Complex Langevin dynamics for SU(3) gauge theory with a $\theta$ term

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Introduction

Test CL with imaginary $\theta$

Simulations at real $\theta$ and (preliminary) results

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Pure Gauge Lagrangian of $SU(3)$:

$$\mathcal{L}_{PG} = -\frac{1}{2} F^a_{\mu \nu} F^a_{\mu \nu} - i \frac{\theta}{32\pi^2} F^a_{\mu \nu} \tilde{F}^a_{\mu \nu};$$

$$\tilde{F}^a_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F^a_{\rho \sigma};$$  \hspace{1cm}  $$F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu;$$

where:

$$\int d^4x \frac{\theta}{32\pi^2} F^a_{\mu \nu} \tilde{F}^a_{\mu \nu} = Q_{top}.$$  

is the topological charge.
Topological density and charge on lattice:

\[ q_L(n) = -\frac{1}{2^4 \times 32\pi^2} \sum_{\mu\nu\rho\sigma}^{\pm 4} \tilde{\epsilon}_{\mu\nu\rho\sigma} \text{Tr}[\Pi_{\mu\nu}(n)\Pi_{\rho\sigma}(n)] \]

\[ Q_L = \sum_n q_L(n) \]
The topological charge density must be corrected by a renormalization factor introduced by the lattice cut-off at the quantum level

\[ q_L(n) \rightarrow a^4 Z_L(g^2)q(x) + O(a^6) . \]

Various methods to take care of \( Z_L \):

- Cooling
- Smearing
- Wilson Flow
- ecc...
The Wilson Flow equation:

\[ \dot{V}_\mu(x, \tau) = -g^2 [\partial_{x,\mu} S(V(\tau))] V_\mu(x, \tau) \]

\[ V_\mu(x, 0) = U_\mu(x) \]

It has some advantages for our purpose:

- Its process can be accurately controlled since associated to a differential equation,
- it can, in principle, be extended to any gauge group
Since

\[ S_\theta = i \theta Q_{\text{top}} \]

is purely imaginary \( \Rightarrow \) SIGN PROBLEM .

Some progress have been made, on the Lattice, in studying the phase diagram of the theory using:

- analytical continuation from imaginary \( \theta \) \( (\theta = \theta_R + i \theta_I) \),
- Reweightening , Taylor expansion,
- large N expansion .

The first two, however, are limited by the small value of \( \theta \), the last is affected from the corrections for \( N=3 \) .
In principle Complex Langevin Dynamics is a method to access the whole phase diagram.
Real Langevin Dynamics

\[ \langle O \rangle = \frac{\int dx \ O(x) \ e^{-S(x)}}{\int dx e^{-S(x)}} \]

- Stochastic process for \( x \):
  \[ \frac{dx}{d\tau} = -\frac{\partial S}{\partial x} + \eta(\tau) \]

  \[ \langle \eta(\tau) \rangle = 0 \quad \langle \eta(\tau)\eta(\tau') \rangle = 2\delta(\tau - \tau') \]

- Averages are calculated along the Langevin trajectory:

  \[ \langle O \rangle = \frac{1}{T} \int_0^T O(x(\tau)) \ d\tau \]

- Fokker-Planck equation for probability distribution \( P(x) \):
  \[ \frac{dP}{d\tau} = \frac{\partial}{\partial x} \left( \frac{\partial P}{\partial x} + P \frac{\partial S}{\partial x} \right) = -H_{FP}P \]

  - real action \( \rightarrow \) positive eigenvalues: \( P(x) \longrightarrow e^{-S(x)} \)

- **Convergence to the correct distribution**
The fields are complexified:

- **real scalar → complex scalar:** \( x ightarrow x + iy \)

\[
\frac{dx}{d\tau} = -\text{Re}\left[ \frac{\partial S(z)}{\partial z} \right]_{z=x+iy} + \eta(\tau)
\]

\[
\frac{dy}{d\tau} = -\text{Im}\left[ \frac{\partial S(z)}{\partial z} \right]_{z=x+iy}
\]

- **gauge group elements:** \( U \in SU(N) \longrightarrow U \in SL(N, \mathbb{C}) \)
  
  \( SL(3, \mathbb{C}) \) is non-compact, \( U^\dagger \neq U^{-1}, \quad \det(U) = 1 \).

- **Analytical continuation of the observables must be considered**

\[
\langle O \rangle = \frac{1}{Z} \int P_{\text{real}}(x, y) \ O(x + iy) \ dx dy
\]
The Fokker-Planck prob. $P(x, y)$ is still \textbf{real} in the complexified variables

$\longrightarrow$ \textbf{NO SIGN PROBLEM}

However

Proof of convergence:

\[
\int dxdy \ P(x, y) \ O(x + iy) = \int dx \ e^{-S_{\text{comp}}(x)} \ O(x)
\]

exist only if $P_{\text{real}}(x, y)$ decays fast enough.
In principle Complex Langevin Dynamics is a method to access the whole phase diagram.

Very careful with the proofs of correctness.

- Compactness of the distribution in the complex plane ,
- agreement of CL with MC methods for $\theta_I$ ,
- smoothness of $\langle O \rangle$ going from $\theta_I$ to $\theta_R$ ,
Compactness

Dynamics:

- 1 Complex Langevin update + several Gauge Cooling steps.
- GC. is a gauge transformation that locally minimize the Unitarity Norm $UN(n) = \sum_\mu \text{Tr}(U_\mu(n)U_\mu^\dagger(n))$, $U \in SL(3, C)$.
- We use GC. to keep the distribution compact, as close as possible to the SU(N) manifold.

Histogram of the distribution of $\langle S \rangle$ for $\theta_L = 2$:
Test dynamics choosing $\theta = i \theta_l$

$\langle Q \rangle_{\theta_L=20i} = -4.93(5)$
$\langle Q \rangle_{\theta_L=-20i} = +4.95(4)$

- Use Complex Langevin evolution ,
- NO unitarization ,
- Gauge cooling to stabilize dynamics ,
- without gc. : explores $SL(3, C)$, and eventually breaks down .

$\Rightarrow$ Test of approach
Exploring Real $\theta$

Preliminary Results for $N = 6^4$.

So far:
- bare lattice parameter $\theta_L$, i.e. not renormalization,
- the lattice version of $F \tilde{F}$ contributes to the eq of motion,
- no renormalization of the topological operators.
We look at the behaviour of the plaquette and the topological charge going from $\theta_I$ to $\theta_R$.

\[ \langle \text{Plaq} \rangle \quad \langle Q_{\text{top}} \rangle, \ (\beta = 6.1) \]

Smooth behaviour of both observables with $\theta$. 

Behaviour of $\langle Q_{top} \rangle$ with $\theta$

$$Z(\theta) = \int D[A] \ e^{-S_{YM}} \ e^{i\theta Q_{top}} = \exp[-VF(\theta)] ;$$

$$F(\theta) = \sum_k \frac{1}{(2k)!} \ F^{2k}(0) \ \theta^{2k} ;$$

The distribution of $\langle Q_{top} \rangle$ with $\theta$ is thus expected to have the form:

- $\langle Q \rangle_{\theta_I} = -V \frac{d}{d\theta_I} F(\theta_I) = -V \chi \ \theta_I (1 - 2b_2 \ \theta_I^2 + 3b_4 \ \theta_I^4 + \ldots) .$
- $\langle Q \rangle_{\theta_R} = i \ V \frac{d}{d\theta_R} F(\theta_R) = i \ V \chi \ \theta_R (1 + 2b_2 \ \theta_R^2 + 3b_4 \ \theta_R^4 + \ldots) .$
Deviation from linear behaviour of $\langle Q \rangle_\theta$ at large $\theta$:

$\beta = 6.1$

lines are fits

$y = b_0 \theta_L (1 + 2 b_2 \theta_L^2)$

$b_0 = 0.026$

$b_2 \sim \pm 1 \cdot 10^{-5}$
Drop of the lattice topological susceptibility $\chi_L$ for increasing values of $\beta$.

The effect will be enhanced including the renormalization factor $Z(\beta)$. 
Conclusions and Outlooks

- We have good control of the CL dynamics at $\theta_R$ for values of $\beta$ high enough ($\beta \gtrsim 5.8$), i.e. satisfaction of the criteria for correctness.

For what concerns the bare theory:

- We showed agreement with some momenta of $Q_{top}$ calculated independently at $\theta_R$ and at $\theta_I$.
- We showed the expected behaviour of the $\chi_{top}$ with $\beta$.

Outlooks:

- Find a way to measure the renormalized topological observables in $SL(3, C)$. 