Tensor renormalization group study of the classical $O(3)$ model

Judah Unmuth-Yockey,
Haiyuan Zou, Yannick Meurice, James Osborn

The Model

- We consider the Hamiltonian: $H = -\sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$.
- Here $\vec{S}$ is a three-component vector, and $\langle ij \rangle$ is a sum over nearest-neighbor pairs in two dimensions.
- The partition function is $Z(\beta) = \sum_{\{\vec{S}\}} e^{-\beta H}$.
- The sum is over spin configurations on a two-dimensional lattice.

Important aspects of the study

- non-Abelian model to test tensor renormalization
- $\langle \vec{S} \rangle = 0$ for all β
- This model is known to be asymptotically free for large β.
- For Monte Carlo calculations, this model has no sign problem.
- In this tensor formulation, some tensor elements are negative
- Blocking methods like Tensor Renormalization appear insensitive to this attribute.
- Other expansion methods avoid this sign problem.

Denbleyker et al. *Phys. Rev. D* **89**, 016008
Wolff, *Nuc. Phys. B* **824**, 254

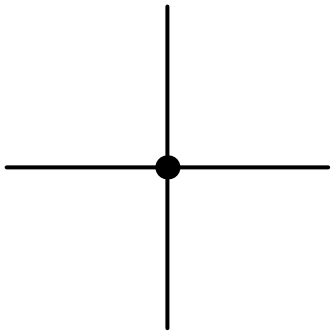
Tensor renormalization

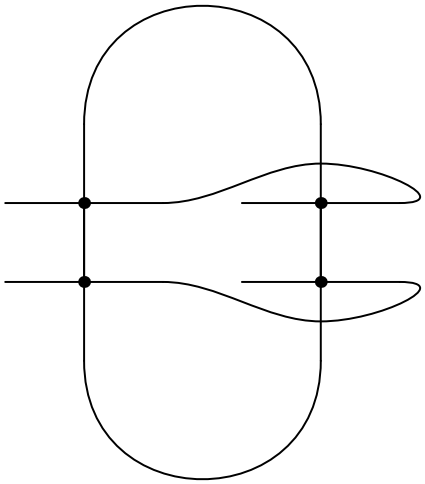
- Proposed by Levin & Nave in '07
- Techniques used here similar to Xie et al. '12
- the Higher Order SVD Tensor Renormalization Group
- Built out of tensor building blocks

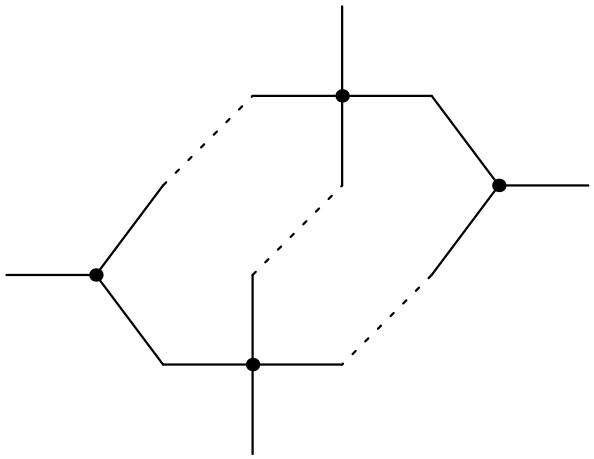
Levin & Nave *Phys. Rev. Lett.* **99**, 120601
Xie et al. *Phys. Rev. B* **86**, 045139

Coarse-graining and Blocking

1. Form initial tensor
2. Decide what to keep
3. Can't keep all the information $\implies$ project what matters

1. 

2. 

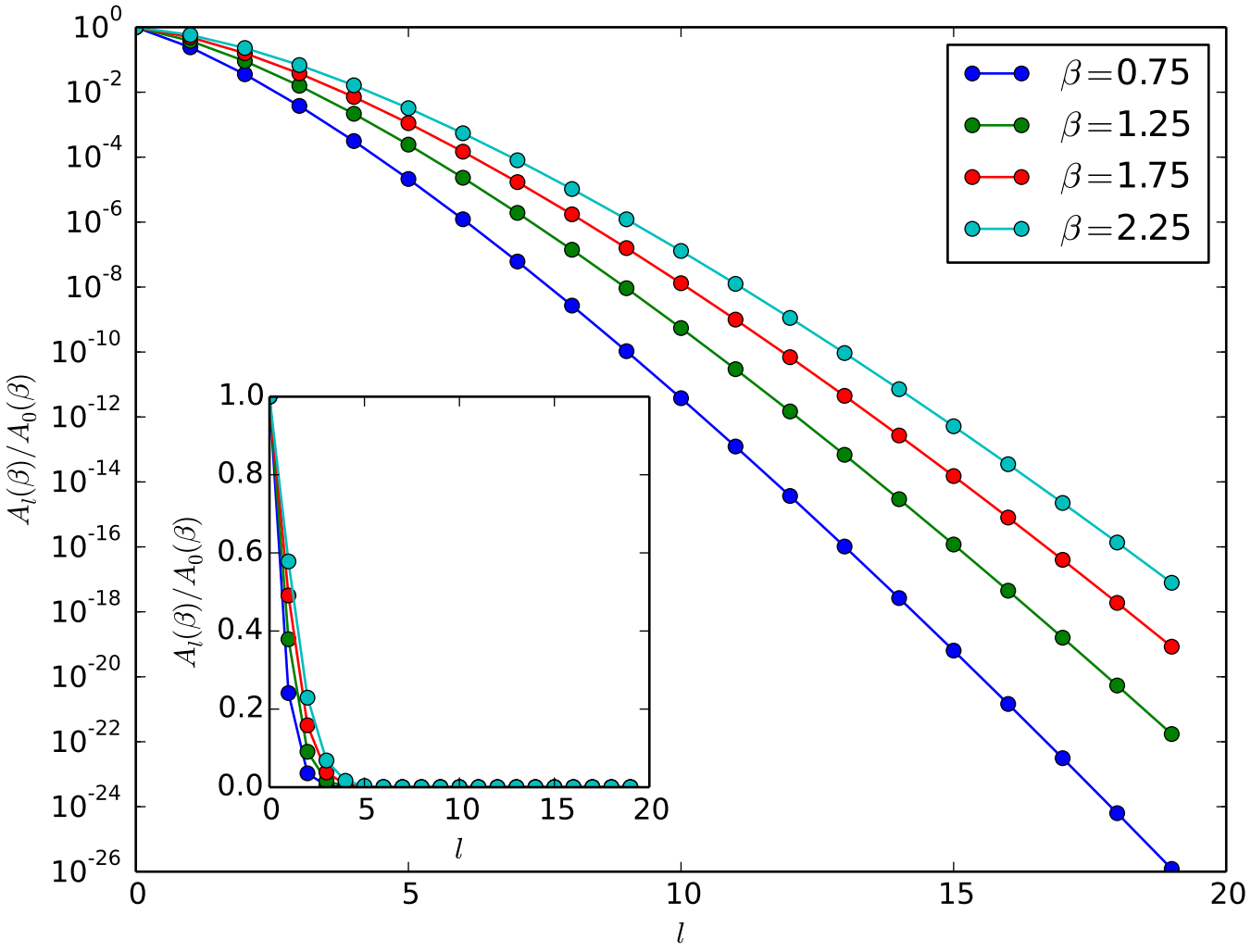
3. 

Tensor Formulation for $O(3)$

- A simple approach is through Harmonic analysis.
- $O(3)$ has two quantum numbers, however the expansion coefficients only depend on one, l.
- $$\exp[\beta \cos \gamma_{ij}] \to \sum_l A_l(\beta) P_l(\cos \gamma_{ij})$$
- Now
$$P_l(\cos \gamma_{ij}) = \frac{4\pi}{2l+1} \sum_m Y_{lm}(\theta_i, \phi_i) Y^*_{lm}(\theta_j, \phi_j)$$
- With the angular dependence decoupled, angular integration can now take place.

Tensor Formulation Cont.

The relative size of the coefficients for various l values.



The $O(3)$ Tensor

- The Spherical Harmonics are associated with pairs of sites, or links.
- Since there are four impinging links per site on the lattice, the angular integration site-wise is

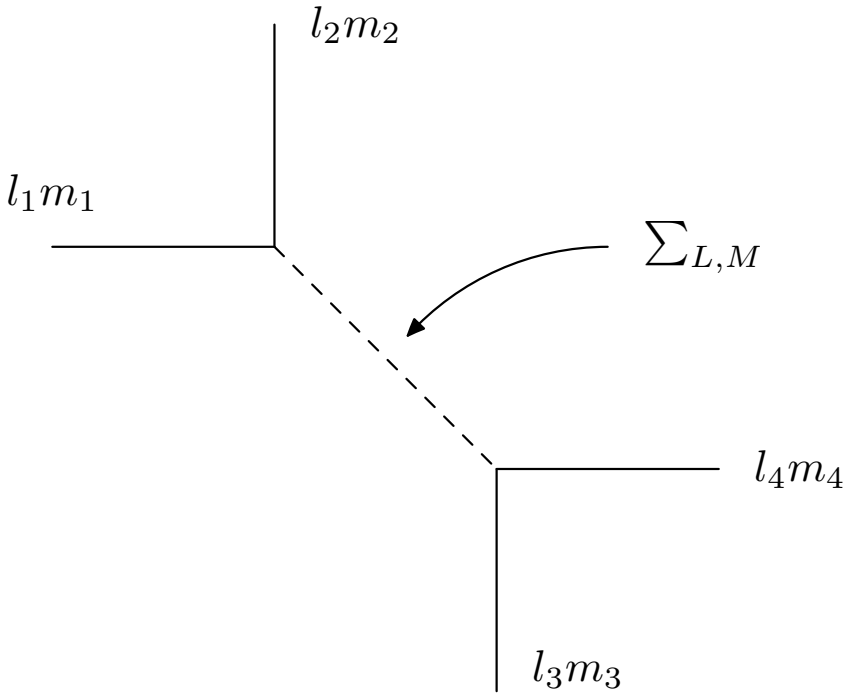
$$\int d\Omega \; Y^*_{lm} Y^*_{l'm'} Y_{l''m''} Y_{l'''m'''}.$$

- The constraint from this is

$$\approx \sum_{L=|l-l'|}^{l+l'} \sum_{M=-L}^{L} C^{LM}_{lml'm'} C^{L0}_{l0l'0} C^{LM}_{l''m''l'''m'''} C^{L0}_{l''0l'''0}.$$

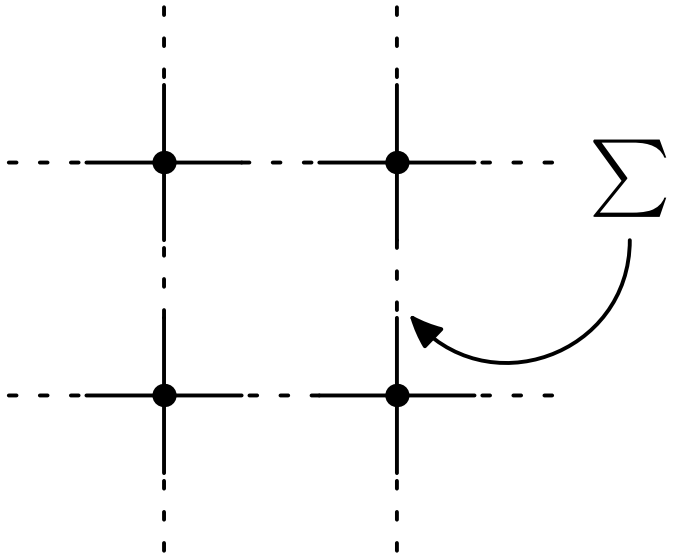
The $O(3)$ Tensor Cont.

- This constraint enforces the triangle in-equalities. The intermediate sum over L and M picks out irreducible representations of the angular momenta.



- Contrast with the Abelian case

Tensor view of the lattice



- The tensor formulation allows one to decouple the lattice at the location of the local constraint.
- The lattice is rebuilt by piecing these tensors together geometrically.

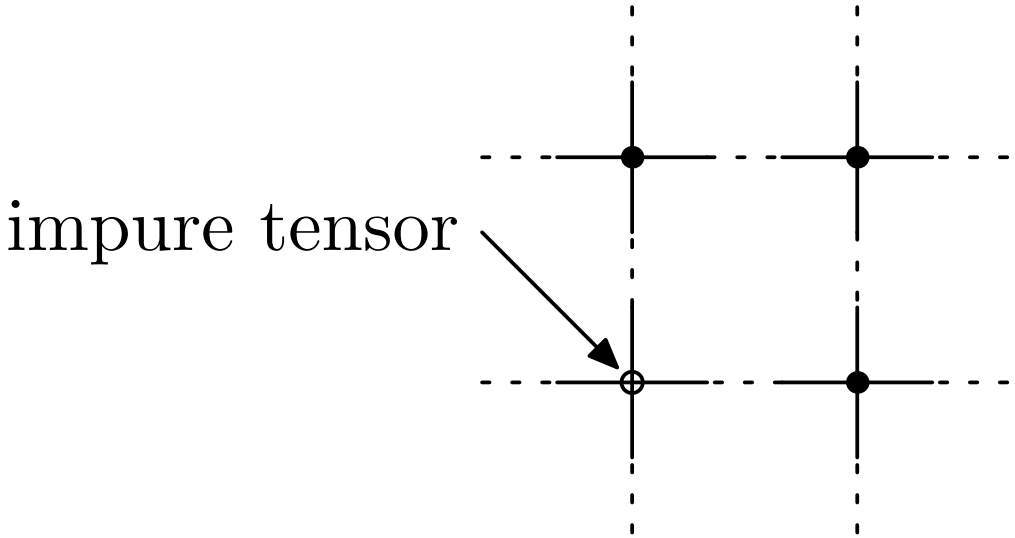
n-point Correlations

- n-point correlations can be realized on the lattice by inserting spin vectors at particular sites.
- These lead to a modified constraint at each site of insertion.
- For $O(3)$, we find an angular integral of the form

$$\int d\Omega \, Y_{l_1 m_1} Y_{l_2 m_2} Y^*_{l_3 m_3} Y^*_{l_4 m_4} Y_{1m}.$$

- This leads to a tensor whose elements are shifted, or an "impure" tensor.

n-point Correlations Cont.



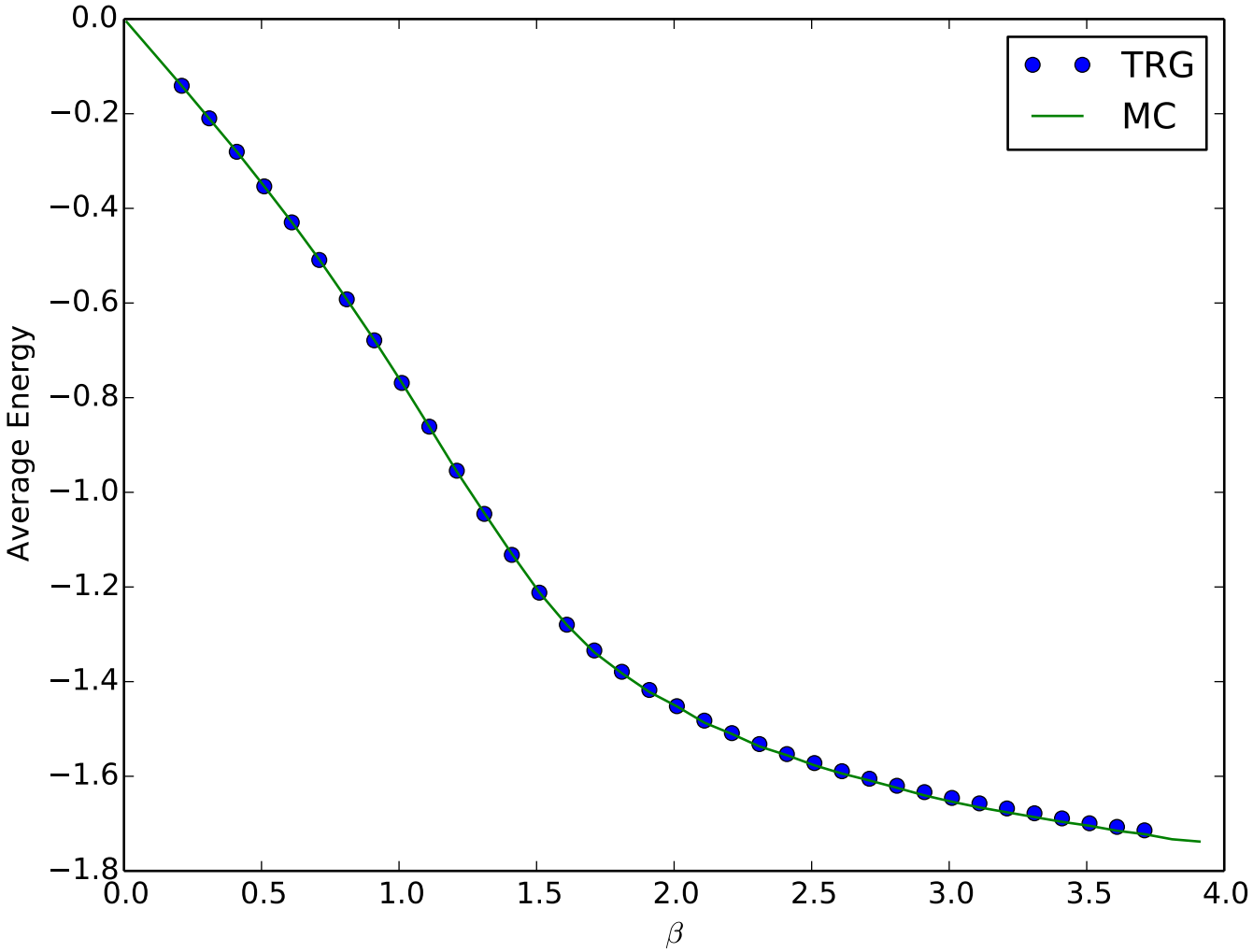
- Average values are computed by contracting impure tensors with pure.
- Same algorithmics

Results

- Monte Carlo comparison
- Infinite volume thermodynamics
- 2-point correlation data/comparison

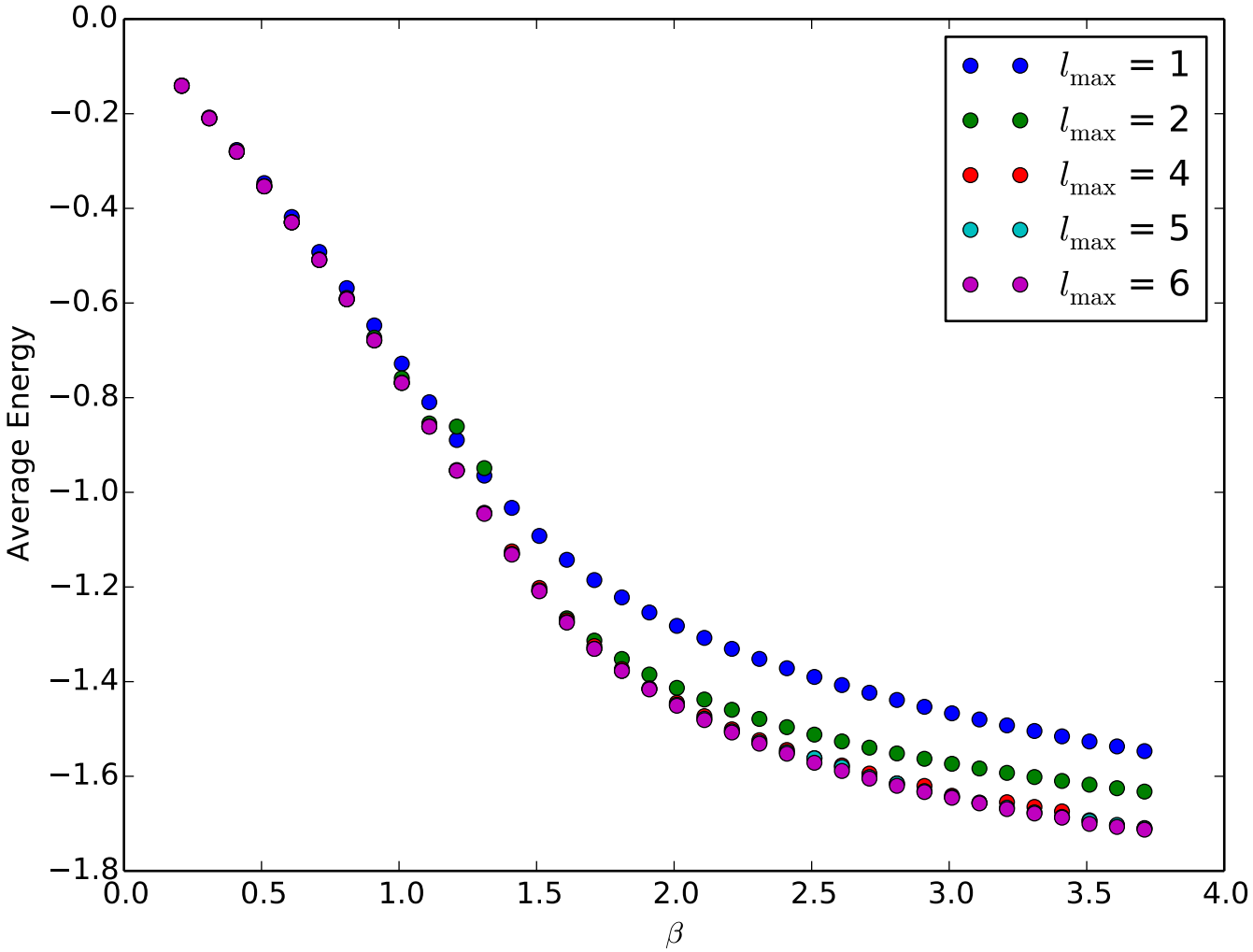
Results Cont.

Monte Carlo comparison: average energy on 32×32 lattice



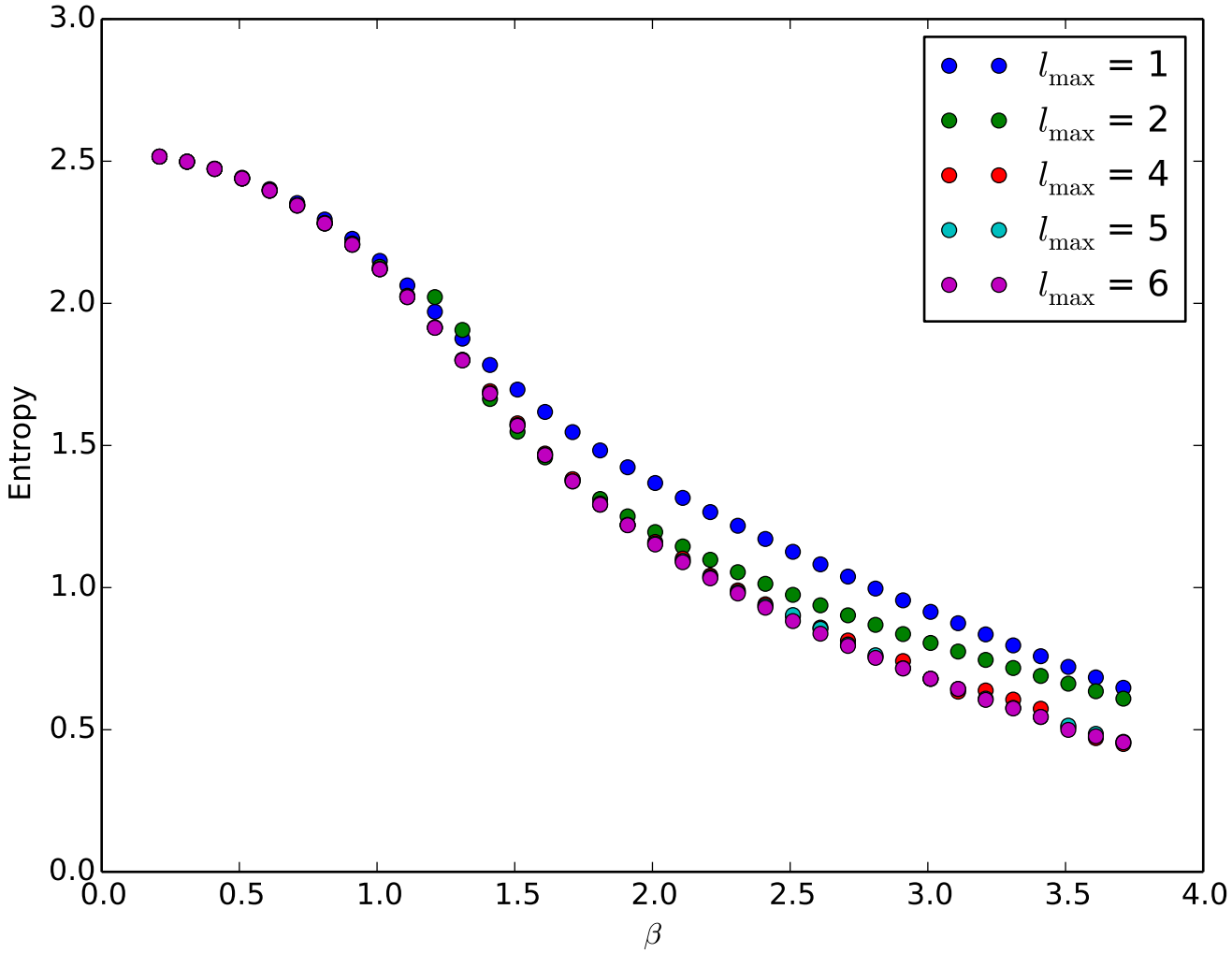
Results Cont.

Infinite volume thermodynamics: energy on $2^{20} \times 2^{20}$ lattice



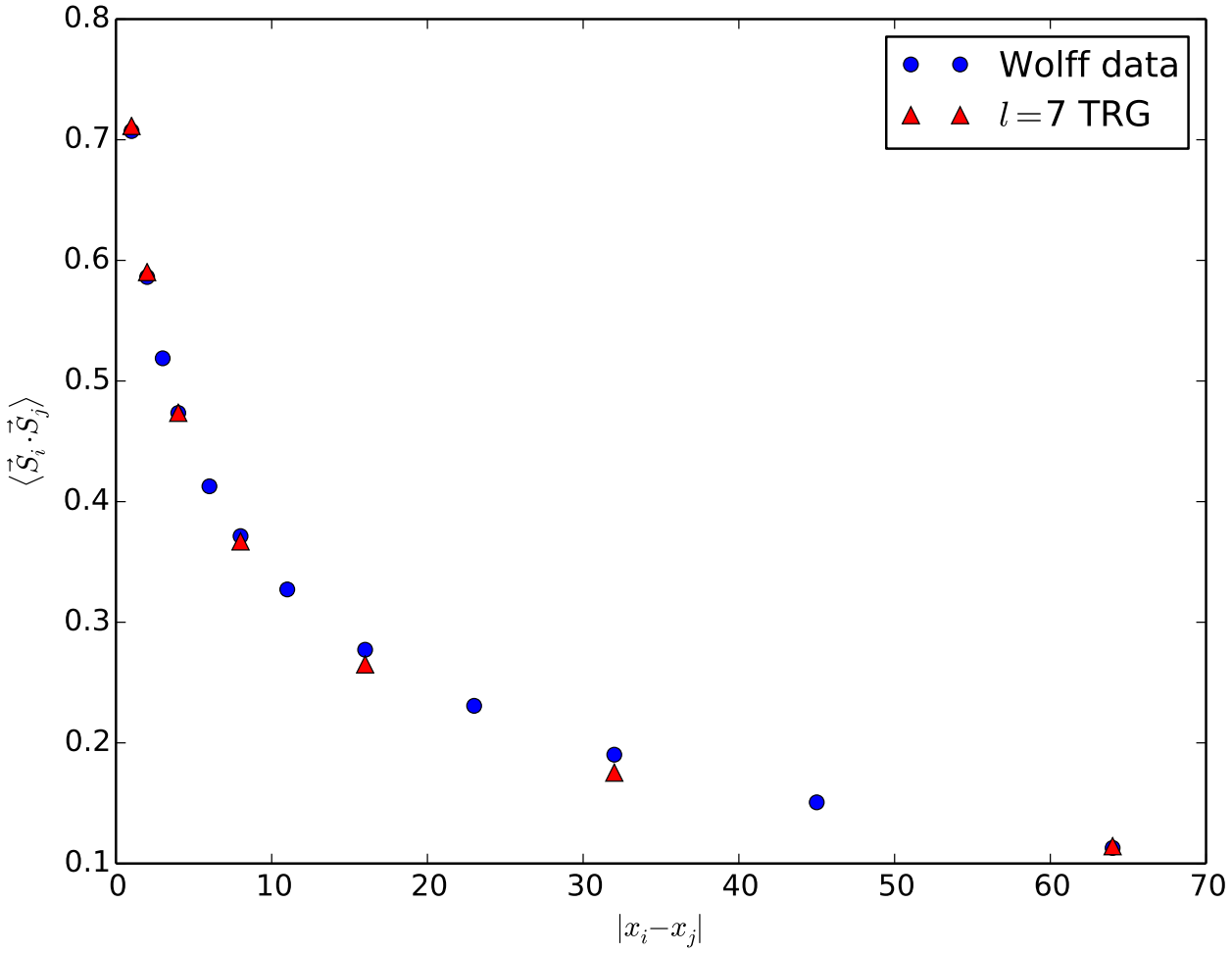
Results Cont.

Infinite volume thermodynamics: entropy on $2^{20} \times 2^{20}$ lattice



Results Cont.

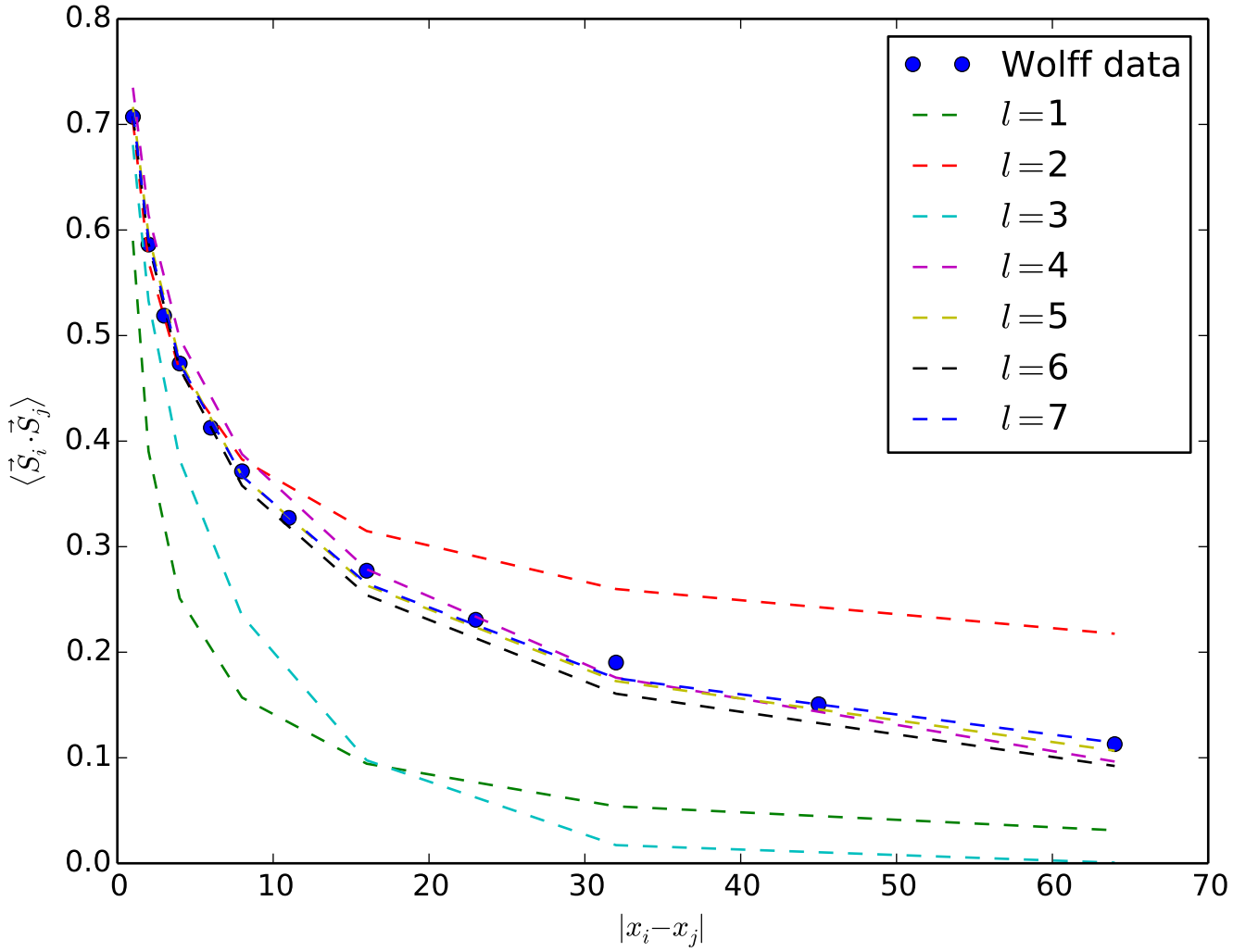
2-point correlation data & comparison on 128×128 lattice



Wolff, *Nuc. Phys. B* **334**, p.581

Results Cont.

2-point correlation data convergence on 128×128 lattice



Tensor Renormalization cost/benefit

Benefits:

- Appears insensitive to sign problems
- **Infinite volume is as easy as finite volume**
- Good qualitative & quantitative results on PCs
- Graphical interface
- **Can be formulated for many popular models**

Liu et al. *Phys. Rev. D* **88**, 056005

Cost:

- Memory scales like D^{2d}
- Computation time scales like D^{4d-1}!
- D is the number of states, and d is the spacetime dimension

Conclusion and Future Work

- Tensor renormalization is a powerful method in 2D.
- Uncover more efficient serial algorithms for tensor renormalization
- Understand phase structures and models through SVD and eigenvalue patterns
- Parallelization of tensor contractions could help the D^7 scaling.
- **Higher dimensions, if the above can be realized**

Thank you!