Volume Effects on the Extraction of Form Factors at Zero Momentum

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Volume Effects at Zero Momentum

I. Rome Method & extension to radii
II. The method in a finite volume
III. Addressing finite volume effects

B.C. Tiburzi, arXiv:1407.????
The Rome Method

**On the extraction of zero momentum form factors on the lattice**

G.M. de Divitiis, R. Petronzio, N. Tantalo (Rome U., Tor Vergata & INFN, Rome2)


- Momentum extrapolation required for many phenomenological applications

\[
C(p_i) = p_i F(p^2)
\]

\[
\frac{\partial C(\vec{p})}{\partial p_i} \bigg|_{\vec{p}=0}
\]

Form Factor

\[
C(p_i)/p_i
\]

Extrapolate

\[
F(0) = \int DU \mathcal{P}[U] C[0, U]
\]

Taylor Series Coefficient

\[
C[\vec{p}, U] = C^{(0)}[U] + p_i C^{(1)}_i [U] + \cdots
\]

Zero Momentum

\[
F(0) = \int DU \mathcal{P}[U] C_i^{(1)}[U]
\]

Taylor Series Coefficient Correlator

\[
p_i = 0
\]

F(0)
Taylor Series Coefficient Correlator

On the extraction of zero momentum form factors on the lattice
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• Momentum expand quark propagator (action dependent)

\[ S(\vec{p}) = S(\vec{0}) - i\vec{p} \cdot S(\vec{0})\tilde{V}S(\vec{0}) + \cdots \]

Form Factor @ Zero Momentum

\[ \frac{\partial C(\vec{p})}{\partial p_k} \bigg|_{\vec{p}=0} = \text{Tr} \left[ S\gamma_5 \frac{\partial S}{\partial p_k} \gamma_k S\gamma_5 \right] = \text{Tr} \left[ S\gamma_5 SV_k S\gamma_k S\gamma_5 \right] \]

Applications:
- form factors of flavor-changing currents @ end point
- hadronic vacuum polarization @ zero momentum

• Vertices often require momentum expansion too (point-split currents)
Applying the Method to Radii

- Measurement of *muonic hydrogen spectrum* leads to an extraction of the proton radius which is 4% smaller than CODATA value (*hydrogen + e-p scattering*).

- Unprecedented precision: $4\% = 7\sigma$

\[
F(q^2) = 1 - \frac{q^2}{6} < r^2 > + \ldots
\]

Experimental electron-proton scattering data

Lattice QCD calculations of proton radius

*Largely limited to “connected”, isovector*

Requires fine lattice

\[
\delta < r^2 > = [0.2 \text{ fm}]^2 \sim (2a)^2 \sim \lambda^2_p
\]

Model small momentum transfer behavior...
Applying the Method to (Meson) Radii

- Current matrix element \( \langle \phi(p')|J_\mu|\phi(p)\rangle = e(p' + p)_\mu F(q^2) \)
  
\[
F(q^2) = 1 - \frac{q^2}{6} <r^2> + \ldots
\]

- Rest frame (similar findings in Breit kinematics)
  
\[
q^2 = \bar{q}^2[1 + O(\bar{q}^2/m_\pi^2)]
\]

- Lattice 3-pt. correlation function (ground-state saturation)
  
\[
C_4(q, \bar{q} | x_4, y_4) = i(E' + m_\pi)F(q^2)|Z|^2 \frac{e^{-E'(x_4-y_4)}e^{-m_\pi y_4}}{2E' 2m_\pi}
\]

\( p = 0 \quad \bar{p}' = \bar{q} \)

Pion form factor is connected
Well-tested lattice calculation
Applying the Method to (Meson) Radii

- Current matrix element \( \langle \phi(\vec{p}') | J_\mu | \phi(\vec{p}) \rangle = e(p' + p)_\mu F(q^2) \)

\[
F(q^2) = 1 - \frac{q^2}{6} < r^2 > + \ldots
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- Rest frame (similar findings in Breit kinematics)

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C_4(q, 0 | x_4, y_4) = i(E' + m_\pi) F(q^2) | Z |^2 \frac{e^{-E'(x_4-y_4)} e^{-m_\pi y_4}}{2E' 2m_\pi}
\]

\[
- \frac{3}{C_4(0)} \frac{\partial^2 C_4}{\partial q^2_1} \bigg|_{q=0} = < r^2 > + \frac{3}{2} \frac{1}{m^2_\pi} + \frac{x_4 - y_4}{m_\pi}
\]

Not impossible, not particularly clean
Applying the Method to (Meson) Radii

- Current matrix element \( \langle \phi(p'') | J_\mu | \phi(p') \rangle = e(p' + p)_\mu F(q^2) \)

  \[ F(q^2) = 1 - \frac{q^2}{6} < r^2 > + \ldots \]

- Current matrix element in arbitrary frame

  \[ q^2 = 2 [E' E - m_\pi^2 - p'' \cdot \bar{p}] \]

- Lattice 3-pt. correlation function (ground-state saturation)

  \[ C_4(p'', \bar{p}|x_4, y_4) = i(E' + E)_4 F(q^2)|Z|^2 e^{\frac{E'(x_4 - y_4) E y_4}{2E' 2E}} \]

\[ \frac{\partial^2 C_4}{\partial p'_1 \partial p_1} \bigg|_{\bar{p}'=\bar{p}=0} \]

\[ \frac{3}{C_4(0)} \frac{\partial^2 C_4}{\partial p'_1 \partial p_1} \bigg|_{\bar{p}'=\bar{p}=0} = < r^2 > \]

Exercise

\[ \frac{1575}{C_4(0)} \frac{\partial^6 C_4}{\partial p'_1 \partial p'_2 \partial p'_3 \partial p_2 \partial p_3} \bigg|_{\bar{p}'=\bar{p}=0} = < r^6 > \]
Rome Method in a Finite Volume

Taylor Series Coefficient
\[ C[p, U] = C^{(0)}[U] + p_i C_i^{(1)}[U] + \cdots \]

\[ F(0) = \int D\bar{U} P[U] C_i^{(1)}[U] \]

Taylor Series Coefficient Correlator

- Infinite volume limit is exact: continuous momenta admit differentiation
- To address finite volume effects must be able to derive expansion on a **fixed-size** lattice

Introduce: active and spectator quarks
\[ \psi(x + L) = e^{i\theta} \psi(x) \]

Continuous twist parameters enable differentiation in a finite volume
\[ C[p, U] = C^{(0)}[U] + p_i C_i^{(1)}[U] + \cdots \]

**Calculated at vanishing twist, i.e. strictly periodic**

- Recipe for volume effects: compute Taylor coeffs. on torus from twisted active quarks
Volume Effects at Zero Momentum

- Example: ascertain volume effect on determination of pion charge radius with ChPT

\[ \frac{\partial^2 C_4}{\partial p_1' \partial p_1} \bigg|_{\bar{p}' = \bar{p} = 0} \]

- Partial twisting is not sick from partial quenching \( m_{\text{sea}} = m_{\text{val}} \)

- Determine pion current matrix element using partially twisted ChPT

Frame dependence: no boost invariance

**Isospin twisted --> rest frame result**


**Breit kinematics** \( \bar{p}' = -\bar{p} = \frac{q}{2} \)

F.-J. Jiang, B.C. Tiburzi PRD\textbf{78}, 037501 (2008)

For method @ zero momentum need arbitrary frame
Finite Volume Computation

\[ \mathcal{M}_\mu(p', \bar{p}) = \langle \pi'^+(0) | J_\mu | \pi^+(0) \rangle \]

\[ J_\mu = \bar{u}' \gamma_\mu u_0 \]

- Time-component of current \( \Delta M_4 = \mathcal{M}_4(L) - \mathcal{M}_4(\infty) \)

\[ \Delta M_4 = (p' + p)_4 \Delta F + q_4 \Delta G \]

\[ \Delta G \text{ exceptionally messy in arbitrary frame} \]

\[ \left. \frac{\partial^2 (q_4 \Delta G)}{\partial p'_1 \partial p_1} \right|_{p'=p=0} = \left. \left( q_4 \frac{\partial^2 \Delta G}{\partial p'_1 \partial p_1} \right) \right|_{p'=p=0} = 0 \]

\[ \Delta F \text{ fortunately simpler} \]

\[ \Delta F \sim \frac{e^{-m_\pi L}}{(m_\pi L)^{3/2}} \quad \Delta r^2 \sim (m_\pi L)^{1/2} e^{-m_\pi L} \]

\[ \Delta r^4 \sim (m_\pi L)^{5/2} e^{-m_\pi L} \quad \Delta r^6 \sim (m_\pi L)^{9/2} e^{-m_\pi L} \]

Originally found in

Origin found in
J. Bijnens, J. Relefors JHEP1405, 015 (2014)

\[ \text{... needed to maintain WTI} \]
Finite Volume Results

\begin{align*}
\Delta r^2_n & \sim (m_\pi L)^{1/2} e^{-m_\pi L} \\
\Delta r^4 & \sim (m_\pi L)^{5/2} e^{-m_\pi L} \\
\Delta r^6 & \sim (m_\pi L)^{9/2} e^{-m_\pi L}
\end{align*}

\begin{align*}
\langle r^2 \rangle_{\text{phys}} &= [0.67 \text{ fm}]^2 \\
\langle r^4 \rangle_{\chi\text{PT}} &= [0.77 \text{ fm}]^4 \\
\langle r^6 \rangle_{\chi\text{PT}} &= [1.5 \text{ fm}]^6
\end{align*}
Summarizing Volume Effects at Zero Momentum

I. Extension of Rome Method to radii:
   Need initial- & final-state quarks to isolate radii cleanly

II. Addressed finite volume effects:
    Twist angle differentiation, evaluate at zero twist

III. Pion radius in chiral perturbation theory:
    SU(5|3) computation shows volume dependence at %-level
I. Extension of *Rome Method* to radii:
   Need *initial*-* & *final-state* quarks to isolate radii cleanly

II. Addressed finite volume effects:
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III. Pion radius in chiral perturbation theory:
   SU(5|3) computation shows volume dependence at %-level

IV. Isovector nucleon magnetic moment + charge & magnetic radii:
   “Straightforward” generalization, SU(6|4), magnetic more volume sensitive

V.Disconnected current insertion:
   Expected to be small, no obvious generalization of method