Thermodynamics in the fixed scale approach with the shifted boundary conditions

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Fixed scale approach to study QCD thermodynamics

Temperature $T=1/(N_t a)$ is varied by $N_t$ at fixed $a$

- Coupling parameters are common at each $T$
  - To study Equation of States
  - $T=0$ subtractions are common
  - beta-functions are common
  - Line of Constant Physics is automatically satisfied

Cost for $T=0$ simulations can be largely reduced

Equation of State in N_f=2+1 QCD

EOS is obtained by temperature integration in the Fixed scale approach

\[
\frac{p}{T^4} = \int_0^T dt \frac{\epsilon - 3p}{t^5}
\]

Some groups adopted the approach
- tmfT, arXiv:1311.1631

However possible temperatures are restricted by integer N_t
Shifted boundary conditions

Thermal momentum distribution from path integrals with shifted boundary conditions

New method to calculate thermodynamic potentials (entropy density, specific heat, etc.)

The method is based on the partition function

\[ Z(\vec{z}) = Tr \left\{ e^{-L_0 \hat{H}} e^{i \hat{p} \vec{z}} \right\} \]

which can be expressed by Path-integral with shifted boundary condition

\[ \phi(L_0, \vec{x}) = \pm \phi(0, \vec{x} + \vec{z}) \]

- L. Giusti and H. B. Meyer, JHEP 01 (2013) 140
Shifted boundary conditions

\[ \phi(L_0, \vec{x}) = \pm \phi(0, \vec{x} + \vec{z}) \]
\[ \vec{z} = a \vec{n} \]

By using the shifted boundary
various T’s are realized with the same lattice spacing

T resolution is largely improved
while keeping advantages of the fixed scale approach
Test in quenched QCD

Simulation setup
- quenched QCD
- $\beta=6.0$
  \[ a \sim 0.1\text{fm} \]
- $32^3 \times N_t$ lattices, $N_t = 3, 4, 5, 6, 7, 8, 9$ $24^4$ (T=0)
- boundary condition
  - spatial: periodic boundary condition
  - temporal: shifted boundary condition
  \[ U_\mu(L_0, \vec{x}) = U_\mu(0, \vec{x} + \vec{z}) \]
- heat-bath algorithm (on SX-8R)
  only “even-shift” to keep even-odd structure
  e.g. $\vec{z}/a = (0,0,0), (1,1,0), (2,0,0), (2,1,1), (2,2,0), (3,1,0), ...$
### Test in quenched QCD

#### Choice of boundary shifts

\[ U_\mu(L_0, \vec{x}) = U_\mu(0, \vec{x} + \vec{z}) \quad \vec{z} = a\vec{n} \]

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Trace anomaly \( \frac{\epsilon - 3p}{T^4} \)

\[
\frac{\epsilon - 3p}{T^4} = \left( \frac{1}{VT^3} \right) a \frac{d\beta}{da} \left\langle \frac{dS}{d\beta} \right\rangle_{sub}
\]

Reference data

S. Borsanyi et al., JHEP 07 (2012) 056

Precision SU(3) lattice thermodynamics for a large temperature range

- \( N_s/N_t = 8 \) near \( T_c \)
- small \( N_t \) dependence at \( T > 1.3T_c \)
- peak height at \( N_t=6 \) is about 7% higher than continuum value
- assuming \( T_c = 294 \text{MeV} \)

The continuum values are referred as “continuum”
Trace anomaly $\left( \frac{\epsilon - 3p}{T^4} \right)$

\[
\frac{\epsilon - 3p}{T^4} = \left( \frac{1}{VT^3} \right) a \frac{d\beta}{da} \left\langle \frac{dS}{d\beta} \right\rangle_{sub}
\]

w/o shifted boundary

![Graph showing the trace anomaly](image)

beta-function: Boyd et al. (1998)
Trace anomaly \( \left( \frac{\epsilon - 3p}{T^4} \right) \)

\[
\frac{\epsilon - 3p}{T^4} = \left( \frac{1}{VT^3} \right) a \frac{d\beta}{da} \left( \frac{dS}{d\beta} \right)_{sub}
\]

\[
T = \frac{1}{a\sqrt{N_t^2 + \bar{n}^2}}
\]

\[
V = \prod_{i=1}^{3} \frac{a N_s}{\sqrt{1 + \left( \frac{n_i}{N_t} \right)^2}}
\]

beta-function: Boyd et al. (1998)
Lattice artifacts from shifted boundaries

- Shifted boundary reduces lattice artifacts of EOS in the non-interacting limit
- We confirmed that the shifted boundary reduces lattice artifacts even in the interacting case numerically.

**Figure 2**: Pressure at finite lattice spacing for the SU(N) Yang–Mills theory in the non-interacting limit. The discretization used is the Wilson action and the ‘clover’ form of the lattice field strength tensor. The inverse temperature is given by $\beta = L_0 \sqrt{1 + \xi^2}$, and $a$ is the lattice spacing.

L. Giusti et al. (2011)
Critical temperature $T_c$

Polyakov loop is difficult to be defined because the compact direction has an angle to the temporal direction.

Dressed Polyakov loop

E. Bilgici et al.,
is defined with light quarks

\[
\Sigma_n(m, V) = \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{e^{-i\phi n}}{V} \langle Tr[(m + D\phi)^{-1}] \rangle_G
\]

FIG. 2 (color online). The dressed Polyakov loop at $m = 100$ MeV in units of GeV$^3$ as a function of the temperature $T$ in MeV.
Critical temperature $T_c$

Plaquette value

$$\langle P \rangle = \frac{1}{6N_s^3N_t} \sum_P \langle 1 - \frac{1}{3} Re Tr U_P \rangle$$

Plaquette susceptibility

$$\chi_P = 6N_s^3N_t \left( \langle P^2 \rangle - \langle P \rangle^2 \right)$$

Plaq. suscep. has a peak around $T = 294$ MeV
Beta-functions (in case of quenched QCD)

\[
\frac{\epsilon - 3p}{T^4} = \left( \frac{1}{VT^3} \right) a \frac{d\beta}{da} \left\langle \frac{dS'}{d\beta} \right\rangle_{\text{sub}}
\]

In order to calculate the beta-function, additional T=0 simulations near the simulation point are necessary.

We are looking for new methods to calculate beta-function:
- Reweighting method
- Shifted boundary / Gradient flow
  entropy density is calculated from only finite temperature configs.

M. Asakawa et al. [FlowQCD Collab.], arXiv:1312.7492
Entropy density from shifted boundaries

- Entropy density at a temperature \((T_0)\) by the new method \(s(T_0)\)

- Entropy density w/o beta-function by the \(T\)-integration

\[
\frac{\epsilon - 3p}{T^4} = \left( \frac{N_t^3}{N_s^3} \right) a \frac{d\beta}{da} \frac{dS}{d\beta}_{sub}
\]

\[
\frac{p}{T^4} = \int_0^T dT' \frac{\epsilon - 3p}{T'^5}
\]

\[
Ts = \epsilon + p
\]

Beta-func is determined by matching of entropy densities at \(T_0\)

FIG. 1 (color online). Scaling behavior of \(s/T^3\); see Eq. (15). The Stefan-Boltzmann value reached in the high-\(T\) limit is also displayed.
Summary & outlook

We presented our study of the QCD Thermodynamics by using Fixed scale approach and Shifted boundary conditions

- **Fixed scale approach**
  - Cost for T=0 simulations can be largely reduced
  - first calculation in N_f=2+1 QCD with Wilson-type quarks

- **Shifted boundary conditions are promising tool**
  - to improve the fixed scale approach
    - fine temperature scan
    - suppression of lattice artifacts at larger shifts
    - Tc determination could be possible
    - New method to calculate beta-functions

- **Test in full QCD** → N_f=2+1 QCD at the physical point
Quark Gluon Plasma in Lattice QCD

Observables in Lattice QCD
- Phase diagram in \((T, \mu, m_{ud}, m_s)\)
- Critical temperature
- Equation of state \((\varepsilon/T^4, p/T^4,\ldots)\)
- Hadronic excitations
- Transport coefficients
- Finite chemical potential
- etc...

http://www.gsi.de/fair/experiments/

Lattice2014
T. Umeda (Hiroshima)
Entropy density from shifted boundaries

Entropy density $s/T^3$

from the cumulant of the momentum distribution


\[
\frac{s(T)}{T^3} = \lim_{a \to 0} \frac{2K(T, \vec{z}, a)}{|\vec{z}|^2 T^5 V}
\]

\[
K(T, \vec{z}, a) = -\ln \frac{Z(T, \vec{z}, a)}{Z(T, \vec{0}, a)}
\]

$Z(T, \vec{z}, a)$ : partition function with shifted boundary

where $\vec{z} = (0, 0, n_z a)$, $n_z$ being kept fixed when $a \to 0$

FIG. 1 (color online). Scaling behavior of $s/T^3$; see Eq. (15). The Stefan-Boltzmann value reached in the high-$T$ limit is also displayed.