Non-perturbative renormalization of bilinear operators with Möbius domain-wall fermions in the coordinate space

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1. Introduction — NPR by X-space method

**Renormalization**

\[ \mathcal{O}^{\text{lat}}(1/a) \rightarrow \mathcal{O}^{\overline{\text{MS}}}(\mu) = Z^{\overline{\text{MS}}/\text{lat}}(1/a \rightarrow \mu) \mathcal{O}^{\text{lat}}(1/a) \]

We investigate NPR of quark bilinears by the X-space method (Martinelli et al ’97, Giménez et al 2004, Cichy et al 2012)

**Advantages**

✧ able to renormalize by gauge invariant quantities
✧ perturbative matching is available up to 4-loop level (Chetyrkin-Maier, 2011)

**Disadvantages**

✧ Suffer from the window problem

Examine the potential of X-space method with the Möbius domain-wall fermions
2. Strategy — sketch of X-space method

- X-space method uses correlation functions of quark bilinears
  - $\Pi_{\Gamma\Gamma}$: current-current correlators
    \[
    \Pi_{PP}(x) = \left\langle P(x)P(0) \right\rangle, \quad \Pi_{SS}(x) = \left\langle S(x)S(0) \right\rangle,
    \Pi_{VV}(x) = \sum_{\mu=1}^{4} \left\langle V_{\mu}(x)V_{\mu}(0) \right\rangle, \quad \Pi_{AA}(x) = \sum_{\mu=1}^{4} \left\langle A_{\mu}(x)A_{\mu}(0) \right\rangle
    \]

- Renormalization condition in the X-space scheme
  \[
  \left( Z_{\Gamma}^{X/\bullet}(\mu_X = 1/|x|) \right)^{2} \Pi_{\Gamma\Gamma}^{\bullet}(x) = \Pi_{\Gamma\Gamma}^{\text{free}}(x)
  \]
  \[
  \rightarrow Z_{\Gamma}^{X/\bullet}(x) = \sqrt{\frac{\Pi_{\Gamma\Gamma}^{\text{free}}(x)}{\Pi_{\Gamma\Gamma}^{\bullet}(x)}} \quad Z_{\Gamma}^{X/\bullet}(x) \equiv Z_{\Gamma}^{X/\bullet}(\mu_X)
  \]

- Matching through the X-space scheme
  \[
  Z_{\Gamma}^{\text{MS}/\text{lat}}(2 \text{ GeV}) = \frac{Z_{\Gamma}^{X/\text{lat}}(x)}{Z_{\Gamma}^{X/\text{MS}}(x, 2 \text{ GeV})} = \sqrt{\frac{\Pi_{\Gamma\Gamma}^{\text{MS}}(x, 2 \text{ GeV})}{\Pi_{\Gamma\Gamma}^{\text{lat}}(x)}}
  \]
2. Strategy — sketch of X-space method

Renormalization constants

\[ Z_{\Gamma}^{\overline{MS}/lat}(2 \text{ GeV}) = \sqrt{\frac{\Pi_{\Gamma \Gamma}^{\overline{MS}}(x, 2 \text{ GeV})}{\Pi_{\Gamma \Gamma}^{\text{lat}}(x)}} \]

to be evaluated at \( m_q = 0 \)

Steps:

1. Lattice calculation \( \rightarrow \) \( \Pi_{\Gamma \Gamma}^{\text{lat}}(x) \) in chiral limit
2. Perturbation \( \rightarrow \) \( \Pi_{\Gamma \Gamma}^{\overline{MS}}(x, 2 \text{ GeV}) \) at \( m_q = 0 \)

Renormalization window

1. needs sufficiently large \( x \) in order to avoid discretization effects
2. needs sufficiently small \( x \) where perturbation theory is reliable

\[ a \ll x \ll \Lambda_{QCD}^{-1} \]
2. Strategy — lattice action & ensembles

Lattice action

✧ 2+1 generalized (Möbius) domain-wall fermions with 3-times stout smearing
✧ Symanzik improved gauge action

Ensembles

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a^{-1}$ [GeV]</th>
<th>Volume</th>
<th>$a m_s$</th>
<th>$a m_{ud}$ (Confs/Trajectory)</th>
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<tbody>
<tr>
<td>4.17</td>
<td>2.4</td>
<td>$32^3$x64</td>
<td>0.040</td>
<td>0.0190(20/10000), 0.0070(20/10000), 0.0120(20/10000), 0.0035(20/10000)</td>
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<td>0.030</td>
<td>0.0190(20/10000), 0.0070(20/10000), 0.0120(20/10000), 0.0035(20/10000)</td>
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<td>4.35</td>
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<td>$48^3$x96</td>
<td>0.025</td>
<td>0.0120(20/10000), 0.0042(20/10000), 0.0080(20/10000), 0.0060(20/4260), 0.0035(20/10000)</td>
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<td></td>
<td>0.018</td>
<td>0.0120(20/10000), 0.0042(20/10000), 0.0080(20/4260), 0.0035(20/10000)</td>
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<td>4.47</td>
<td>4.5</td>
<td>$32^3$x64</td>
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<td>0.0090(10/10000), 0.0040(10/10000), 0.0060(10/10000), 0.0030(10/10000)</td>
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<td>0.015</td>
<td>0.0090(10/10000), 0.0040(10/10000), 0.0060(10/10000), 0.0030(10/10000)</td>
</tr>
</tbody>
</table>

64³x128 on the product run
3. Lattice Part — short-distance correlator

Discretization effect is very large
3. Lattice Part — short-distance correlator

- Discretization effect is very large

- “lattice, interacting” and “lattice, free” are strongly correlated

  ⇒ discretization effect is similar
3. Lattice Part — free theory correction

We use

\[ \Pi_{\Gamma\Gamma}(x) \rightarrow \Pi'_{\Gamma\Gamma}(x) \]

\[ = \Pi_{\Gamma\Gamma}^{lat}(x) + \Pi_{\Gamma\Gamma}^{cont,free}(x) - \Pi_{\Gamma\Gamma}^{lat,free}(x) \]
3. Lattice Part — democratic cut


- $\theta$: angle between $x$ and (1,1,1,1)
- Correlators at large $\theta$ are more distorted
- Free correlators at small $\theta$ are closer to those in continuum theory
- We neglect correlators at $\theta > 30^\circ$

![Diagram](image)

```
x

\log(a^6\Pi_{PP}^{lat}(x))
```

![Graph](image)
4. Continuum Part — procedure

① Perturbative expansion of correlation functions up to 4-loop order (Chetyrkin-Maier, 2011)

\[
\Pi_{PP,SS}^{\overline{MS}}(x, \mu) = \Pi_{PP,SS}^{\overline{MS}}(x, \tilde{\mu}) = \frac{3}{\pi^4 x^6} \left( 1 + \sum_{n=1}^{\infty} \tilde{C}_n^{S} \tilde{a}_s^n \right)
\]

\[
\Pi_{VV,AA}^{\overline{MS}}(x) = \Pi_{VV,AA}^{\overline{MS}}(x) = \frac{6}{\pi^4 x^6} \left( 1 + \sum_{n=1}^{\infty} \tilde{C}_n^{V} \tilde{a}_s^n \right)
\]

\[
\tilde{a}_s = \frac{\alpha_s^{\overline{MS}}(\tilde{\mu} = 1/x)}{\pi} = \frac{\alpha_s^{\overline{MS}}(\mu = 2e^{-\gamma_E}/x)}{\pi}
\]

② Scale evolution

\[
\Pi_{SS,PP}^{\overline{MS}}(x, \tilde{\mu}') = \frac{c(a_s(\tilde{\mu}'))}{c(a_s(\tilde{\mu}))} \Pi_{SS,PP}^{\overline{MS}}(x, \tilde{\mu}) \leftarrow \text{RG equation}
\]

\[
c(x) : \text{known up to 4-loop order (Chetyrkin '97, Vermasren et al '97)}
\]

\[
\Pi_{VV,AA}^{\overline{MS}}(x) : \text{scale independent} \leftarrow \text{WTI}
\]

① + ② → \[
\Pi_{\Gamma \Gamma}^{\overline{MS}}(x, 2 \text{ GeV})
\]
4. Continuum Part — convergence

Convergence of the perturbative series

Numerical coefficients of $\tilde{\alpha}_s^4$ are relatively large

$$\Pi_{SS}^{\overline{\text{MS}}}(x, \tilde{\mu}) = \frac{3}{\pi^4 x^6} (1 + 0.67\tilde{\alpha}_s - 16.3\tilde{\alpha}_s^2 - 31\tilde{\alpha}_s^3 + 497\tilde{\alpha}_s^4)$$

$$\Pi_{VV}^{\overline{\text{MS}}}(x) = \frac{6}{\pi^4 x^6} (1 + \tilde{\alpha}_s - 4\tilde{\alpha}_s^2 - 1.9\tilde{\alpha}_s^3 + 94\tilde{\alpha}_s^4)$$

\[\alpha_s(\mu) \leftrightarrow N_f = 3, \quad \Lambda_{\text{QCD}} = 340 \text{ MeV}\]
4. Continuum Part — improve convergence

- Scale evolution of \( a_s \) \( \mapsto \) a new perturbative series
  - \( a_s(\tilde{\mu}) = a_s(\tilde{\mu}') \left( 1 + \sum_{n=1}^{\infty} \delta_n(l) a_s^n(\tilde{\mu}') \right) \)
  - We know \( \delta_1, \delta_2, \delta_3, \delta_4 \) as functions of \( l = 2 \ln(\tilde{\mu}'/\tilde{\mu}) \)
    \[ \mapsto \Pi_{\overline{\text{MS}}}^{\overline{\text{MS}}}(x, \tilde{\mu}) = \frac{3}{\pi^4 x^6} \left( 1 + \sum_{n=1}^{\infty} C_n^{\Gamma} a_s^n(\tilde{\mu}') \right) \text{ up to } n = 4 \]

- Scale evolution of correlation function (for scalar channel)
  - \( \Pi_{\overline{\text{MS}}}^{\overline{\text{SS}}}(x, \tilde{\mu}') = \frac{c(a_s(\tilde{\mu}'))}{c(a_s(\tilde{\mu}))} \Pi_{\overline{\text{SS}}}^{\overline{\text{MS}}}(x, \tilde{\mu}) \) as a polynomial of \( a_s(\tilde{\mu}') \)
  - Perform the scale evolution : \( (\overline{\text{MS}}, \tilde{\mu}') \rightarrow (\overline{\text{MS}}, 2 \text{ GeV}) \)

- We use BLM scale for vector channel (Brodsky-Lepage-Mackenzie ’83)
  \[ \tilde{\mu}' = \tilde{\mu}^* = \tilde{\mu} \exp \left( - \frac{11}{6} + 2\zeta(3) \right) \simeq 1.8\tilde{\mu} \]
  \[ \tilde{a}_s^* \equiv a_s(\tilde{\mu}^*) \]

- Improved perturbative series
  \[ \Pi_{\overline{\text{MS}}}^{\overline{\text{SS}}}(x, \tilde{\mu}^*) = \frac{3}{\pi^4 x^6} \left( 1 + 2.9\tilde{a}_s^* + 1.1\tilde{a}_s^{*2} - 42\tilde{a}_s^{*3} + 24\tilde{a}_s^{*4} \right) \]
  \[ \Pi_{\overline{\text{VV}}}^{\overline{\text{MS}}}(x) = \frac{6}{\pi^4 x^6} \left( 1 + \tilde{a}_s^* + 0.083\tilde{a}_s^{*2} - 6\tilde{a}_s^{*3} + 18\tilde{a}_s^{*4} \right) \]
4. Continuum Part — consistency between lattice & PT

- Perturbative series become much better
- Correlators on lattice & PT are roughly same valued
5. Result — plot for $\beta = 4.35$, $a m_s = 0.025$

\[ Z^{\text{MS/lat}}_{\Gamma} (2 \text{ GeV}) = \sqrt{\frac{\Pi^{\text{MS}}_{\Gamma\Gamma}(x, 2 \text{ GeV})}{\Pi^{\text{lat}}_{\Gamma\Gamma}(x)}} \] are ideally independent of $x$

Extract RCs from renormalization window where “$x$-dependence” of $Z^{\text{MS}}_{\Gamma} (2 \text{ GeV})$ is roughly absent (i.e. plateau)

\[ Z^{\text{MS/lat}}_{S} (2 \text{ GeV}) = 0.924 \pm 0.021 \pm 0.003 \]

\[ Z^{\text{MS/lat}}_{V} = 0.990 \pm 0.011 \pm 0.007 \]
5. Result — preliminary values

- Systematic error $\lesssim 1\%$

- 20 confs $\rightarrow$ 200 confs: statistical error $\rightarrow < 1\%$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a^{-1}$ [GeV]</th>
<th>$a m_s$</th>
<th>$Z_{S^{\overline{MS}/lat}}$ (2 GeV)</th>
<th>$Z_{V^{\overline{MS}/lat}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.17</td>
<td>2.4</td>
<td>0.040</td>
<td>1.144(12)(8)</td>
<td>1.033(3)(11)</td>
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<td>0.030</td>
<td>1.073(28)(13)</td>
<td>1.003(9)(5)</td>
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<tr>
<td>4.35</td>
<td>3.6</td>
<td>0.025</td>
<td>0.924(21)(3)</td>
<td>0.990(11)(7)</td>
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<td></td>
<td>0.018</td>
<td>0.975(15)(13)</td>
<td>1.009(4)(3)</td>
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<tr>
<td>4.47</td>
<td>4.5</td>
<td>0.018</td>
<td>on going</td>
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The table shows the results for different values of $\beta$ and $a^{-1}$, with $a m_s$ values and corresponding $Z_{S^{\overline{MS}/lat}}$ and $Z_{V^{\overline{MS}/lat}}$ values. The systematic error and statistical error are indicated.
We investigate NPR of quark bilinears using X-space method

- Discretization effect is reduced by applying the free theory correction and democratic cut
- Convergence of perturbative series of correlation functions become better by expanding correlators in a polynomials of coupling $\tilde{a}_s^*$ at an appropriate $\tilde{\mu}^*$

Goal is to obtain RCs within 1% precision

- Statistical error is currently larger than systematic error
- $20 \text{ confs} \rightarrow 200 \text{ confs}$: statistical error would become $< 1\%$

Furthermore we will try to

- extract $\alpha_s$, $\langle \bar{q}q \rangle$, \ldots from short distance correlators
- apply to heavy quark physics