}_a] + \sum_{m \neq a,b} [\bar{\mathcal{D}}_m, \mathcal{D}_m] \]

The equation $Q_{ab} d_{ab} = 0$ requires the EOM.
R_a and renormalization

- Returning to

\[ Q\text{Tr}\{\alpha_1 \chi_{ab} F_{ab} + \alpha_2 \eta [\overline{D}_a, D_a] - \frac{\alpha_3}{2} \eta d\} \]

\[ -\frac{\alpha_4}{4} \epsilon_{abcde} \chi_{de} \overline{D}_c \chi_{ab} + \beta \{ \cdots \} \]

- Eliminate auxiliary

\[ \text{Tr}\{ -\alpha_1 \overline{F}_{ab} F_{ab} + \frac{\alpha_2^2}{2\alpha_3} [\overline{D}_a, D_a]^2 - \alpha_1 \chi_{ab} D[a \psi_b] \}

\[ -\alpha_2 \eta \overline{D}_a \psi_a - \frac{\alpha_4}{4} \epsilon_{abcde} \chi_{de} \overline{D}_c \chi_{ab} + \beta \{ \cdots \} \]
• Apply $R_a$ to this and demand invariance

• In bosonic sector terms are interchanged, requiring

\[ \alpha_1 = \frac{\alpha_2}{\alpha_3}, \quad \beta = 0 \]

• In fermionic sector terms are interchanged, requiring

\[ \alpha_1 = \alpha_2 = \alpha_4, \quad \beta = 0 \]

• Thus $R_a$ invariance forces SUSY long distance theory.
• Recall

\[ \mathcal{U}_a = e^{\mathcal{A}_a}, \quad \overline{\mathcal{U}}_a = e^{-\mathcal{A}_a} \]

• Implies under \( R_a \)

\[ \mathcal{U}_a \rightarrow \mathcal{U}_a, \quad \overline{\mathcal{U}}_a \rightarrow \overline{\mathcal{U}}_a, \quad \mathcal{U}_b \rightarrow \overline{\mathcal{U}}_b^{-1}, \quad \overline{\mathcal{U}}_b \rightarrow \mathcal{U}_b^{-1} \]

• Thus a simple test of \( R_a \) restoration, and hence full N=4 SUSY restoration is

\[
\langle \text{Tr}\{\mathcal{U}_a(x)\mathcal{U}_b(x + e_a)\overline{\mathcal{U}}_a(x + e_b)\overline{\mathcal{U}}_b(x)\} \rangle \\
= \langle \text{Tr}\{\mathcal{U}_a(x)\overline{\mathcal{U}}_b^{-1}(x + e_a)\overline{\mathcal{U}}_a(x + e_b)\mathcal{U}_b^{-1}(x)\} \rangle
\]
• Amazing
• Due to exact symmetries of lattice theory
$\mathcal{N} = 4$ SYM, U(2)

$8^3 \times 24$

$2\langle \frac{\mathcal{P}_- - \mathcal{P}_+}{\mathcal{P}_+ + \mathcal{P}_-} \rangle$

$\lambda_{\text{lat}}$

- $\mu = 0.2, \kappa = 0.6$
- $\mu = 0.2, \kappa = 0.8$
- $\mu = 0.4, \kappa = 0.6$
- $\mu = 0.4, \kappa = 0.8$
- $\mu = 0.8, \kappa = 0.6$
- $\mu = 0.8, \kappa = 0.8$
Blocking

- The arguments about the long distance effective action only hold if there is a real space renormalization group which preserves the lattice structure.
- This means that $Q, S_5$, gauge invariance and geometric interpretation of fields should survive the flow.
- Here we provide an explicit construction.
• The original lattice $\Lambda$ may be described by

$$\Lambda = \{a \sum_{\mu=1}^{4} n_{\mu} e_{\mu} | n \in \mathbb{Z}^{4}\}$$

• where the $e_{\mu}$ are the first four of the five (degenerate) basis vectors of the $A_{4}^{*}$ lattice described above.

• The blocked lattice will merely be doubled in every direction:

$$\Lambda' = \{2a \sum_{\mu=1}^{4} n_{\mu} e_{\mu} | n \in \mathbb{Z}^{4}\}$$

• From this point forward we will work in lattice units, setting $a = 1$
• The blocked fields will be denoted by primes.
• They must begin and end on sites of the blocked lattice $\Lambda'$.
• We want the geometric interpretation to survive the blocking.
• For example, $\chi'_{ab}(x)$ must begin on site $x + 2e_a + 2e_b$ and end on site $x$. 

\[
\chi_{ab}(x) \rightarrow x + e_a + e_b \rightarrow \chi_{ab}(x) \\
\chi'_{ab}(x) \rightarrow x + 2e_a + 2e_b \rightarrow \chi_{ab}(x)
\]
One choice that achieves this is the following:

\[
\begin{align*}
U'_a(x) &= U_a(x)U_a(x + e_a), \quad \overline{U}'_a(x) = \overline{U}_a(x + e_a)\overline{U}_a(x) \\
d'(x) &= d(x), \quad \eta'(x) = \eta(x) \\
\psi'_a(x) &= \psi_a(x)U_a(x + e_a) + U_a(x)\psi_a(x + e_a) \\
\chi'_{ab}(x) &= \frac{1}{2} \left[ \overline{U}_a(x + e_a + 2e_b)\overline{U}_b(x + e_a + e_b)\chi_{ab}(x) \\
&\quad + \overline{U}_b(x + 2e_a + e_b)\overline{U}_a(x + e_a + e_b)\chi_{ab}(x) \right] \\
&\quad + \left[ \overline{U}_a(x + e_a + 2e_b)\chi_{ab}(x + e_b)\overline{U}_b(x) \\
&\quad + \overline{U}_b(x + 2e_a + e_b)\chi_{ab}(x + e_a)\overline{U}_a(x) \right] \\
&\quad + \frac{1}{2} \left[ \chi_{ab}(x + e_a + e_b)\overline{U}_a(x + e_b)\overline{U}_b(x) \\
&\quad + \chi_{ab}(x + e_a + e_b)\overline{U}_b(x + e_a)\overline{U}_a(x) \right]
\end{align*}
\]
This choice preserves the Q algebra, namely

\[ Q \mathcal{U}'_a = \psi_a \mathcal{U}_a + \mathcal{U}_a \psi_a = \psi'_a \]
\[ Q \psi'_a = -\psi_a \psi_a + \psi_a \psi_a = 0 \]
\[ Q \mathcal{U}'_a = 0 \quad Q \eta' = d = d' \quad Q d' = 0 \]
\[ Q \chi'_{ab} = \mathcal{F}'_{ab} \]

\[ \mathcal{F}'_{ab}(x) = \mathcal{U}'_b(x + 2e_a)\mathcal{U}'_a(x) - \mathcal{U}'_a(x + 2e_b)\mathcal{U}'_b(x) \]

The last result, for \( Q \chi'_{ab} \), is the only one that requires any significant computation.
Future directions

- RSRG calculations: MCRG
- CTs, finite parts, two loops
- Other 15 SUSYs after RSRG, fine-tuning
- Strong coupling issues