String tension from smearing and Wilson flow methods

M. Okawa* with A. Gonzalez-Arroyo

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*Hiroshima University

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Last year, we proposed a new method to extract string tension from 4 dimensionally smeared Creutz ratios (PL B718 (2013) 1524).

After reviewing this method, we first show that the same physical results can be obtained replacing the 4-d smearing technique by Wilson flow for sufficiently small time steps $\Delta t$.

We then demonstrate the practical advantage of our method by applying it to the calculation of the Creutz ratio of SU(3) Yang-Mills theory in the continuum limit.
● Creutz ratio from 4-dimensional smearing method

It is well known that Wilson loops \( W(R, T) \) with large \( R, T \) are quite noisy.

Our proposal is to calculate Creutz ratio with 4-dimensional Ape smearing

\[
U_{n, \mu}^{\text{smeared}} = \text{Proj}_{SU(3)} \left[ (1 - f)U_{n, \mu} + \frac{f}{6} \sum_{v \neq \mu = \pm 1}^{\pm 4} U_{n, v} U_{n+v, \mu} U_{n+\mu, v}^\dagger \right]
\]

This form of Ape smearing has been introduced by Narayanan and Neuberger, JHEP 0603 (2006) 064.

Let \( t = f \ast n_s / 6 \) with \( n_s \) the number of smearing steps.
Wilson loop and potential have huge $t$ dependences, which makes 4-d Ape smearing almost useless for these quantities. See, however, Lohmayer and Neuberger, JHEP 1208 (2012) 102.

It is crucial to consider Creutz ratio

$$\chi(R, T) = -\log \frac{W(R + 1/2, T + 1/2)W(R - 1/2, T - 1/2)}{W(R + 1/2, T - 1/2)W(R - 1/2, T - 1/2)}$$

which is free from ultraviolet divergences and its $t$ dependence is quite well fitted by

$$\chi(t) = a \left( 1 - \exp \left( \frac{-b}{t + c} \right) \right)$$

This is the method proposed in Phys. Lett. B718 (2013) 1524.

We demonstrate this method using SU(3) LGT on a $32^4$ lattice at $\beta = 6.17$ with $f = 0.1$. 
dependence of Wilson loop $W(R, T)$

$t = f * n_s / 6$ dependence of Wilson loop $W(R, T)$

$W(6, 6), 32^4, \beta = 6.17, f = 0.1$
\[ t = f \times n_s / 6 \text{ dependence of Potential} \]

\[ V(R, T) = -\log \frac{W(R, T + 1/2)}{W(R, T - 1/2)} \]

\[ V(6, 5.5), \ 32^4, \ \beta = 6.17, \ f = 0.1 \]
dependence of Creutz ratio

$t = f \ast n_s / 6$

\[
\chi(R,T) = -\log \frac{W(R+1/2, T+1/2)W(R-1/2, T-1/2)}{W(R+1/2, T-1/2)W(R-1/2, T+1/2)}
\]

\[
\chi(5.5,5.5), \quad 32^4, \quad \beta = 6.17, \quad f = 0.1
\]

\[
\chi(t) = a \left(1 - \exp\left(-\frac{b}{(t+c)}\right)\right)
\]
4-d Ape smearing and Wilson flow

4-d Ape smearing

\[ U_{n,\mu}^{\text{smeared}} = \text{Proj}_{SU(3)} \left[ (1 - f)U_{n,\mu} + \frac{f}{6} \sum_{v \neq \mu = \pm 1}^{\pm 4} U_{n,v}U_{n+v,\mu}U_{n+\mu,v}^\dagger \right] \]

with \( t = f \times n_s / 6 \) is equivalent to Wilson flow

\[ \frac{dV_{n,\mu}(t)}{dt} = -g_0^2 \{\partial_{n,\mu}S_W\}V_{n,\mu}(t), \quad V_{n,\mu}(t = 0) = U_{n,\mu} \]

provided \( f \) is sufficiently small.

In our method, we use data typically at \( t \leq \frac{(R - 1/2)^2}{25} \) for \( \chi(R, R) \)
with \( f \sim 0.1 \).

For Wilson flow, we use 3rd order Runge-Kutta with \( \Delta t = 0.01 \).
For small $t$, 4-d smearing and Wilson flow results are essentially identical.
Actually, for small $t$, 4-d smearing with $f = 0.1, 0.2, 0.4$ gives essentially the same result for Creutz ratios.

$\chi(5.5, 5.5), \beta = 6.17$
For large $t$, we need some care. \[
\left\{ \frac{t^2 \langle E \rangle}{t=t_0} \right\} = 0.3
\]
Creutz ratio in the continuum limit

We made simulations at three lattices having almost same physical volume \((La)^4\)

<table>
<thead>
<tr>
<th>Lattice</th>
<th>(\beta)</th>
<th>(N_{\text{cnfg}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>24(^4)</td>
<td>5.96</td>
<td>1200</td>
</tr>
<tr>
<td>32(^4)</td>
<td>6.17</td>
<td>600</td>
</tr>
<tr>
<td>48(^4)</td>
<td>6.42</td>
<td>100</td>
</tr>
</tbody>
</table>

In this talk, we concentrate on the diagonal \(\chi(R,R)\), although there are a lot of interesting physics in off-diagonal \(\chi(R,T)\).
\[ \chi(R, R) \]

\[ L = 48, \beta = 6.42 \]
\[ L = 32, \beta = 6.17 \]
\[ L = 24, \beta = 5.96 \]
Introducing a scale $\bar{r}$ and writing $r = Ra$, dimensional analysis implies that there should be $O(a^2)$ lattice artifact in $1/r^4$ term.

$$\left(\frac{\bar{r}}{a}\right)^2 \chi(R, R) = \sigma \bar{r}^2 + 2\gamma \left(\frac{\bar{r}}{r}\right)^2 + 4 \left(\frac{\bar{r}}{r}\right)^4 \left(c + d \left(\frac{a}{\bar{r}}\right)^2\right)$$

$\tilde{F}(r)$ defined by

$$\bar{r}^2 \tilde{F}(r) = \sigma \bar{r}^2 + 2\gamma \left(\frac{\bar{r}}{r}\right)^2 + 4c \left(\frac{\bar{r}}{r}\right)^4$$

is the Creutz ratio in the continuum limit.

We can fix the scale a la Sommer as

$$\bar{r}^2 \tilde{F}(\bar{r}) = \sigma \bar{r}^2 + 2\gamma + 4c = 1.65$$
Eliminating $c$ from the expression and replacing $r$ to $Ra$, we finally find a fitting function

$$\left( \frac{r}{a} \right)^2 \chi(R, R) = \sigma r^2 + 2\gamma \left( \frac{r}{a} \right)^2 \frac{1}{R^2} + 4 \left( \frac{r}{a} \right)^4 \frac{1}{R^4} \left( c + d \left( \frac{a}{r} \right)^2 \right)$$

$$4c = 1.65 - \sigma r^2 - 2\gamma$$

with 6 fitting parameters

$$\sigma r^2, \gamma, d, \frac{a(\beta = 5.96)}{\bar{r}}, \frac{a(\beta = 6.17)}{\bar{r}}, \frac{a(\beta = 6.42)}{\bar{r}}$$

The resultant fit has $\chi^2 / (#\ of\ freedom) = 1.05$ with

$$\sigma r^2 = 1.159(6) \quad \gamma = 0.250(3), \quad d = 0.24(2)$$

$$a(\beta = 5.96) / \bar{r} = 0.2117(5) ,$$

$$a(\beta = 6.17) / \bar{r} = 0.1499(2) ,$$

$$a(\beta = 6.42) / \bar{r} = 0.1052(3)$$
\frac{\bar{r}^2}{a^2} \chi(R,R)
\[
\bar{r}^2 \tilde{F}(r) = \sigma \bar{r}^2 + 2\gamma \left( \frac{\bar{r}}{r} \right)^2 + 4c \left( \frac{\bar{r}}{r} \right)^4
\]
relation between $\bar{r}$ and $t_0$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a / \bar{r}$</th>
<th>$t_0 / a^2$</th>
<th>$\sqrt{8t_0} / \bar{r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.96</td>
<td>0.2117(5)</td>
<td>2.794(3)</td>
<td>1.001(2)</td>
</tr>
<tr>
<td>6.17</td>
<td>0.1499(2)</td>
<td>5.506(7)</td>
<td>0.995(2)</td>
</tr>
<tr>
<td>6.42</td>
<td>0.1052(3)</td>
<td>11.17(3)</td>
<td>0.994(3)</td>
</tr>
</tbody>
</table>

In the continuum limit

\[
\sqrt{8t_0} / \bar{r} = 0.990(3)
\]

\[
\sigma \bar{r}^2 = 1.159(6)
\]

\[
\therefore \sqrt{8t_0 \sigma} = 1.066(4)
\]
comparison with 3-d smearing methods

In the continuum limit, \( \sqrt{8t_0} / r_0 = 0.948(6) \). Lüscher JHEP08(2010) 071.

Then, our result is converted to \( r_0 \sqrt{\sigma} = 1.124(8) \)

Previous results of \( r_0 \sqrt{\sigma} \) derived from 3-d smeared Potential are almost consistent with the value \( r_0 \sqrt{\sigma} = \sqrt{1.65 - \pi / 12} = 1.178 \)

It is not clear how to interpret this 5% difference, however, they are obtained from quite different geometries of Wilson loops!

3-d smearing, \( W(r,t) \) with finite \( r \) and \( t = \infty \)

\[
r_0^2 F(r) = r_0^2 \sigma + \frac{\pi r_0^2}{12 r^2} \approx r_0^2 \sigma + 0.2612 \frac{r_0^2}{r^2}
\]

4-d smearing, \( W(r,t) \) with finite \( r \approx t \)

\[
\bar{r}^2 \tilde{F}(r) = \bar{r}^2 \sigma + 2\gamma \frac{\bar{r}^2}{r^2} + 4c \frac{\bar{r}^2}{r^4} \approx \bar{r}^2 \sigma + 0.499 \frac{\bar{r}^2}{r^2} - 0.008 \frac{\bar{r}^4}{r^4}
\]
● relation between $\bar{r}$ and $\Lambda_{\overline{\text{MS}}}$

We use $g_E^2 = 3(1 - u_P)$ for

$$a\Lambda_{\overline{\text{MS}}} = a\Lambda_E \left( \frac{\Lambda_{\overline{\text{MS}}}}{\Lambda_E} \right)$$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a / \bar{r}$</th>
<th>$a\Lambda_{\overline{\text{MS}}}$</th>
<th>$\bar{r}\Lambda_{\overline{\text{MS}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.96</td>
<td>0.2117(5)</td>
<td>0.11523</td>
<td>0.5443(13)</td>
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<tr>
<td>6.17</td>
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<td>0.08520</td>
<td>0.5684(8)</td>
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<td>6.42</td>
<td>0.1052(3)</td>
<td>0.06105</td>
<td>0.5803(17)</td>
</tr>
</tbody>
</table>

In the continuum limit

$$\bar{r}\Lambda_{\overline{\text{MS}}} = 0.5924(16)$$

$$\sigma \bar{r}^2 = 1.159(6)$$

$$\therefore \frac{\Lambda_{\overline{\text{MS}}}}{\sqrt{\sigma}} = 0.550(2)$$
● Conclusion

Creutz ratios in the continuum limit can be evaluated precisely by 4-d Ape smearing, giving rather reliable determination of string tension.

Wilson flow technique gives the same physical results once the time steps $\Delta t$ is sufficiently small.

It is worth calculating Wilson loops during Wilson flow updating.

They should give fruitful physics!