Volume Effects on the Extraction of Form Factors at Zero Momentum

Brian C.Tiburzi

Volume Effects at Zero Momentum

I. *Rome Method* & extension to radii
II. The method in a finite volume
III. Addressing finite volume effects

B.C. Tiburzi, arXiv:1407.????

The Rome Method

On the extraction of zero momentum form factors on the lattice
G.M. de Divitiis, R. Petronzio, N. Tantalo (Rome U.,Tor Vergata & INFN, Rome2)
Published in **Phys.Lett. B718 (2012) 589-596**

- Momentum extrapolation required for many phenomenological applications

$$C(\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \int D\mathcal{U}\, \mathcal{P}[\mathcal{U}] C[x,\mathcal{U}] \equiv \int D\mathcal{U}\, \mathcal{P}[\mathcal{U}] C[\vec{p},\mathcal{U}]$$

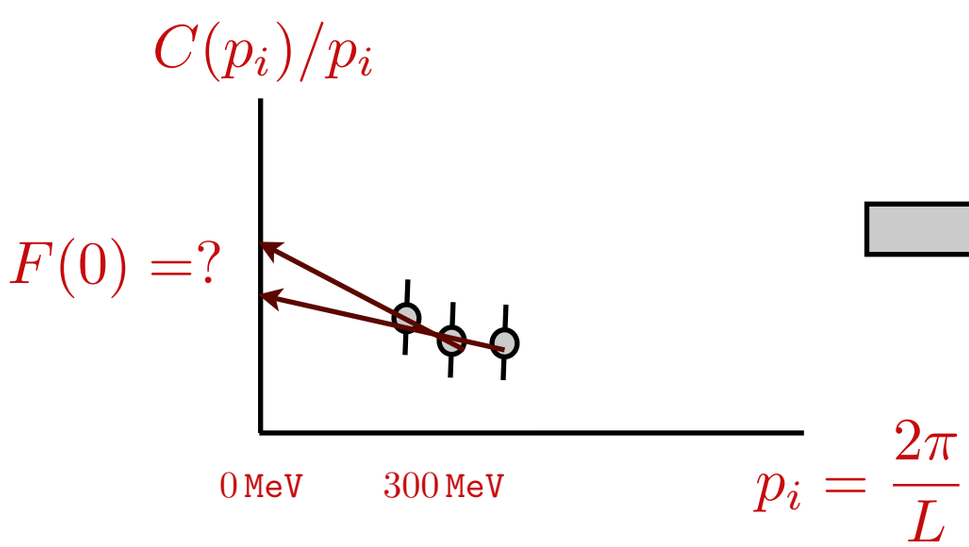
Form Factor

$$C(p_i) = p_i F(p^2)$$

$$\frac{\partial C(\vec{p})}{\partial p_i}\Big|_{\vec{p}=0}$$

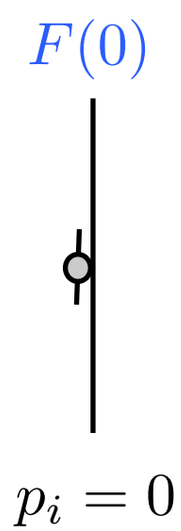
Taylor Series Coefficient

$$C[\vec{p},\mathcal{U}] = C^{(0)}[\mathcal{U}] + p_i C_i^{(1)}[\mathcal{U}] + \cdots$$



$$F(0) = \int D\mathcal{U}\, \mathcal{P}[\mathcal{U}]\, C_i^{(1)}[\mathcal{U}]$$

Taylor Series Coefficient Correlator



Extrapolate

Zero Momentum

Taylor Series Coefficient Correlator

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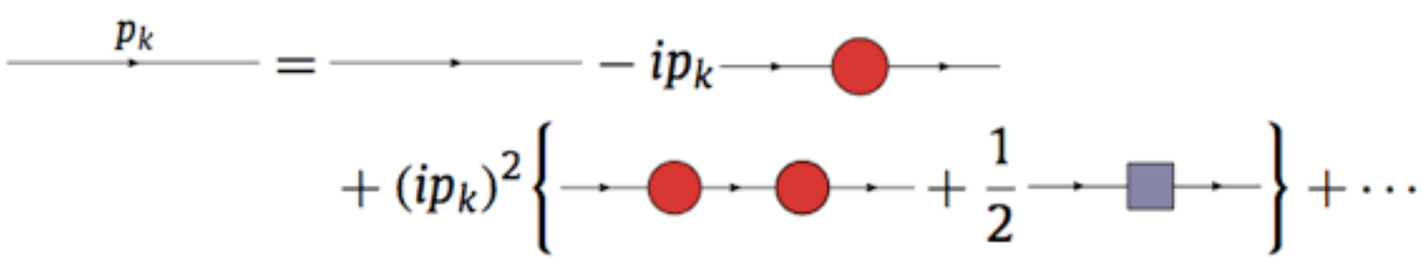
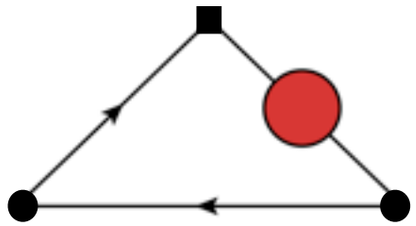
- Momentum expand quark propagator (action dependent)

$$S(\vec{p}) = S(\vec{0}) - i\vec{p}\cdot S(\vec{0})\vec{V}S(\vec{0}) + \cdots$$



Form Factor @ Zero Momentum



$$\frac{\partial C(\vec{p})}{\partial p_k}\Big|_{\vec{p}=0} = \mathtt{Tr}\left[S\gamma_5\frac{\partial S}{\partial p_k}\gamma_k S\gamma_5\right] = \mathtt{Tr}\left[S\gamma_5 S V_k S\gamma_k S\gamma_5\right]$$

Applications:

- form factors of flavor-changing currents @ end point
- hadronic vacuum polarization @ zero momentum

- Vertices often require momentum expansion too (point-split currents)

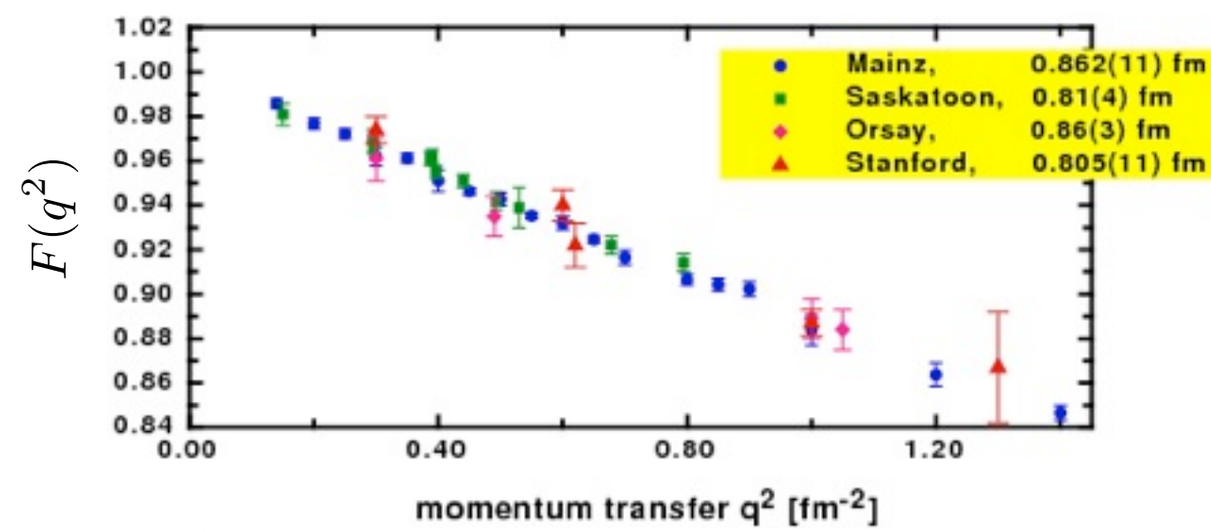
Applying the Method to Radii

- Measurement of *muonic hydrogen spectrum* leads to an extraction of the proton radius which is 4% smaller than CODATA value (*hydrogen + e-p scattering*)

- Unprecedented precision: 4% = 7σ

$$F(q^2) = 1 - \frac{q^2}{6} < r^2 > + \ldots$$

Experimental electron-proton scattering data



Lattice QCD calculations of proton radius

Largely limited to "connected", isovector

Requires fine lattice $\delta < r^2 >= [0.2\,\mathtt{fm}]^2 \sim (2a)^2 \sim \lambda\!\!\!^{-}_p^2$

Model small momentum transfer behavior...

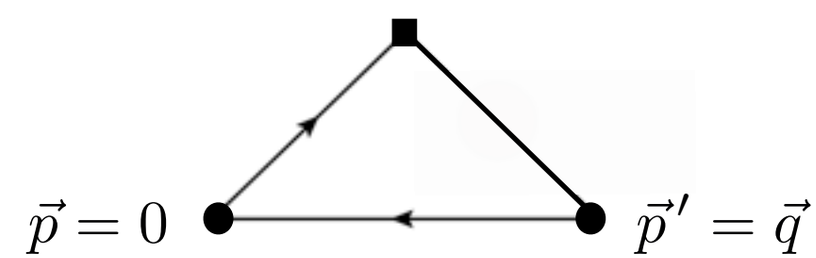
Experimental momentum problem is similar to lattice

Applying the Method to (Meson) Radii

- Current matrix element $\langle \phi(\vec{p}\,')|J_\mu|\phi(\vec{p})\rangle = e(p'+p)_\mu F(q^2)$

Pion form factor is connected
Well-tested lattice calculation

$$F(q^2) = 1 - \frac{q^2}{6} < r^2 > + \ldots$$



- Rest frame (similar findings in Breit kinematics)

$$q^2 = \vec{q}^{\,2}[1 + O(\vec{q}^{\,2}/m_\pi^2)]$$

- Lattice 3-pt. correlation function (ground-state saturation)

$$C_4(\vec{q}, \vec{0}\,|x_4, y_4) = i(E' + m_\pi)_4 F(q^2)|Z|^2 \frac{e^{-E'(x_4-y_4)} e^{-m_\pi y_4}}{2E'\, 2m_\pi}$$

Applying the Method to (Meson) Radii

- Current matrix element $\langle\phi(\vec{p}\,')|J_\mu|\phi(\vec{p})\rangle = e(p'+p)_\mu F(q^2)$ *Pion form factor is connected*
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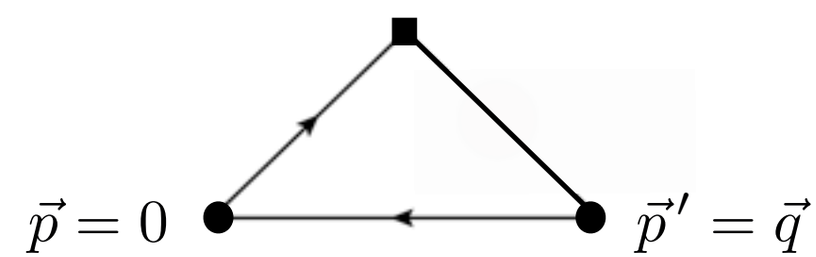

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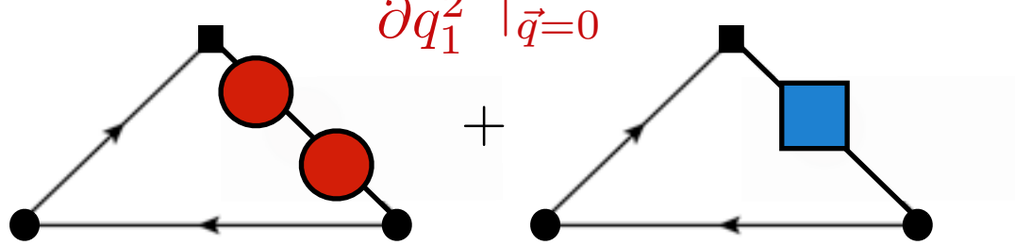
$$\frac{\partial^2 C_4}{\partial q_1^2}\Big|_{\vec{q}=0}$$



$$-\frac{3}{C_4(0)} \frac{\partial^2 C_4}{\partial q_1^2}\Big|_{\vec{q}=0} =< r^2 > + \frac{3}{2}\frac{1}{m_\pi^2} + \frac{x_4 - y_4}{m_\pi}$$

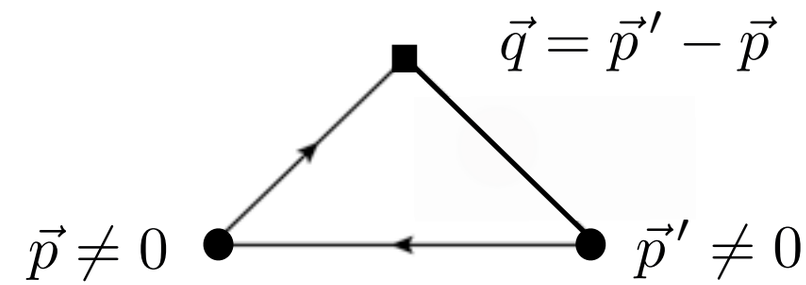
Not impossible, not particularly clean

Applying the Method to (Meson) Radii

- Current matrix element $\langle\phi(\vec{p}\,')|J_\mu|\phi(\vec{p})\rangle = e(p'+p)_\mu F(q^2)$

Pion form factor is connected
Well-tested lattice calculation

$$F(q^2) = 1 - \frac{q^2}{6} < r^2 > + \ldots$$



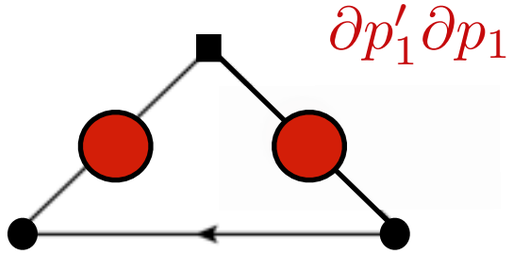
- Current matrix element in arbitrary frame

$$q^2 = 2[E'E - m_\pi^2 - \vec{p}\,' \cdot \vec{p}]$$

- Lattice 3-pt. correlation function (ground-state saturation)

$$C_4(\vec{p}\,', \vec{p}|x_4, y_4) = i(E'+E)_4 F(q^2)|Z|^2 \frac{e^{-E'(x_4-y_4)}e^{-Ey_4}}{2E'\,2E}$$

$$\frac{\partial^2 C_4}{\partial p_1' \partial p_1}\bigg|_{\vec{p}\,'=\vec{p}=0}$$



$$\frac{3}{C_4(0)} \frac{\partial^2 C_4}{\partial p_1' \partial p_1}\bigg|_{\vec{p}\,'=\vec{p}=0} =< r^2 >$$

Direct access to radius at zero momentum

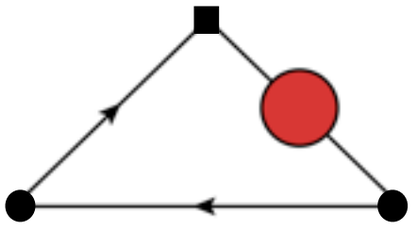
Exercise

$$\frac{1575}{C_4(0)} \frac{\partial^6 C_4}{\partial p_1' \partial p_1 \partial p_2' \partial p_2 \partial p_3' \partial p_3}\bigg|_{\vec{p}\,'=\vec{p}=0} =< r^6 >$$

Rome Method in a Finite Volume

Taylor Series Coefficient

$$C[\vec{p},\mathcal{U}] = C^{(0)}[\mathcal{U}] + p_i C_i^{(1)}[\mathcal{U}] + \cdots$$



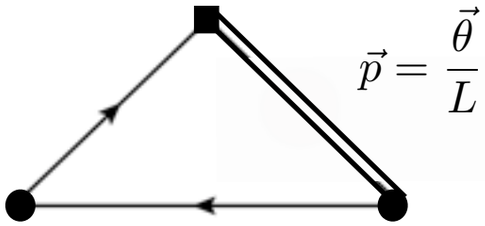
$$F(0) = \int D\mathcal{U}\, \mathcal{P}[\mathcal{U}]\, C_i^{(1)}[\mathcal{U}]$$

Taylor Series Coefficient Correlator

- Infinite volume limit is exact: continuous momenta admit differentiation
- To address finite volume effects must be able to derive expansion on a ***fixed-size*** lattice

Introduce: active and spectator quarks

$$\psi(x+L) = e^{i\theta}\psi(x)$$



Continuous twist parameters enable differentiation in a finite volume

$$C[\vec{p},\mathcal{U}] = C^{(0)}[\mathcal{U}] + p_i C_i^{(1)}[\mathcal{U}] + \cdots$$

Taylor Series Coefficient Correlator

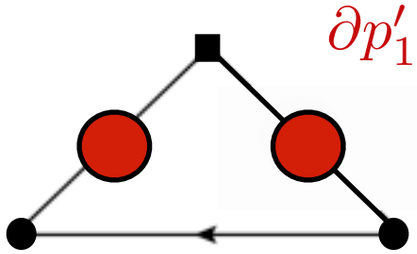
Calculated at vanishing twist, i.e. *strictly periodic*

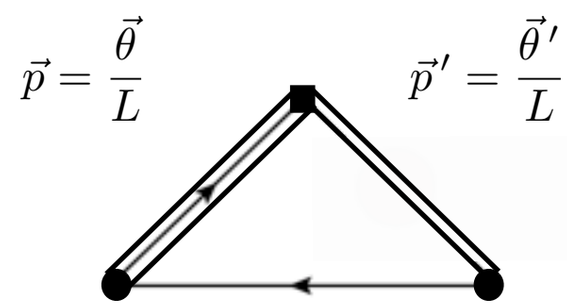
- Recipe for volume effects: compute Taylor coeffs. on torus from twisted active quarks

Volume Effects at Zero Momentum

- Example: ascertain volume effect on determination of pion charge radius with ChPT

$$\left.\frac{\partial^2 C_4}{\partial p_1' \partial p_1}\right|_{\vec{p}\,'=\vec{p}=0}$$





2 active valence quarks, one spectator
--> partially twisted SU(5|3) ChPT

- Partial twisting is not sick from partial quenching $m_{\rm sea} = m_{\rm val}$

- Determine pion current matrix element using partially twisted ChPT

Frame dependence:
no boost invariance

Isospin twisted --> rest frame result

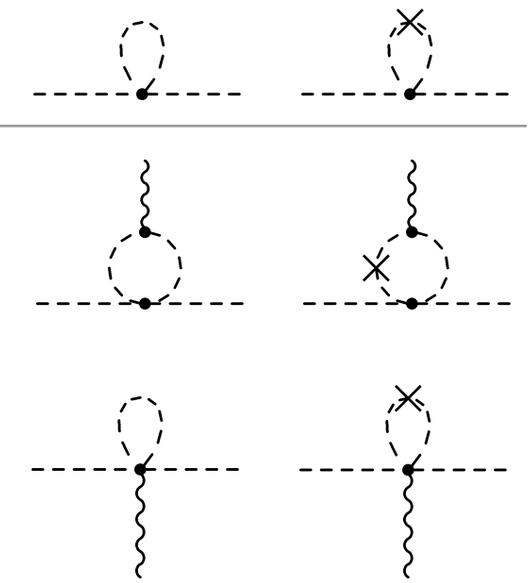
F.-J. Jiang, B.C. Tiburzi PLB**645**, 314 (2007)

Breit kinematics $\vec{p}\,' = -\vec{p} = \frac{\vec{q}}{2}$

F.-J. Jiang, B.C. Tiburzi PRD**78**, 037501 (2008)

For method @ zero momentum need arbitrary frame

Finite Volume Computation



Fourier momenta

$$\mathcal{M}_\mu(\vec{p}\,',\vec{p}) = \langle \pi'^{+}(0)|J_\mu|\pi^{+}(0)\rangle$$

$$J_\mu = \overline{u}'_{\theta'}\gamma_\mu u_\theta$$

Flavor changing current

- Time-component of current $\Delta\mathcal{M}_4 = \mathcal{M}_4(L) - \mathcal{M}_4(\infty)$

Extra Form Factor

$$\Delta\mathcal{M}_4 = (p'+p)_4\,\Delta F + q_4\,\Delta G$$

ΔG *exceptionally* messy in arbitrary frame

$$\frac{\partial^2(q_4\Delta G)}{\partial p'_1\partial p_1}\Big|_{p'=p=0} = \left(q_4\frac{\partial^2\Delta G}{\partial p'_1\partial p_1}\right)\Big|_{p'=p=0} = 0$$

[it is needed in rest frame...]

Originally found in

F.-J. Jiang, B.C. Tiburzi PLB**645**, 314 (2007)

Origin found in

J. Bijnens, J. Relefors JHEP**1405**, 015 (2014)

.... needed to maintain WTI

ΔF fortunately simpler

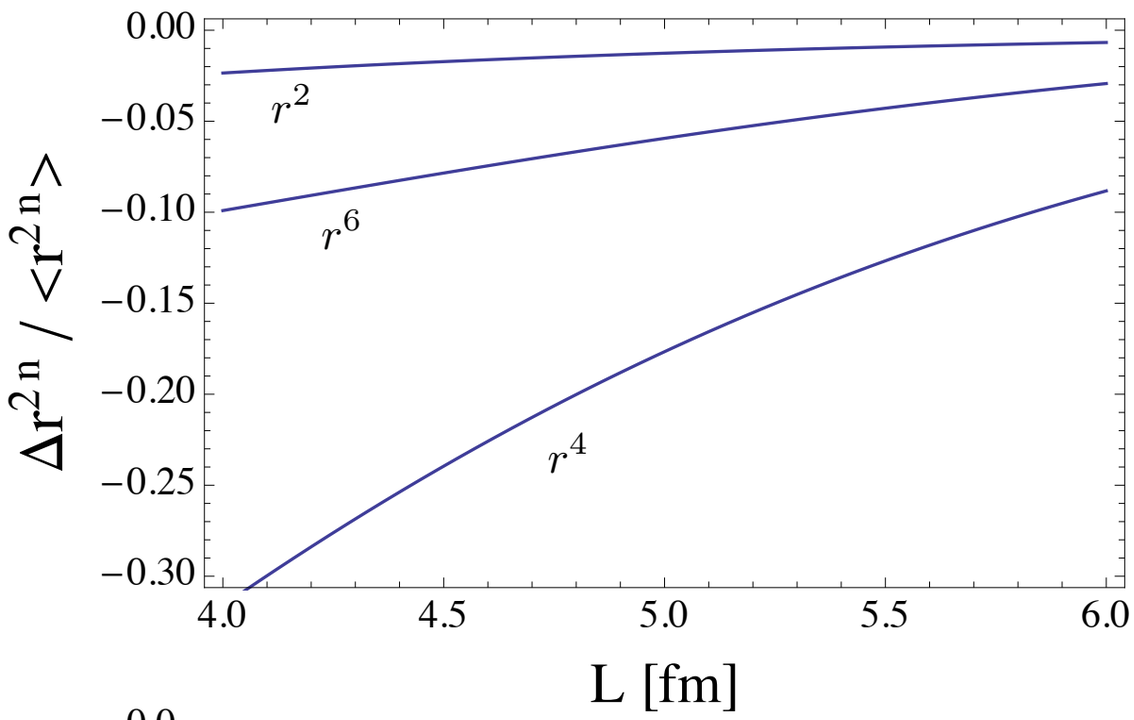
$$\Delta F \sim \frac{e^{-m_\pi L}}{(m_\pi L)^{3/2}}$$

$$\Delta r^2 \sim (m_\pi L)^{1/2} e^{-m_\pi L}$$

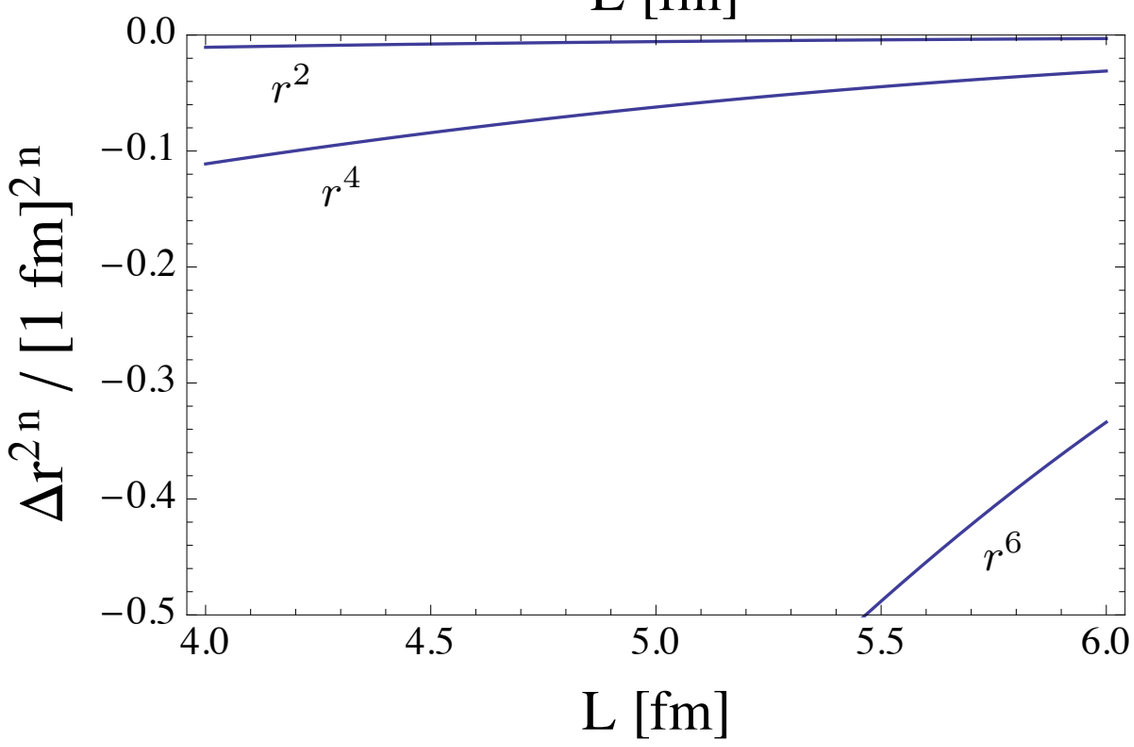
$$\Delta r^4 \sim (m_\pi L)^{5/2} e^{-m_\pi L}$$

$$\Delta r^6 \sim (m_\pi L)^{9/2} e^{-m_\pi L}$$

Finite Volume Results



$$< r^2 >_{\rm phys} = [0.67\,\mathtt{fm}]^2$$
$$< r^4 >_{\chi\rm PT} = [0.77\,\mathtt{fm}]^4$$
$$< r^6 >_{\chi\rm PT} = [1.5\,\mathtt{fm}]^6$$



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Summarizing Volume Effects at Zero Momentum

I. Extension of *Rome Method* to radii:
Need *initial- & final-state* quarks to isolate radii cleanly

II. Addressed finite volume effects:
Twist angle differentiation, evaluate at zero twist

III. Pion radius in chiral perturbation theory:
SU(5|3) computation shows volume dependence at %-level

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IV. Isovector nucleon magnetic moment + charge & magnetic radii:
"Straightforward" generalization, SU(6|4), magnetic more volume sensitive

V. Disconnected current insertion:
Expected to be small, no obvious generalization of method