Complex Langevin dynamics for SU(3) gauge theory with a θ term

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Introduction

Pure Gauge Lagrangian of $SU(3)$:

$$\mathcal{L}_{PG} \;=\; -\frac{1}{2}F^a_{\mu\nu}F^a_{\mu\nu} \;-\; i\;\frac{\theta}{32\pi^2}\;F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}\;;$$

$$\tilde{F}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^a_{\rho\sigma}\;; \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\;f^{abc}A^b_\mu A^c_\nu\;;$$

where :

$$\int d^4x\;\frac{\theta}{32\pi^2}\;F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} \;=\; Q_{top}\;.$$

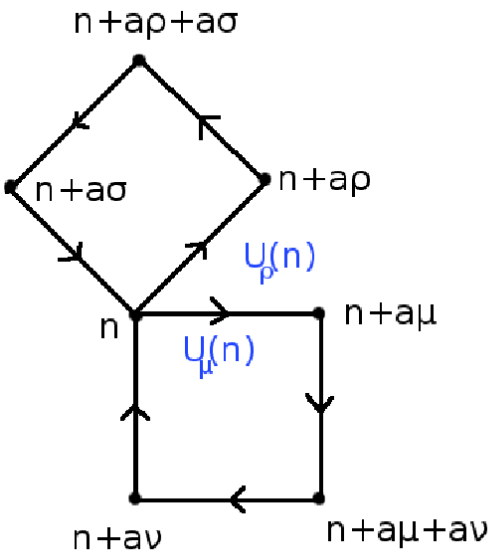
is the *topological charge* .

Discretization on the Lattice

Topological density and charge on lattice :

$$q_L(n) \;=\; -\frac{1}{2^4 \times 32\pi^2} \sum_{\mu\nu\rho\sigma=\pm 1}^{\pm 4} \tilde{\epsilon}_{\mu\nu\rho\sigma}\, Tr[\Pi_{\mu\nu}(n)\Pi_{\rho\sigma}(n)]$$

$Q_L = \sum_n q_L(n)$



Renormalization

The topological charge density must be corrected by a renormalization factor introduced by the lattice cut-off at the quantum level

$$q_L(n) \to a^4 Z_L(g^2) q(x) + O(a^6) \ .$$

Various methods to take care of Z_L :

- Cooling
- Smearing
- Wilson Flow
- ecc...

Wilson Flow

The Wilson Flow equation :

$$\dot{V}_\mu(x,\tau) \;= -g^2 \;\left[\partial_{x,\mu} S(V(\tau))\right] V_\mu(x,\tau)$$

$$V_\mu(x,0) \;=\; U_\mu(x)$$

It has some advantages for our purpose :

- Its process can be accurately controlled since associated to a differential equation,
- it can, in principle, be extended to any gauge group

Sign Problem

Since

$$\mathcal{S}_\theta \;=\; i\theta Q_{top}$$

is purely imaginary $\Rightarrow$ SIGN PROBLEM .

Some progress have been made, on the Lattice, in studying the phase diagram of the theory using :

- analytical continuation from imaginary θ $\quad (\theta = \theta_R + i\theta_I)$,
- Reweightening , Taylor expansion,
- large N expansion .

The first two, however, are limited by the small value of θ , the last is affected from the corrections for N=3 .

Complex Langevin Dynamics

In principle Complex Langevin Dynamics is a method to access the whole phase diagram .

Real Langevin Dynamics

$$\langle O \rangle \;=\; \frac{\int dx\; O(x)\; e^{-S(x)}}{\int dx e^{-S(x)}}$$

- Stochastic process for x : $\frac{dx}{d\tau} = -\frac{\partial S}{\partial x} + \eta(\tau)$

$$\langle \eta(\tau) \rangle = 0 \qquad \langle \eta(\tau)\eta(\tau') \rangle = 2\delta(\tau - \tau')$$

- Averages are calculated along the Langevin trajectory :

$$\langle O \rangle \;=\; \frac{1}{T}\int_0^T O(x(\tau))\; d\tau$$

- Fokker-Planck equation for probability distribution P(x) :
$\frac{dP}{d\tau} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x} + P\frac{\partial S}{\partial x}\right) = -H_{FP}P$
- real action $\rightarrow$ positive eigenvalues: $P(x) \longrightarrow e^{-S(x)}$
- **Convergence to the correct distribution**

Complex Langevin Dynamics

The fields are complexified :

- real scalar $\rightarrow$ complex scalar: $x \longrightarrow x + iy$

$$\frac{dx}{d\tau} = -Re[\frac{\partial S(z)}{\partial z}]_{z=x+iy} + \eta(\tau)$$

$$\frac{dy}{d\tau} = -Im[\frac{\partial S(z)}{\partial z}]_{z=x+iy}$$

- gauge group elements: $U \in SU(N) \longrightarrow U \in SL(N, C)$
  $SL(3, C)$ is non-compact, $U^\dagger \neq U^{-1}$, $det(U) = 1$.

- Analytical continuation of the observables must be consider

$$\langle O \rangle \;=\; \frac{1}{Z} \int P_{real}(x, y)\; O(x + iy)\; dxdy$$

- The Fokker-Planck prob. $P(x, y)$ is still **real** in the complexified variables

$$\longrightarrow \textbf{NO SIGN PROBLEM}$$

However

- Proof of convergence :

$$\int dxdy \; P(x, y) \; O(x + iy) \; = \; \int \; dx \; e^{-S_{comp(x)}} \; O(x)$$

exist only if $P_{real}(x, y)$ decays fast enough.

Complex Langevin

In principle Complex Langevin Dynamics is a method to access the whole phase diagram .

Very careful with the proofs of correctness.

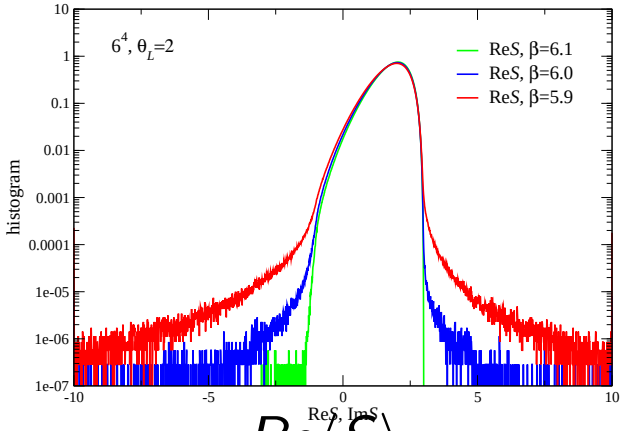
- Compactness of the distribution in the complex plane ,
- agreement of CL with MC methods for θ_I ,
- smoothness of $\langle O \rangle$ going from θ_I to θ_R ,

Compactness

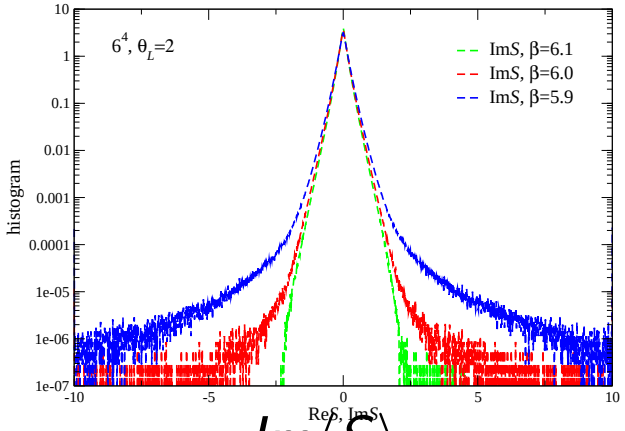
Dynamics :

- 1 Complex Langevin update + several Gauge Cooling steps .
- GC. is a gauge transformation that locally minimize the Unitarity Norm $\mathrm{UN}(n) = \sum_\mu \mathrm{Tr}(U_\mu(n) U_\mu^\dagger(n))$, $U \in SL(3, C)$.
- We use GC. to keep the distribution compact, as close as possible to the SU(N) manifold .

Histogram of the distribution of $\langle S \rangle$ for $\theta_L = 2$:

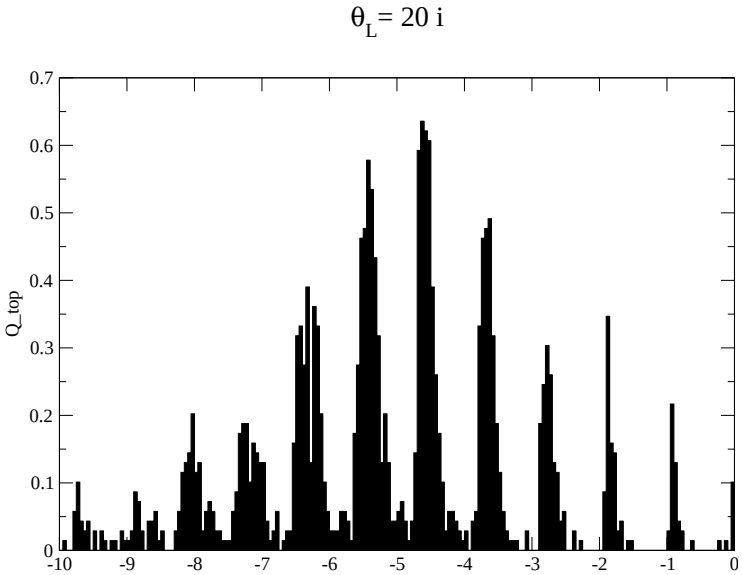


$Re\langle S \rangle$

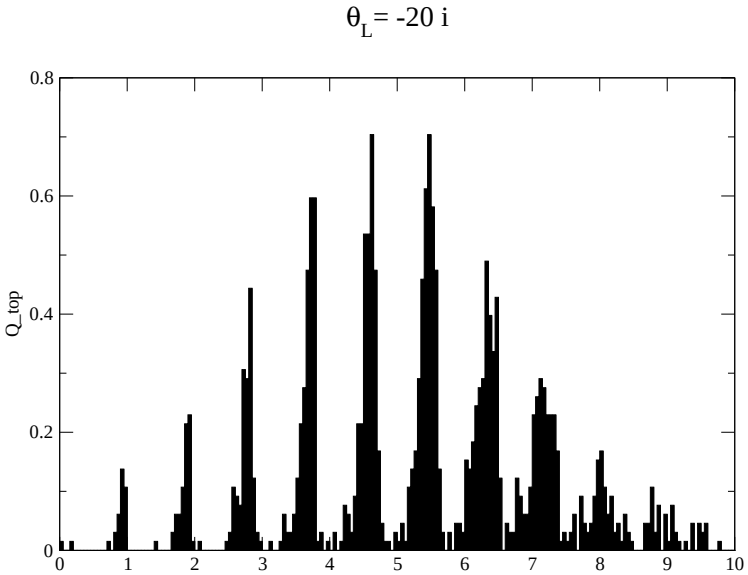


$Im\langle S \rangle$

Test dynamics choosing $\theta = \mathrm{i}\ \theta_l$



$\langle Q \rangle_{\theta_L = 20i} = -4.93(5)$



$\langle Q \rangle_{\theta_L = -20i} = +4.95(4)$

- Use Complex Langevin evolution ,
- NO unitarization ,
- Gauge cooling to stabilize dynamics ,
- without gc. : explores $SL(3, C)$, and eventually breaks down .

$\Longrightarrow$ Test of approach

Exploring Real θ

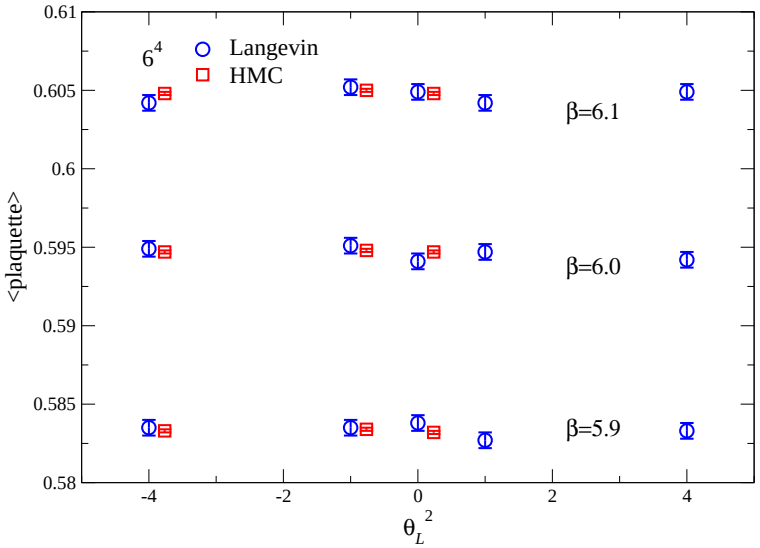
Preliminary Results for $N = 6^4$.

So far :

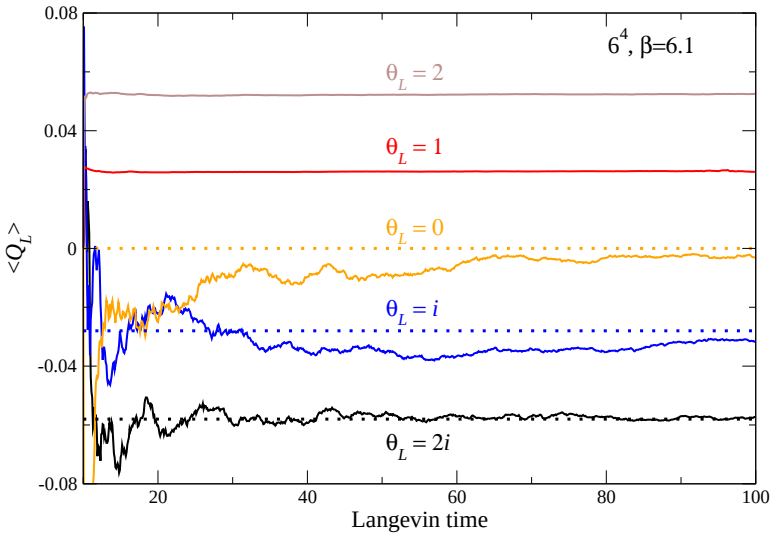
- bare lattice parameter θ_L , i.e. not renormalization ,
- the lattice version of $F\tilde{F}$ contributes to the eq of motion ,
- no renormalization of the topological operators .

Smoothness

We look at the behaviour of the plaquette and the topological charge going from θ_I to θ_R.



$\langle Plaq \rangle$



$\langle Q_{top} \rangle$, $(\beta = 6.1)$

Smooth behaviour of both observables with θ .

Behaviour of $\langle Q_{top} \rangle$ with θ

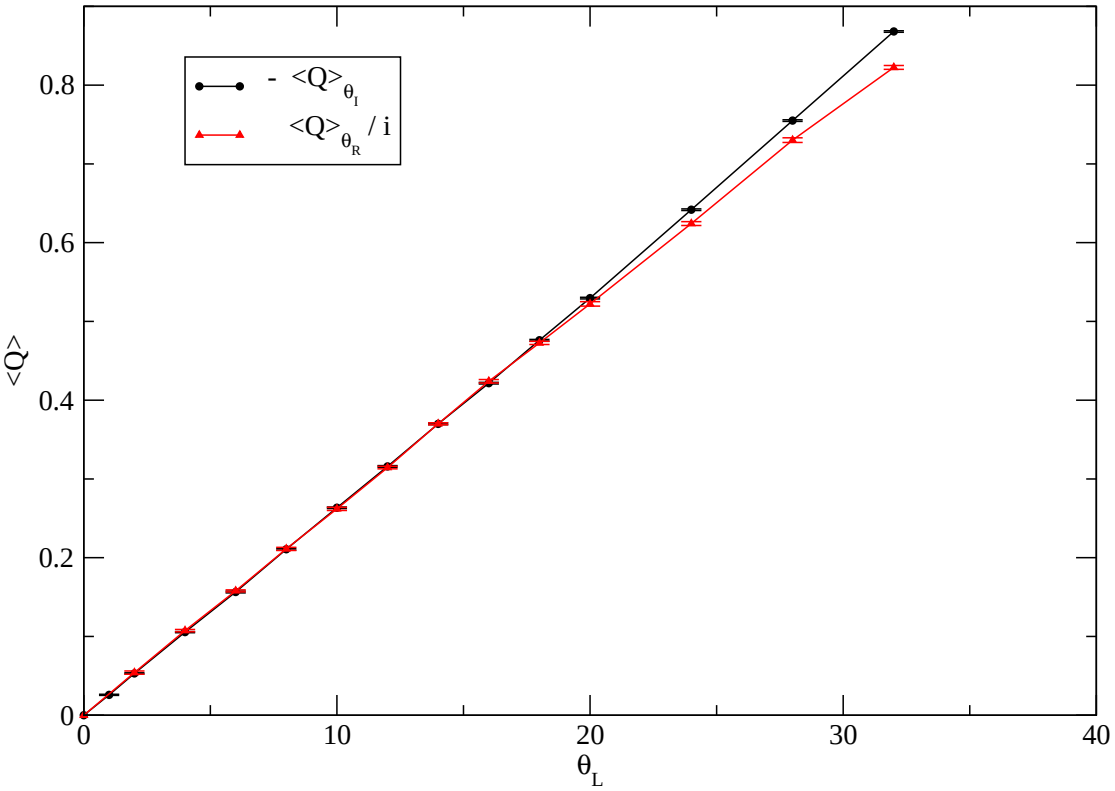
$$Z(\theta) \;=\; \int D[A]\; e^{-S_{YM}}\; e^{i\theta Q_{top}} \;=\; \exp[-VF(\theta)]\;;$$

$$F(\theta) \;=\; \sum_k \frac{1}{(2k)!}\; F^{2k}(0)\; \theta^{2k}\;;$$

The distribution of $\langle Q_{top} \rangle$ with θ is thus is expected to have the form :

- $\langle Q \rangle_{\theta_I} = -\, V \dfrac{d}{d\theta_I} F(\theta_I) = -\, V\chi\, \theta_I (1 \,-\, 2b_2\, \theta_I^2 + 3b_4\, \theta_I^4 + \ldots)\;.$
- $\langle Q \rangle_{\theta_R} = i\; V \dfrac{d}{d\theta_R} F(\theta_R) = i\; V\chi\, \theta_R (1 \,+\, 2b_2\, \theta_R^2 + 3b_4\, \theta_R^4 + \ldots)\;.$

Deviation from linear behaviour of $\langle Q \rangle_\theta$ at large θ :



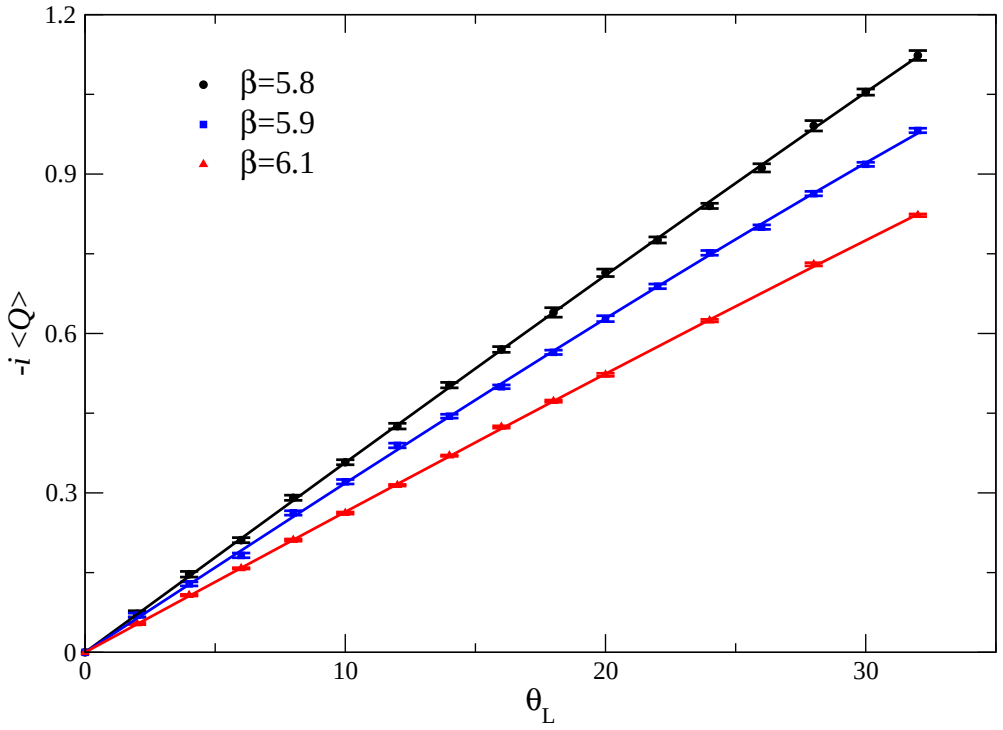
$$\beta = 6.1$$

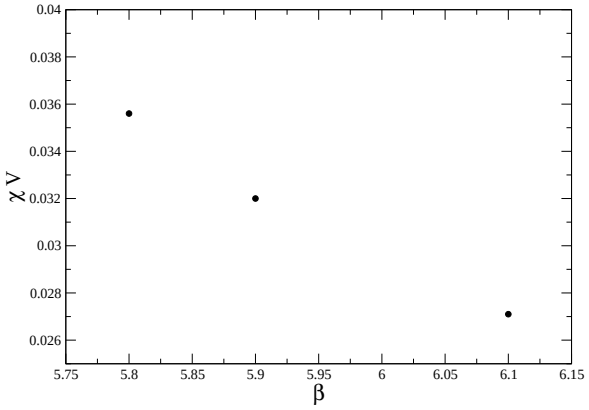
lines are fits

$$y = b_0 \theta_L (1 + 2 b_2 \theta_L^2)$$

$$b_0 = 0.026$$
$$b_2 \sim \pm 1 \cdot 10^{-5}$$

Drop of the lattice topological susceptibility χ_L for increasing values of β





The effect will be enhanced including the renormalization factor $Z(\beta)$.

Conclusipons and Outlooks

- We have good control of the CL dynamics at θ_R for values of β high enough ($\beta \gtrsim 5.8$), i.e. satisfaction of the criteria for correctness .

For what concerns the bare theory :

- We showed agreement with some momenta of Q_{top} calculated independently at θ_R and at θ_I .
- We showed the expected behaviour of the χ_{top} with β .

Outlooks :

- Find a way to measure the renormalized topological observables in $SL(3, C)$.