Confinement, the Abelian Decomposition, and the Contribution of Topology to the Static Quark Potential

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- We intend to investigate confinement using the CDG Abelian decomposition
- Our eventual aim is to demonstrate
  1. That the static quark potential can be expressed entirely in terms of an Abelian restricted field (Mesons are colour singlets)
  2. To identify topological objects which cause or partially cause confinement
  3. To show that and under what conditions these objects may appear in QCD
- The CDG Abelian decomposition is a means of extracting an Abelian component from the gauge field
- Our method needs no gauge fixing, or arbitrary cuts of the field, or additional path integrals.
- Why do this? The Abelian field requires no path ordering – much simpler to analyse.

- What is the gauge-invariant Abelian decomposition?
- We start by choosing a field $\theta(x) \in SU(N)$
- We then construct $n_a = \theta\lambda_a\theta^\dagger$
- We select the Abelian directions $n_j \equiv n_3, n_8, \dots$
- We choose fields $\hat{A}_\mu$ and X_μ so that

$$A_\mu = \hat{A}_\mu + X_\mu \qquad D_\mu[\hat{A}]n_j = 0 \qquad \text{tr}(n_j X_\mu) = 0$$

- This has a known solution,

$$\begin{aligned}\hat{A}_\mu =& \frac{1}{2} n_j \text{tr}(n_j A_\mu) + \frac{i}{4g}[n_j, \partial_\mu n_j]\\ F_{\mu\nu}[\hat{A}] =& \frac{n_j}{2}\big[\partial_\mu \text{tr}(n_j A_\nu) - \partial_\nu \text{tr}(n_j A_\mu)\big] + \frac{i}{8g} n_j \text{tr}(n_j[\partial_\mu n_k, \partial_\nu n_k]).\end{aligned}$$

- On the lattice, we can write this as

$$U_{\mu,x} = \hat{X}_{\mu,x}\hat{U}_{\mu,x},$$
$$\hat{U}_{\mu,x}n_{j,x+\hat{\mu}}\hat{U}^{\dagger}_{\mu,x} - n_{j,x} = 0 \quad \mathrm{tr}(n_{j,x}(\hat{X}_{\mu,x} - \hat{X}^{\dagger}_{\mu,x})) = 0$$

- Choose solution where tr$\hat{X}_\mu$ is maximised
- $U \equiv$ standard gauge link constructed from A
- $\hat{U} \equiv$ gauge link constructed from $\hat{A}$
- $\hat{X}$ corresponds to X
- The gauge transformations are $(\Lambda_x \in \mathrm{SU}(N))$

$$U_{\mu,x} \to \Lambda_x U_{\mu,x+\hat{\mu}}\Lambda^{\dagger}_{x+\hat{\mu}} \qquad \theta_x \to \Lambda_x\theta_x,$$
$$\hat{U}_{\mu,x} \to \Lambda_x \hat{U}_{\mu,x+\hat{\mu}}\Lambda^{\dagger}_{x+\hat{\mu}} \qquad \hat{X}_{\mu,x} \to \Lambda_x\hat{X}_{\mu,x}\Lambda^{\dagger}_x$$

- Paths of gauge links constructed from $\hat{U}$ are gauge covariant.

- So how do we choose θ?
- We will extract the static potential from the Wilson Loop
- We choose θ so that along the Wilson Loop $\hat{U}_\mu = U_\mu$.
- This θ_x contains the eigenvectors of the Wilson Loop operator starting and ending at position x.
- This guarantees that the Wilson Loop for the restricted field is identical to the Wilson Loop for the non-Abelian field.
- Furthermore,

$$\hat{U}_\mu(x) n_{j,x+\hat{\mu}} \hat{U}_\mu(x)^\dagger - n_{j,x} = 0 \quad \Leftrightarrow \quad [\theta_x^\dagger \hat{U}_{\mu,x} \theta_{x+\hat{\mu}}, \lambda_j] = 0,$$

- $\theta_x^\dagger \hat{U}_{\mu,x} \theta_{x+\hat{\mu}}$ is Abelian and gauge invariant – no need for path ordering.
- The coloured field X does not contribute to confinement – mesons are colour-neutral.

$$\hat{A}_\mu = \frac{1}{2} n_j \mathrm{tr}(n_j A_\mu) + \frac{i}{4g}[n_j, \partial_\mu n_j] = \frac{1}{2} n_j \mathrm{tr}(n_j A_\mu - i\theta\partial_\mu\theta^\dagger) + \frac{i}{g}\theta\partial_\mu\theta^\dagger$$

$$F_{\mu\nu}[\hat{A}] = \frac{n_j}{2}\left[\partial_\mu \mathrm{tr}(n_j A_\nu) - \partial_\nu \mathrm{tr}(n_j A_\mu)\right] + \frac{i}{8g} n_j \mathrm{tr}(n_j[\partial_\mu n_k, \partial_\nu n_k])$$

- Both $F_{\mu\nu}[\hat{A}]$ and $\hat{A}_\mu$ depend on two terms:
  1. A function of both A_μ and θ (the Maxwell term)
  2. A function of θ alone (the Topological term) (Topological field strength $= H^{3,8}_{\mu,\nu}$).
- Parametrise the SU(2) θ in terms of a, c, d

$$\theta = \begin{pmatrix} \cos a & i \sin a e^{ic} \\ i \sin a e^{-ic} & \cos a \end{pmatrix} \begin{pmatrix} e^{id} & 0 \\ 0 & e^{-id} \end{pmatrix}, \quad \bar{\phi} = \begin{pmatrix} 0 & ie^{ic} \\ -ie^{-ic} & 0 \end{pmatrix}$$

- d makes no contribution ($n_3 = \theta\lambda_3\theta^\dagger$) – fix it to zero
- $\theta^\dagger\partial_\mu\theta = \lambda_3 \sin^2 a\partial_\mu c + \bar{\phi}\cos 2a\partial_\mu a$
- $\mathrm{tr}(n_3[\partial_\mu n_3, \partial_\nu n_3]) = \partial_\mu a\partial_\nu c - \partial_\nu a\partial_\mu c$.

- We want to map a and c to E^4; (Wilson Loop in xt plane)
- $(t, x, y, z) = r(\cos\psi_3, \sin\psi_3\cos\psi_2,$
  $\sin\psi_3\sin\psi_2\cos\psi_1, \sin\psi_3\sin\psi_2\sin\psi_1).$
- There are two types of topological object available
  1. The Wang-Yu Monopole (π_2 topology):
     $a = \psi_1/2, c = \nu_{WY}\psi_2$.
  2. Another object (π_1 topology) with $a(r), c = \nu_T\psi_3$,
     Appears at $a \sim 0$ or $a \sim \pi/2$.
- ν_{WY} and ν_T are integer winding numbers.
- Both winding numbers are invariant under continuous gauge transformations and deformations of the gauge field.

- We apply Stoke's theorem to our Abelian representation of the Wilson Loop
  1. A surface integral over the continuous part of $F_{\mu\nu}[\hat{A}]$
  2. Line integrals around each topological singularity
- This line integral resembles $\oint dx_\mu (\sin^2 a)\partial_\mu c$.
- It is proportional to the winding number ν_T (the monopoles do not directly contribute).
- Since the number of these objects is proportional to the area, we expect an area law scaling for the Wilson Loop.

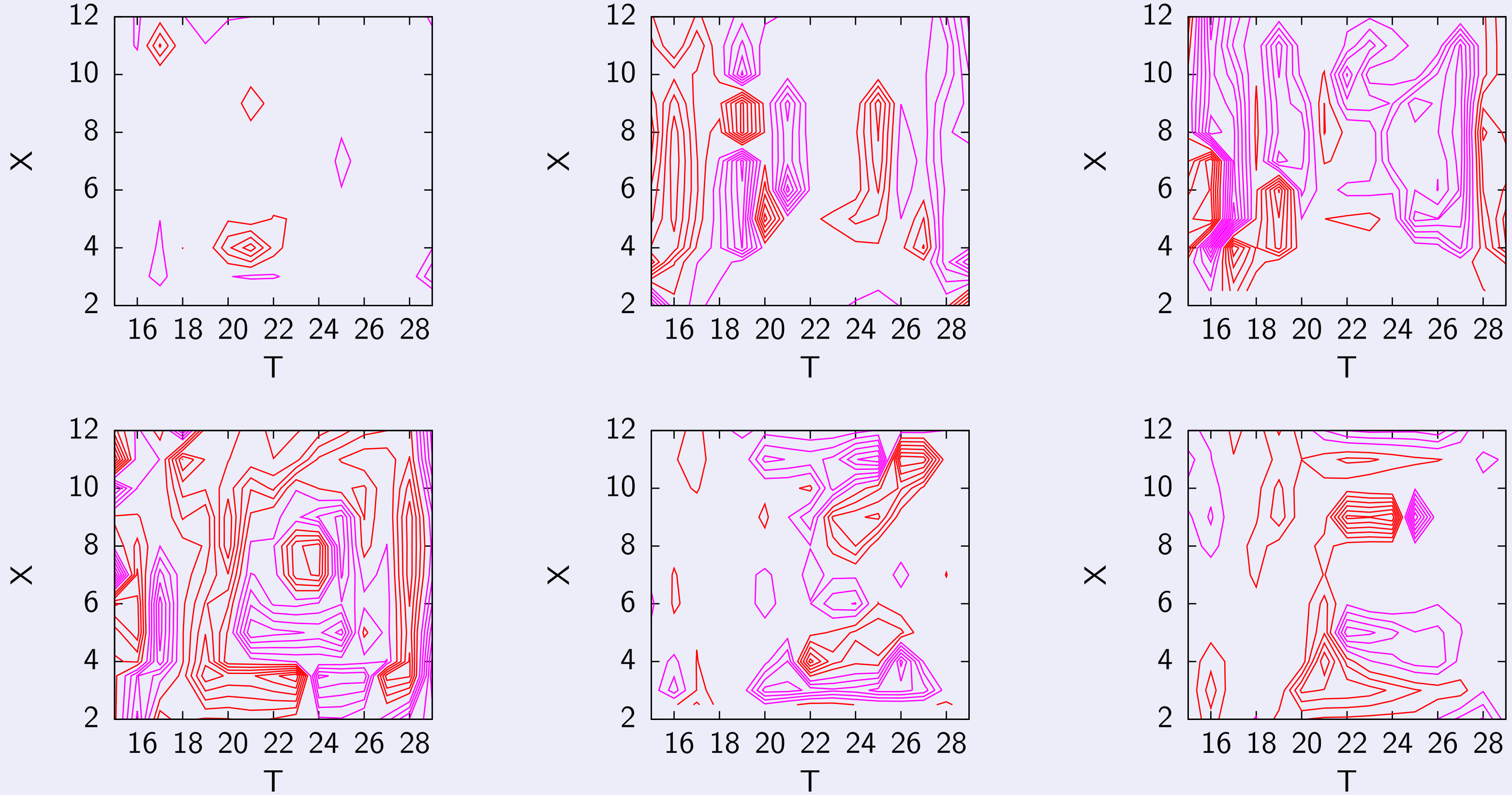
- It is easy to calculate the topological field strength surrounding each of these objects.
- We characterise $H^{3,8}_{\mu\nu}$ in terms of 'Electric' and 'Magnetic' fields.

1. **Monopoles**: Nothing in the E_x field
   1-D lines of high field strength in B_x, B_y and B_z parallel to the T axis; or
   1-D lines in B_x, E_y and E_z parallel to the X axis

2. π_1 **objects**: 0-D Points in the E_x field, accompanied by some of
   1-D lines in B_x, B_y and B_z parallel to the T axis; and/or
   1-D lines in B_x, E_y and E_z parallel to the X axis

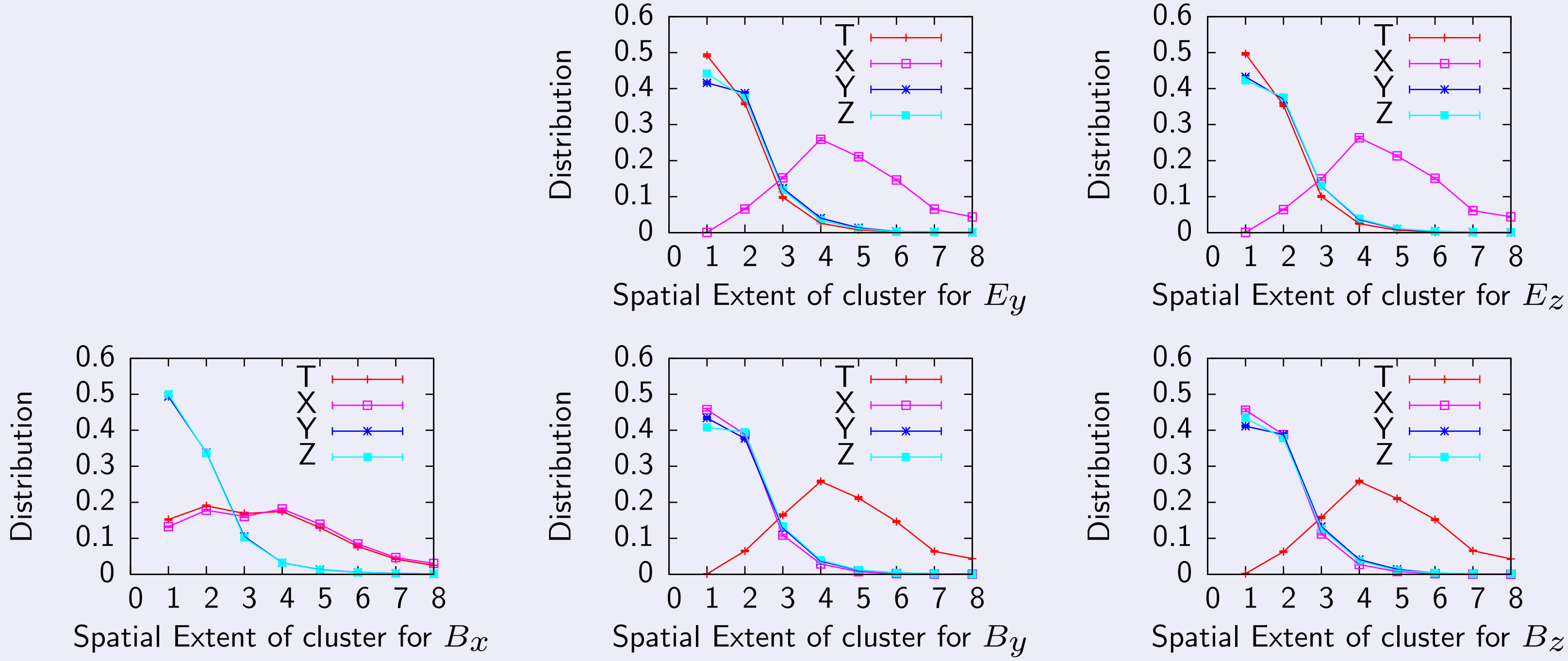
$\beta 8.52$ TILW gauge action quenched, $a \sim 0.095\text{fm}$.

The X (left) Y (middle) and Z (right) components of the Electric (top) and Magnetic (bottom) Abelian Field Strengths



The plots show the field strength $H^8_{\mu\nu}$ (red and purple contour lines) on a slice of the lattice in the XT plane.

These plots show the extent in each spatial direction of structures of connected high field strength for the $H^8_{\mu\nu}$ field.



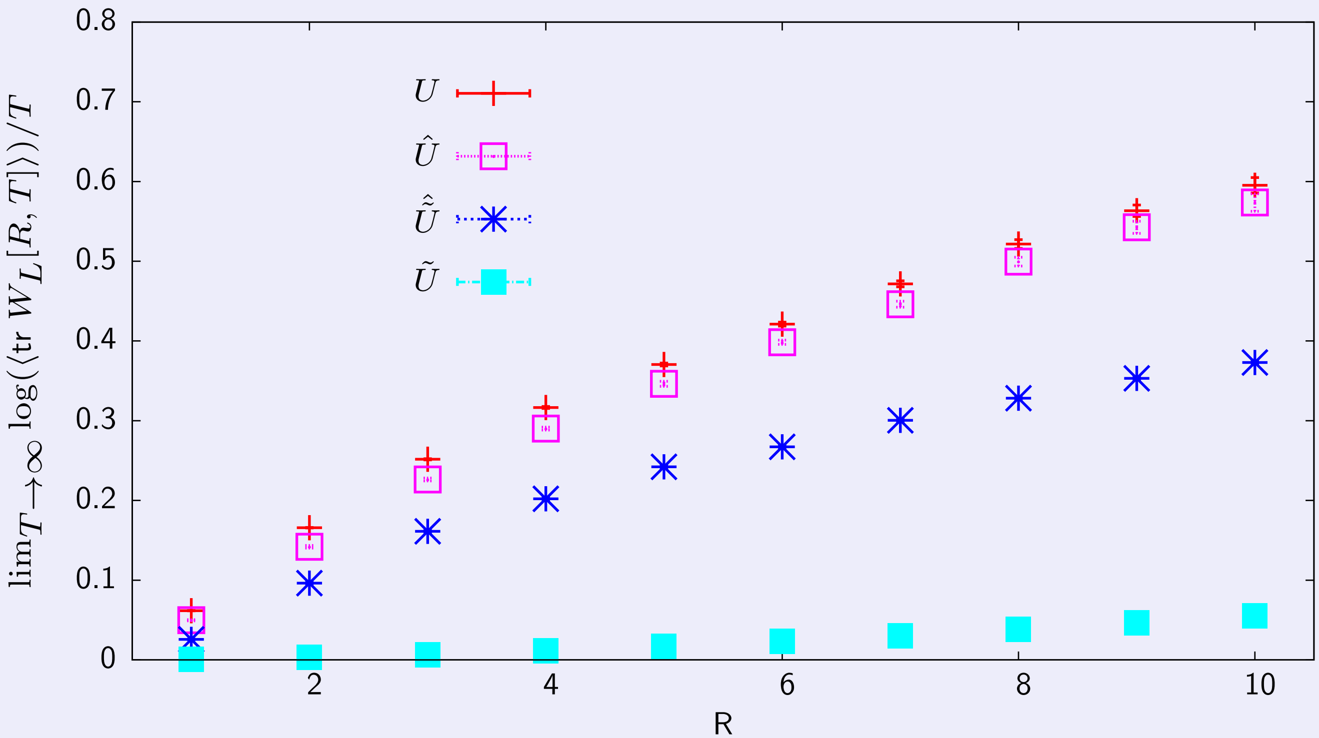
The location of the strings are correlated with the spatial location of the points in the E_x field.
So it seems like the topological vacuum does indeed contain both monopoles and the π_1 topological objects.

- So does the topological part dominate the string tension ρ calculated from the restricted Abelian field $\hat{A}$?
- Previous results, including our own, suggested that it does (accounting for $\sim 90\%$)
- None of these studies had the exact relationship between the restricted and Yang-Mills string tensions.
- How do we extract the topological part of the string tension?
- Other approaches have extracted the part of the field strength which gives the magnetic charges – this observable is not gauge invariant, so requires gauge fixing

- We use a gauge-invariant approach, calculating the string tension from:

$$\hat{\tilde{A}}_\mu = \frac{n_j}{2}\text{tr}(n_j \tilde{A}) + \frac{i}{4g}[n_j, \partial_\mu n_j]$$

- We have replaced A with a stout-smeared gauge field $\tilde{A}$
- Attempt to apply enough smearing sweeps that its contribution is negligible
- Leaves us with just the topological term $+$ gauge transformation
- In practice, we still found dependence of the string tension on the number of smearing sweeps even after 2500 sweeps
- From 600 to 2500 smearing sweeps, we also observed that $\rho_{\theta,\tilde{A}} = \text{constant} + \rho_{\tilde{A}}$
- This allowed us to extract the topological string tension ρ_{Top} as though $\rho_{\tilde{A}}$ were zero.



β	8.0	8.3	8.52	$8.3L$
U	0.0976(22)	0.0600(11)	0.0420(8)	0.0598(8)
$\hat{U}$	0.0922(38)	0.0578(19)	0.0418(18)	0.0629(27)
$\hat{\tilde{U}}_{2500}$	0.0378(14)	0.0256(9)	0.0221(7)	0.0276(12)
$(\hat{\tilde{U}}_{2500} - \tilde{U}_{2500})/U$	0.27(1)	0.28(1)	0.32(1)	0.32(2)

Concluding Questions

- So we have a cute little model which predicts confinement:
  - Area Law scaling of the Wilson Loop
  - Why Mesons are colour neutral
- The field strength seems to back up the model – we see the objects we expect
- However, the string tension does not show that the topological θ term is dominant
- The Maxwell term seems to also contribute significantly to confinement
- Why is this the case?
- Why the discrepancy with other results (including our own earlier results) using different choices for θ?