# Effective Polyakov line actions

## and their solution at finite chemical potential

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# An “indirect” approach to the sign problem

Finite chemical potential $\rightarrow$ straightforward importance sampling is impossible.

Our approach: First map the gauge-matter theory onto a theory with fewer d.o.f. – a **Polyakov line action (or “SU(3) spin”) model**.

There is still a sign problem that must be faced. I will deal with that in two ways:

## Methods

1. mean field theory *(Splittorff and JG)*
2. complex Langevin equation *(Aarts and James)*

We will find that these methods sometimes agree perfectly, and sometimes not. I will discuss who is right – or who is wrong – in the latter case.

# Effective Polyakov Line Action

Start with lattice gauge theory and integrate out all d.o.f. subject to the constraint that the Polyakov line holonomies are held fixed. In temporal gauge

$$e^{S_P[U_{\mathbf{x}}]} = \int DU_0(\mathbf{x},0)DU_k D\phi \left\{ \prod_{\mathbf{x}} \delta[U_{\mathbf{x}} - U_0(\mathbf{x},0)] \right\} e^{S_L}$$

Avoid dynamical fermion simulations for now, work instead with an SU(3) gauge-Higgs model with a fixed modulus Higgs

$$\begin{aligned} S_L &= \frac{\beta}{3}\sum_p \mathrm{ReTr}[U(p)] \\ &\quad + \frac{\kappa}{3}\sum_x \sum_{\mu=1}^{4} \mathrm{Re}\left[\Omega^\dagger(x)U_\mu(x)\Omega(x+\hat{\mu})\right] \end{aligned}$$

# From $\mu = 0$ to finite $\mu$

If we can derive $S_P$ at $\mu = 0$, then (in principle) we also have $S_P$ at $\mu > 0$ by the following identity:

$$S_P^{\mu}[U_{\mathbf{x}}, U_{\mathbf{x}}^{\dagger}] = S_P^{\mu=0}\left[e^{N_t\mu} U_{\mathbf{x}}, e^{-N_t\mu} U_{\mathbf{x}}^{\dagger}\right]$$

which is true to all orders in the strong coupling/hopping parameter expansion.

# The Relative Weights Method

Let $S'_L$ be the lattice action in temporal gauge with $U_0(\mathbf{x},0)$ fixed to $U'_{\mathbf{x}}$. It is not so easy to compute

$$\exp\Big[S_P[U'_{\mathbf{x}}]\Big] = \int DU_k D\phi \; e^{S'_L}$$

directly. But the ratio ("relative weights")

$$e^{\Delta S_P} = \frac{\exp[S_P[U'_{\mathbf{x}}]]}{\exp[S_P[U''_{\mathbf{x}}]]}$$

is easily computed as an expectation value

$$\begin{aligned}
\exp[\Delta S_P] &= \frac{\int DU_k D\phi \; e^{S'_L}}{\int DU_k D\phi \; e^{S''_L}} \\
&= \frac{\int DU_k D\phi \; \exp[S'_L - S''_L] e^{S''_L}}{\int DU_k D\phi \; e^{S''_L}} \\
&= \Big\langle \exp[S'_L - S''_L] \Big\rangle''
\end{aligned}$$

where $\langle ... \rangle''$ means the VEV in the Boltzman weight $\propto e^{S''_L}$.

Suppose $U_{\mathbf{x}}(\lambda)$ is some path through configuration space parametrized by $\lambda$, and suppose $U'_{\mathbf{x}}$ and $U''_{\mathbf{x}}$ differ by a small change in that parameter, i.e.

$$U'_{\mathbf{x}} = U_{\mathbf{x}}(\lambda_0 + \tfrac{1}{2}\Delta\lambda) \quad , \quad U''_{\mathbf{x}} = U_{\mathbf{x}}(\lambda_0 - \tfrac{1}{2}\Delta\lambda)$$

Then the relative weights method gives us the derivative of the true effective action $S_P$ along the path:

$$\left(\frac{dS_P}{d\lambda}\right)_{\lambda=\lambda_0} \approx \frac{\Delta S}{\Delta\lambda}$$

The question is: which derivatives will help us to determine $S_P$ itself?

# Fourier components of $P_x$

We compute derivatives of $S_P$ w.r.t. Fourier components $a_{\mathbf{k}}$ of the Polyakov lines

$$P_{\mathbf{x}} \equiv \frac{1}{N_c}\mathrm{Tr}\,U_{\mathbf{x}} = \sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

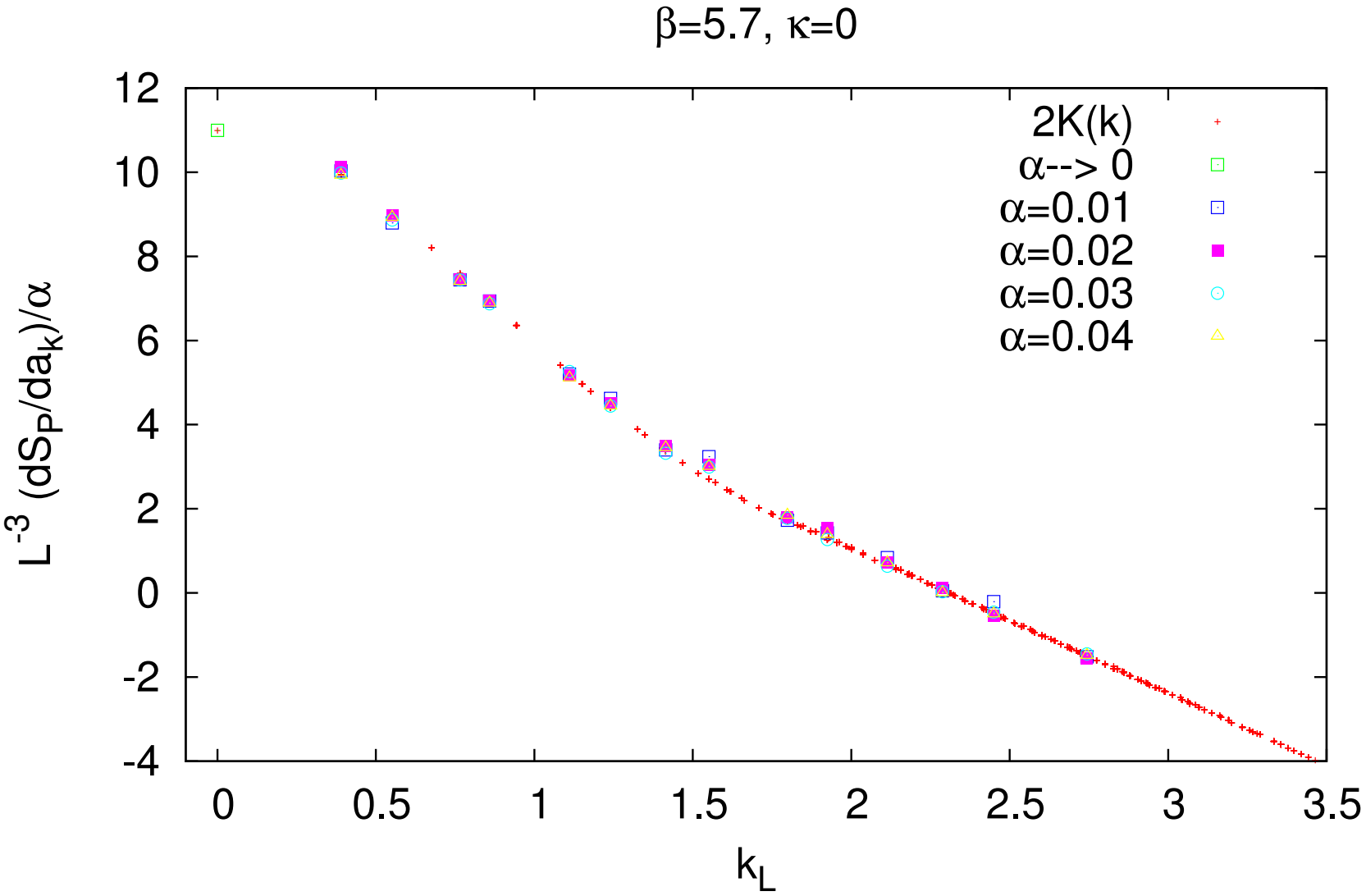
For a pure gauge theory, the part of $S_P$ bilinear in $P_x$ is constrained to have the form

$$S_P = \sum_{\mathbf{xy}} P_{\mathbf{x}} P^{\dagger}_{\mathbf{y}} K(\mathbf{x}-\mathbf{y})$$

Then, going over to Fourier modes

$$\frac{1}{\alpha}\frac{1}{L^3}\left(\frac{\partial S_P}{\partial a^R_{\mathbf{k}}}\right)_{a_{\mathbf{k}}=\alpha} = 2\widetilde{K}(\mathbf{k})$$

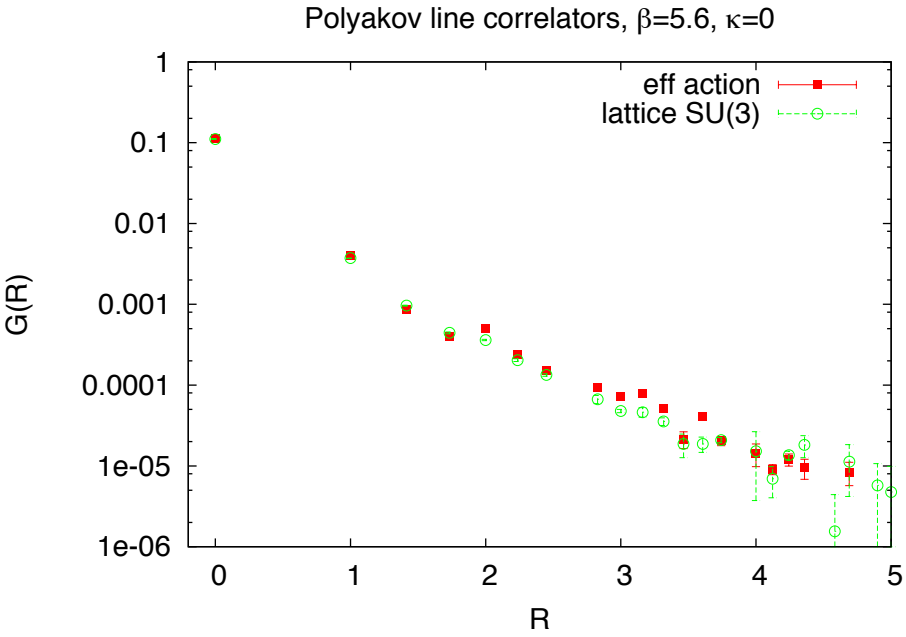
The red points are the Fourier transform of $K(\mathbf{x}-\mathbf{y})$, which gives us the effective action $S_P$



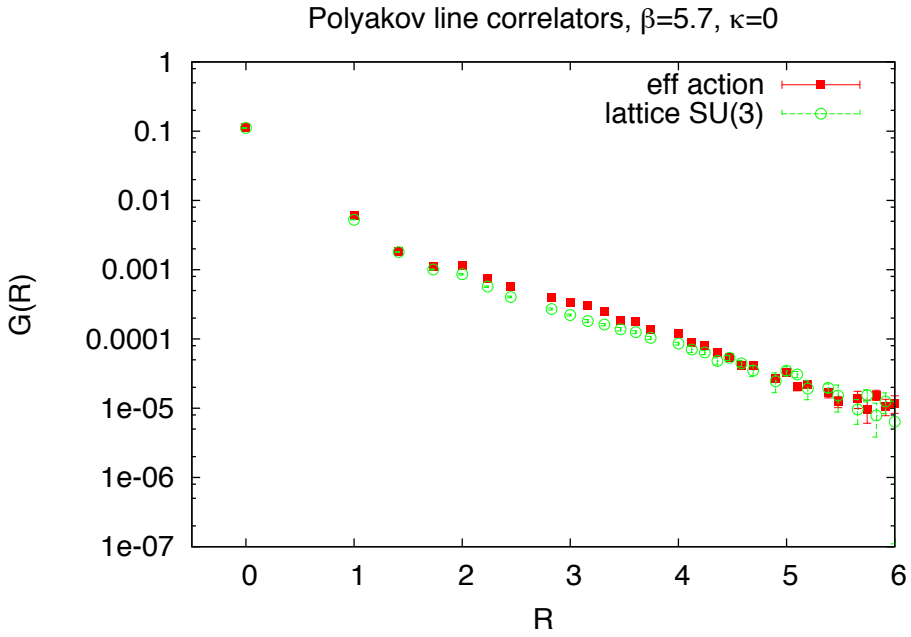
# Correlator comparisons at $\beta = 5.6, 5.7$

$$S_P = \sum_{\mathbf{xy}} P_{\mathbf{x}} P^\dagger_{\mathbf{y}} K(\mathbf{x} - \mathbf{y})$$

Simulate the effective theory in the usual way, and compare the Polyakov line correlators in the effective theory with the correlators in the underlying pure gauge theory




$\beta = 5.6$




$\beta = 5.7$

# $S_P$ for the SU(3) gauge-Higgs model

Including linear and bilinear center symmetry-breaking terms, it can be shown that at finite chemical potential

$$\begin{aligned} S_P &= \sum_{\mathbf{xy}} P_{\mathbf{x}} P_{\mathbf{y}}^\dagger K(\mathbf{x}-\mathbf{y}) + \sum_{\mathbf{xy}} (P_{\mathbf{x}} P_{\mathbf{y}} Q(\mathbf{x}-\mathbf{y},\mu) + P_{\mathbf{x}}^\dagger P_{\mathbf{y}}^\dagger Q(\mathbf{x}-\mathbf{y};-\mu)) \\ &\quad + \sum_{\mathbf{x}} \left\{ (d_1 e^{\mu/T} - d_2 e^{-2\mu/T}) P_{\mathbf{x}} + (d_1 e^{-\mu/T} - d_2 e^{2\mu/T}) P_{\mathbf{x}}^\dagger \right\} \end{aligned}$$

where

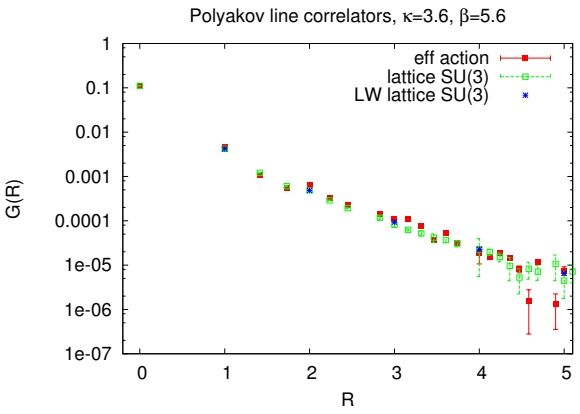
$$Q(\mathbf{x}-\mathbf{y};\mu) = Q^{(1)}(\mathbf{x}-\mathbf{y}) e^{-\mu/T} + Q^{(2)}(\mathbf{x}-\mathbf{y}) e^{2\mu/T} + Q^{(4)}(\mathbf{x}-\mathbf{y}) e^{-4\mu/T}$$

To determine $d_1, d_2, Q(\mathbf{x}-\mathbf{y};\mu)$ it is helpful to compute $dS_P/da_{\mathbf{k}}$ at imaginary chemical potential $\mu/T = i\theta$.

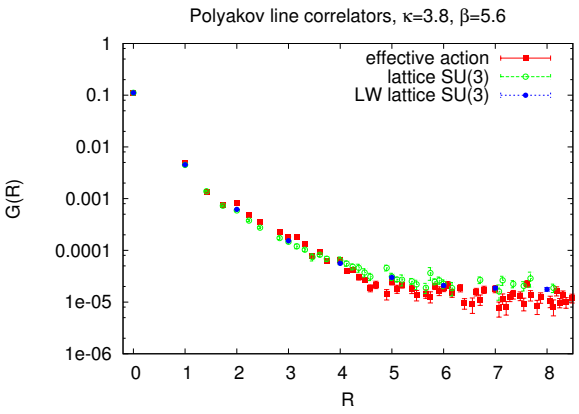
# Gauge-Higgs Correlator Comparison

## Effective action vs. lattice gauge theory

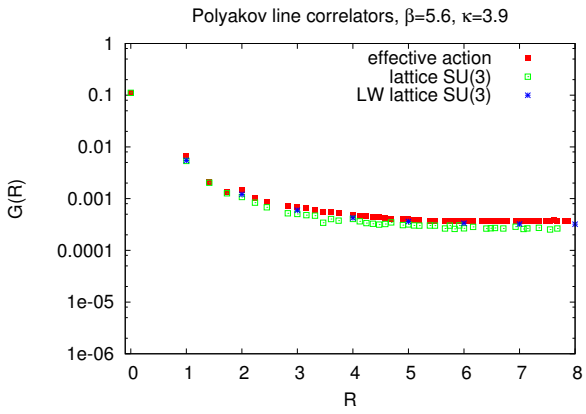
The underlying lattice gauge-Higgs theory is at $\beta = 5.6, \mu = 0$ and $\kappa = 3.6, 3.8, 3.9$ on a $16^3 \times 6$ lattice volume.




(a) $\kappa = 3.6$




(b) $\kappa = 3.8$




(c) $\kappa = 3.9$

# There is still a sign problem...

**Comparison of complex Langevin**

**and mean field methods**

**applied to effective actions at $\mu > 0$**

**Gauge-Higgs at $\kappa = 3.8, 3.9$**

$$
\begin{aligned}
S_P &= \frac{1}{9}\sum_{xy} \mathrm{Tr}[U_{\mathbf{x}}]\mathrm{Tr}[U_{\mathbf{y}}^\dagger] K(\mathbf{x}-\mathbf{y}) \\
&+ \frac{1}{3}\sum_{x} \Big\{ (d_1 e^{\mu/T} - d_2 e^{-2\mu/T})\mathrm{Tr}[U_{\mathbf{x}}] + (d_1 e^{-\mu/T} - d_2 e^{2\mu/T})\mathrm{Tr}[U_{\mathbf{x}}^\dagger] \Big\}
\end{aligned}
$$

The $d_2$ dependent terms must originate from "double-winding "terms

$$d_2 e^{2\mu/T}\mathrm{Tr}[U_{\mathbf{x}}^2] + d_2 e^{-2\mu/T}\mathrm{Tr}[U_{\mathbf{x}}^{\dagger 2}]$$

via the SU(3) identities

$$\mathrm{Tr}[U_{\mathbf{x}}^2] = \mathrm{Tr}[U_{\mathbf{x}}]^2 - 2\mathrm{Tr}[U_{\mathbf{x}}^\dagger] \quad , \quad \mathrm{Tr}[U_{\mathbf{x}}^{\dagger 2}] = \mathrm{Tr}[U_{\mathbf{x}}^\dagger]^2 - 2\mathrm{Tr}[U_{\mathbf{x}}] \ ,$$

With that motivation, we also consider

**A model with a double-winding term**

$$
\begin{aligned}
S_P \quad = \quad & \frac{1}{9}\sum_{xy} \mathrm{Tr}[U_{\mathbf{x}}]\mathrm{Tr}[U^\dagger_{\mathbf{y}}]K(\mathbf{x}-\mathbf{y}) \\
& + \frac{1}{3}\sum_{x}\left\{ d_1\, e^{\mu/T}\mathrm{Tr}[U_{\mathbf{x}}] + d_1\, e^{-\mu/T}\mathrm{Tr}[U^\dagger_{\mathbf{x}}]\right\} \\
& + \frac{1}{6}\sum_{x}\left\{ d_2\, e^{2\mu/T}\mathrm{Tr}[U^2_{\mathbf{x}}] + d_2\, e^{-2\mu/T}\mathrm{Tr}[U^{\dagger 2}_{\mathbf{x}}]\right\}
\end{aligned}
$$

# Heavy Quark model

Hopping parameter very small, chemical potential $\mu$ very large. In temporal gauge, the lattice action is simply

$$e^{S_L} = \prod_{\mathbf{x}} \det\left[1 + he^{\mu/T} U_0(\mathbf{x}, 0)\right]^p \det\left[1 + he^{-\mu/T} U^\dagger(\mathbf{x}, 0)\right]^p e^{S_{plaq}}$$

$p = 1$ for staggered fermions, $p = 2N_f$ for Wilson fermions. If we know the Polyakov line action for the pure gauge theory $S_P^{pg}$, then

$$e^{S_P} = \prod_{\mathbf{x}} \det\left[1 + he^{\mu/T} U_{\mathbf{x}}\right]^p \det\left[1 + he^{-\mu/T} U_{\mathbf{x}}^\dagger\right]^p e^{S_P^{pg}}$$

# Complex Langevin for the effective actions

We follow the approach of *Aarts and James (2012)*.

Effective Polyakov line models depend only on the eigenvalues $\exp[i\theta_a(x)]$ of $U_{\mathbf{x}}$. In particular

$$\mathrm{Tr}[U_{\mathbf{x}}] = e^{i\theta_1(\mathbf{x})} + e^{i\theta_2(\mathbf{x})} + e^{-i(\theta_1(\mathbf{x})+\theta_2(\mathbf{x}))}$$

Treat $\theta_{1,2}(\mathbf{x})$ as the dynamical variables. Then the Haar integration measure must be incorporated into the action

$$S_P \quad \longrightarrow \quad S'_P = S_P + \sum_{\mathbf{x}} \log \left[ \sin^2\left(\frac{\theta_1(\mathbf{x}) - \theta_2(\mathbf{x})}{2}\right) \times \sin^2\left(\frac{2\theta_1(\mathbf{x}) + \theta_2(\mathbf{x})}{2}\right) \sin^2\left(\frac{\theta_1(\mathbf{x}) + 2\theta_2(\mathbf{x})}{2}\right) \right]$$

The prescription is then to complexify the angles $\theta_{1,2}(\mathbf{x})$, and solve the complex Langevin equation.

# The Møllgård-Splittorff Warning (2013)

**Beware!** Logarithms have branch cuts along the negative real axis. Complex Langevin can go wrong if

1. there is a logarithmic term in the action (e.g. the log of a measure or a fermion determinant), and
2. Langevin evolution frequently crosses the branch cut.

To check this, we keep track of the argument of the logarithm

$$\text{Arg} = \sin^2\left(\frac{\theta_1(\mathbf{x}') - \theta_2(\mathbf{x}')}{2}\right)\sin^2\left(\frac{2\theta_1(\mathbf{x}') + \theta_2(\mathbf{x}')}{2}\right)\sin^2\left(\frac{\theta_1(\mathbf{x}') + 2\theta_2(\mathbf{x}')}{2}\right)$$

at an arbitrarily chosen lattice site $\mathbf{x}'$.

# Mean Field Theory for the effective actions

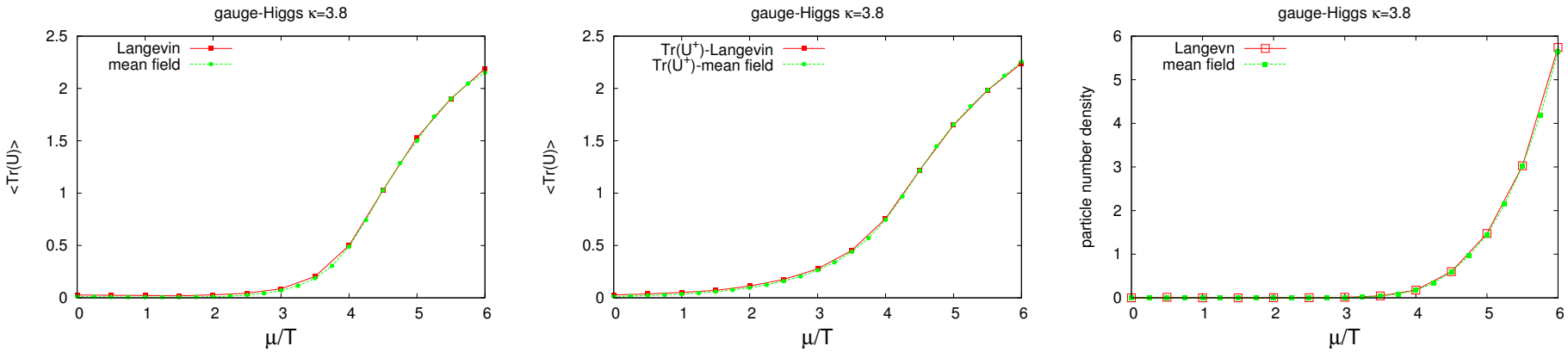
The mean field treatment of SU(3) spin models at finite $\mu$ is a minor variation of standard mean field theory at zero chemical potential.

Two "magnetizations" are introduced; one for $\mathrm{Tr}U$ and one for $\mathrm{Tr}U^\dagger$. These are determined, as usual, by minimizing the free energy.

For details, see *Splittorff and JG (2012)*.

# Results I - Gauge-Higgs at $\kappa = 3.8$

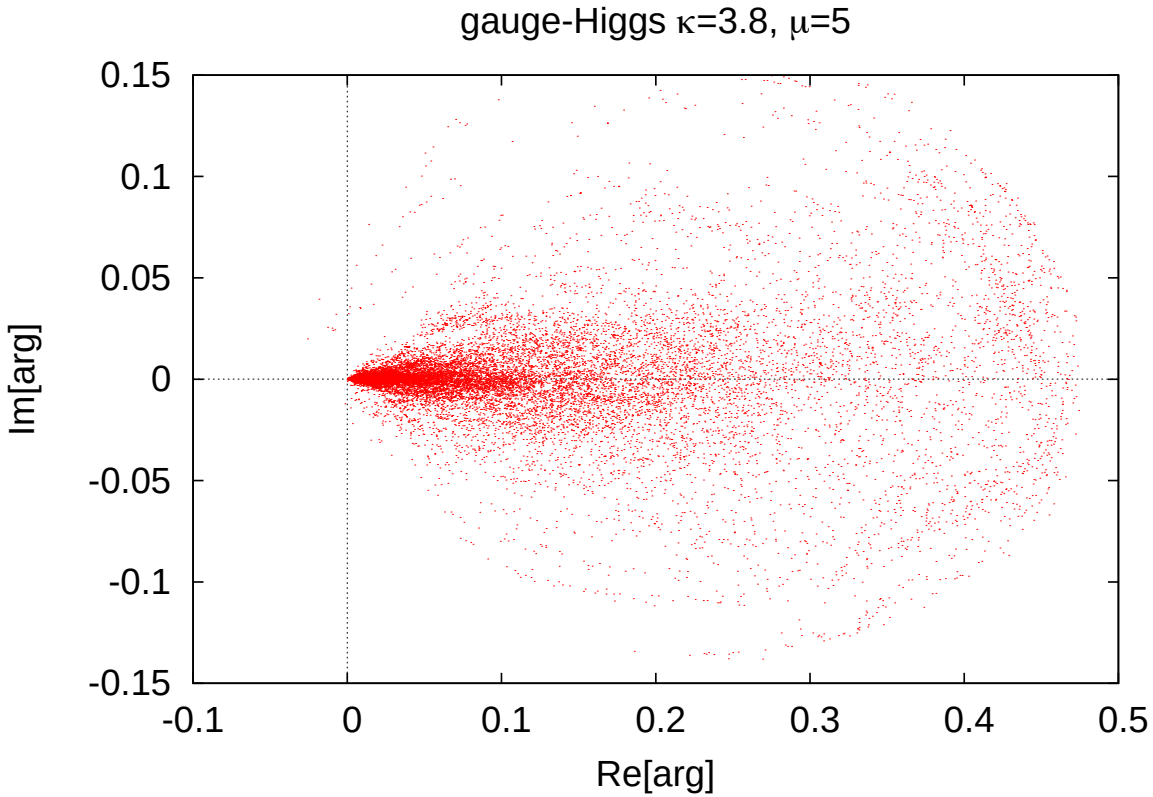
Here are the results for the Polyakov lines and the number density, derived from complex Langevin and mean field ($\beta = 5.6, 16^3 \times 6$ lattice as before):




(a) $\langle \mathrm{Tr}(U) \rangle$ (b) $\langle \mathrm{Tr}(U^\dagger) \rangle$ (c) density

It is hard to even detect a difference between the two methods.

Here is a plot of the argument of the logarithm in the complex plane, at a fixed lattice site, at each Langevin time step for $\mu = 5$:

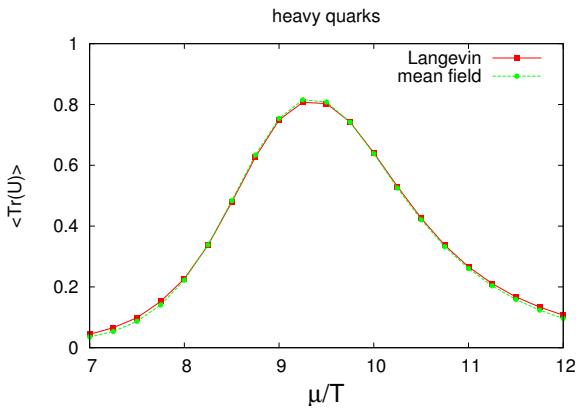



There seems to be no branch-cut crossing problem.

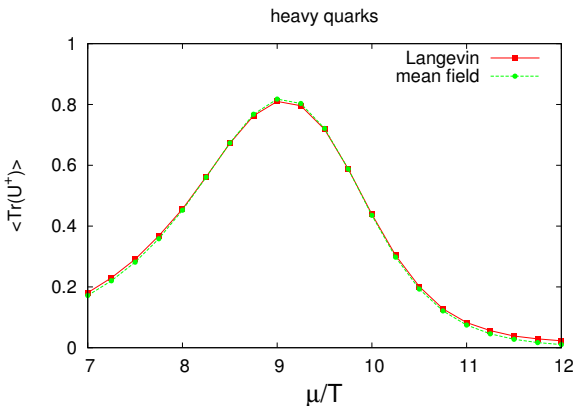
# Results II - The Heavy Quark Model

$(p = 1,\ \beta = 5.6,\ h = 10^{-4},\ 16^3 \times 6)$

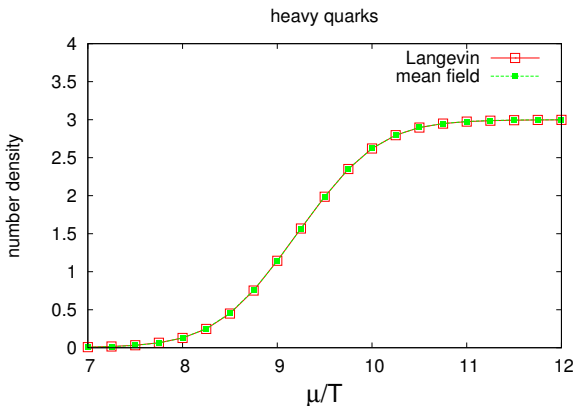
Again, near-perfect agreement:




(a) $\langle \mathrm{Tr}(U) \rangle$

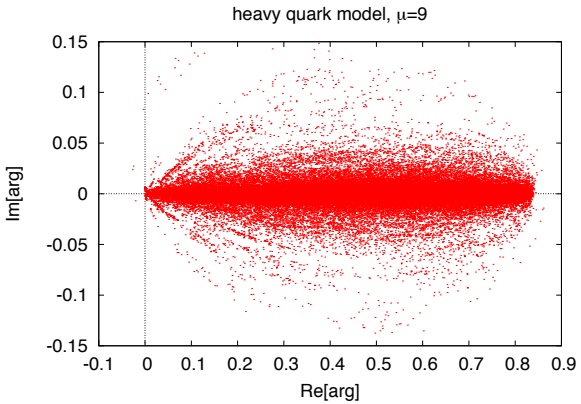



(b) $\langle \mathrm{Tr}(U^\dagger) \rangle$



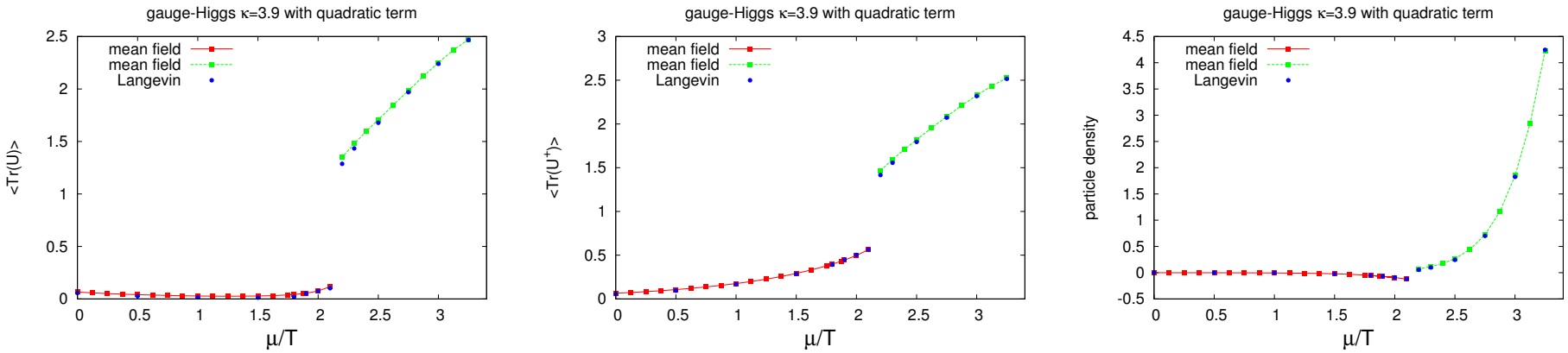

(c) density

and no branch-cut crossing problem



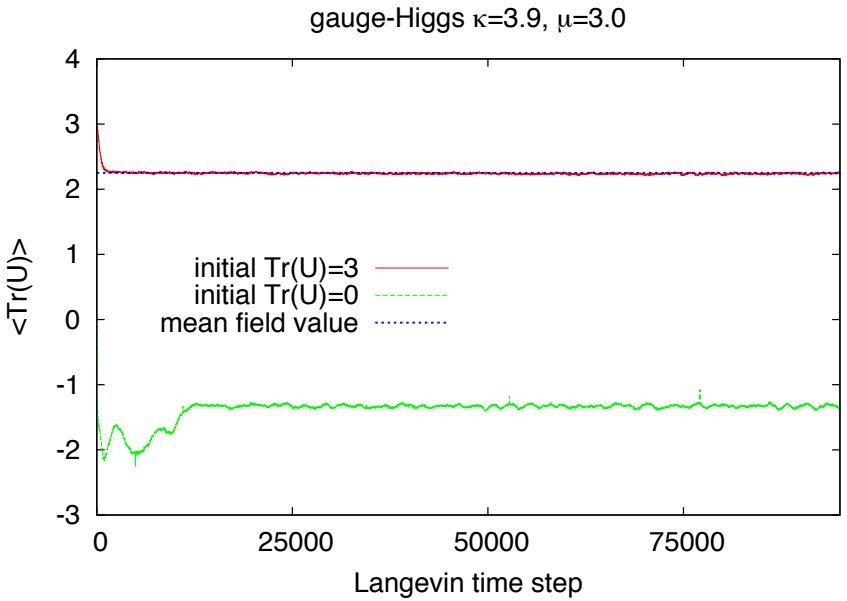
# Results III - Gauge-Higgs with double winding term

Here we see a phase transition

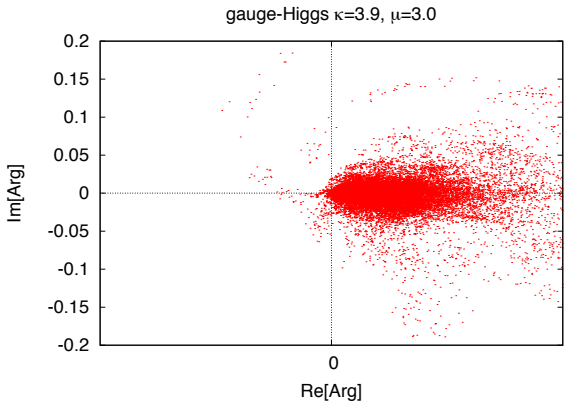



(a) $\langle \mathrm{Tr}(U) \rangle$ (b) $\langle \mathrm{Tr}(U^\dagger) \rangle$ (c) density

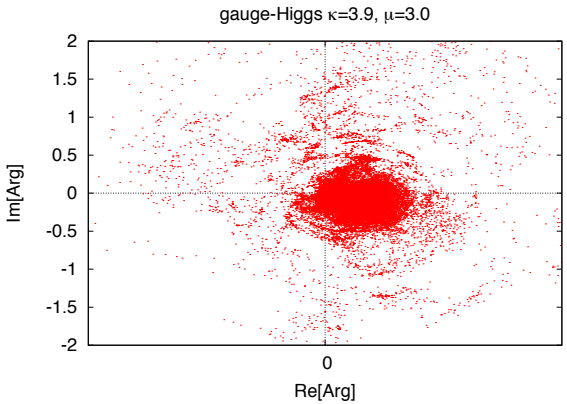
Great agreement. However, complex Langevin has at least two solutions above the transition, depending on initialization, and only one agrees with mean field.




Which to prefer? It may be determined by the branch-cut crossing problem:

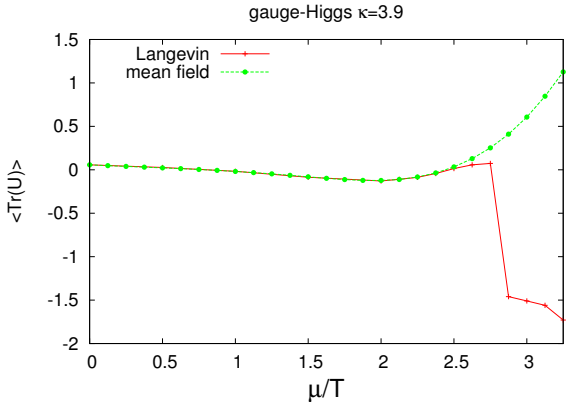



(a) initialize $\mathrm{Tr}U = 3$

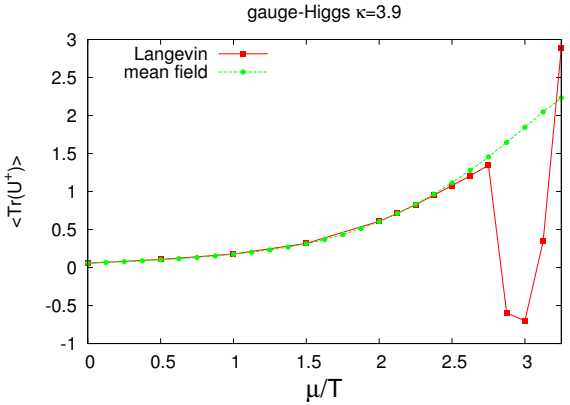



(b) initialize $\mathrm{Tr}U = 0$

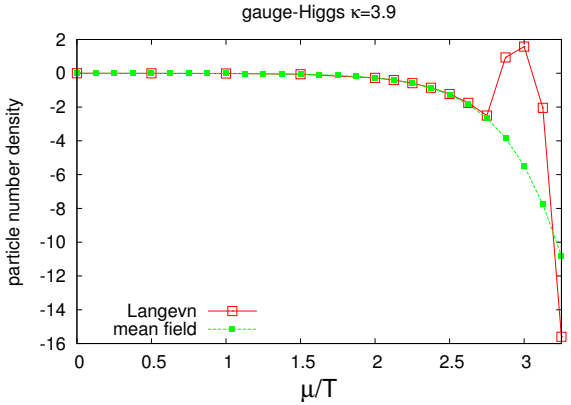
# Results IV - Gauge-Higgs $\kappa = 3.9$, no double winding term




(a) $\langle \mathrm{Tr}(U) \rangle$



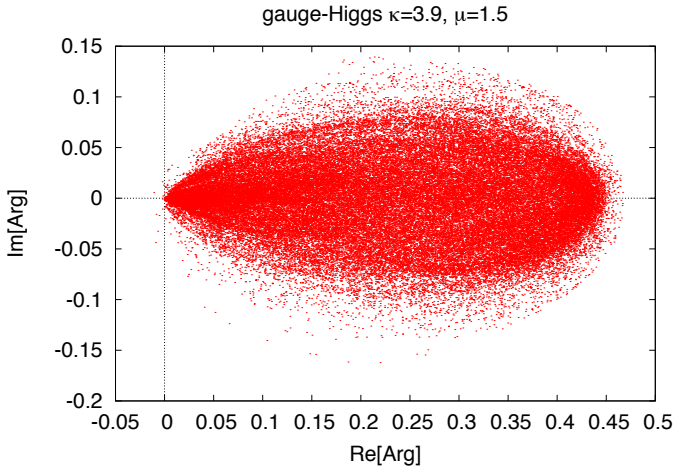

(b) $\langle \mathrm{Tr}(U^\dagger) \rangle$



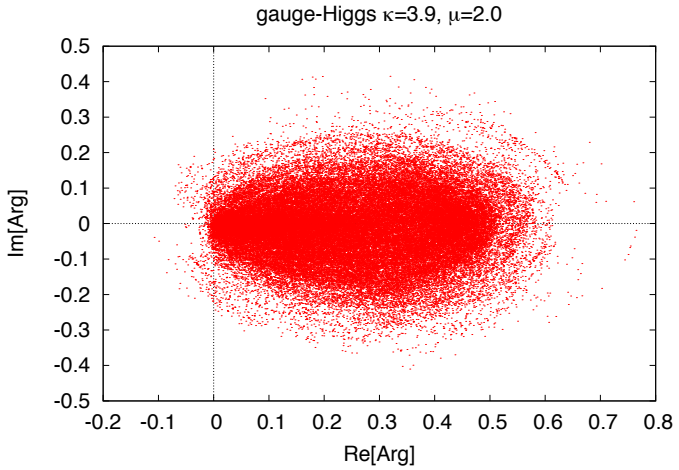

(c) density

Here there is a very strong disagreement between mean field and complex Langevin at $\mu \geq 2.75$.

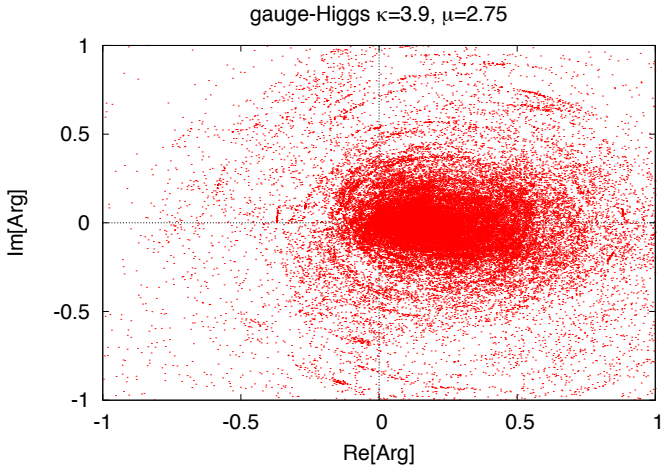
But where the results differ, complex Langevin evolution has a branch-cut problem.



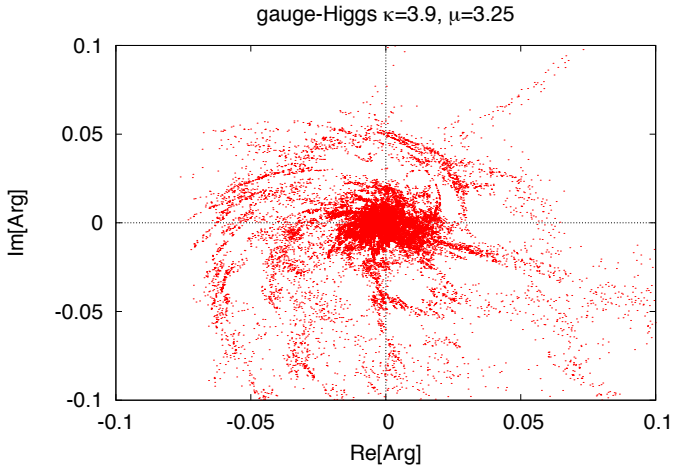

(a) $\mu/T = 1.5$




(b) $\mu/T = 2.0$




(c) $\mu/T = 2.75$

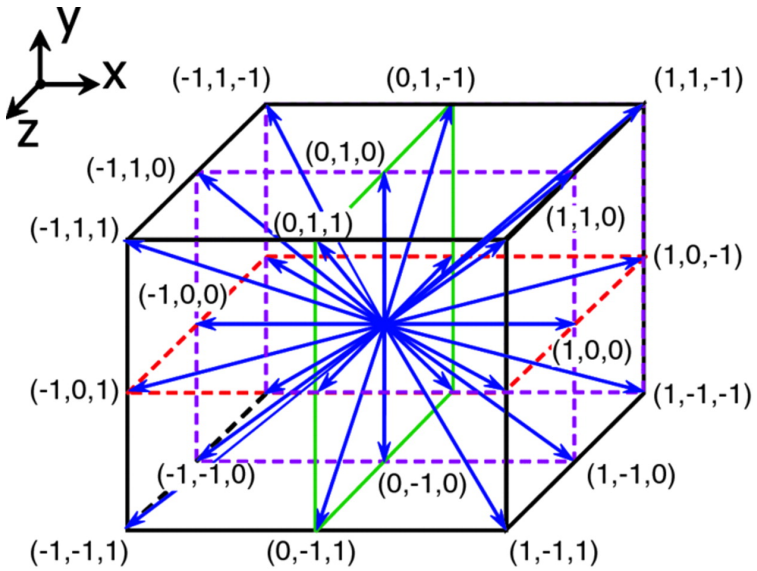



(d) $\mu/T = 3.25$

## *Why is mean field so good?*

Perhaps because many spins – not just nearest neighbors – are coupled to a given spin, through the non-local kernel $K(\mathbf{x} - \mathbf{y})$.

The basic idea behind mean field theory, i.e. that each spin is effectively coupled to the average spin on the lattice, may be a very good approximation to the true situation



# CONCLUSIONS

- We have developed a method for determining the effective Polyakov line action.
- At $\mu = 0$ there is excellent agreement for the Polyakov line correlators computed in the effective theory and underlying lattice gauge theory.
- At $\mu > 0$ we can solve the effective theory by either mean field or complex Langevin methods.
- Where the two methods agree, they agree almost perfectly. Where they disagree, complex Langevin has a Møllgård-Splittorff branch cut crossing problem.

# Next Steps

1. Determine the quadratic, quasi-local center symmetry-breaking terms. They may be important at finite chemical potential.
2. Go on to dynamical fermions.

**Prescription:**

*Find the effective action via relative weights, solve by mean field.*

Given $S_P$, there may be no need to resort to any further numerical simulation at finite $\mu$.