Non-perturbative Renormalization of Four-Fermion Operators Relevant to B_K with Staggered Quarks

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Introduction

- We present matching factors for the four-fermion operators obtained using the non-perturbative renormalization (NPR) method in RI-MOM scheme for improved staggered fermions on the MILC asqtad lattices ($N_f = 2 + 1$).
- Using $20^3 \times 64$ lattice ($a \approx 0.12fm, am_\ell/am_s = 0.01/0.05$), we obtain the matching factor of B_K operator.
- We compare NPR results with those of one-loop perturbative matching.

Four-Fermion Operator Renormalization

- $\widetilde{p}$ is the momentum in reduced Brillouin zone.

$$p \in (-\frac{\pi}{a}, \frac{\pi}{a}]^4, \qquad \widetilde{p} \in (-\frac{\pi}{2a}, \frac{\pi}{2a}]^4, \qquad p = \widetilde{p} + \pi_B$$

  where $\pi_B (\equiv \frac{\pi}{a} B)$ is cut-off momentum in hypercube.
- a : lattice spacing.
- B : vector in hypercube. Each element is 0 or 1
  ex) $B = (0, 0, 1, 1)$

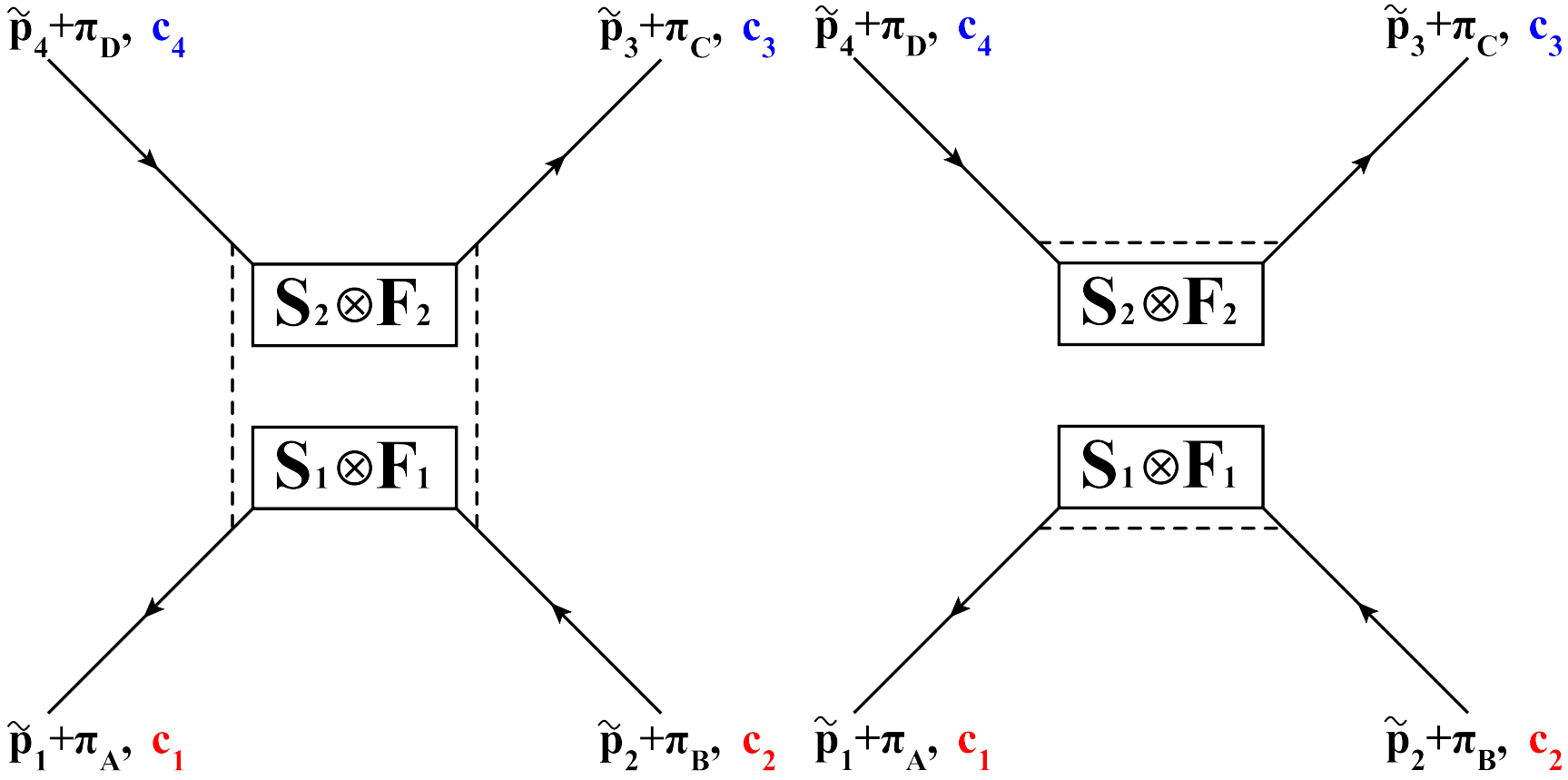
The one-color trace four-fermion operator

$$O_{\mathrm{I}}(x) = \left[\overline{\chi}_{c_1}(x_A)\overline{(\gamma_{S_1} \otimes \xi_{F_1})}_{AB}\chi_{c_2}(x_B)\right]\left[\overline{\chi}_{c_3}(x_C)\overline{(\gamma_{S_2} \otimes \xi_{F_2})}_{CD}\chi_{c_4}(x_D)\right] \times U_{AD;c_1c_4}(x)U_{BC;c_2c_3}(x) \quad (1)$$

The two-color trace four-fermion operator

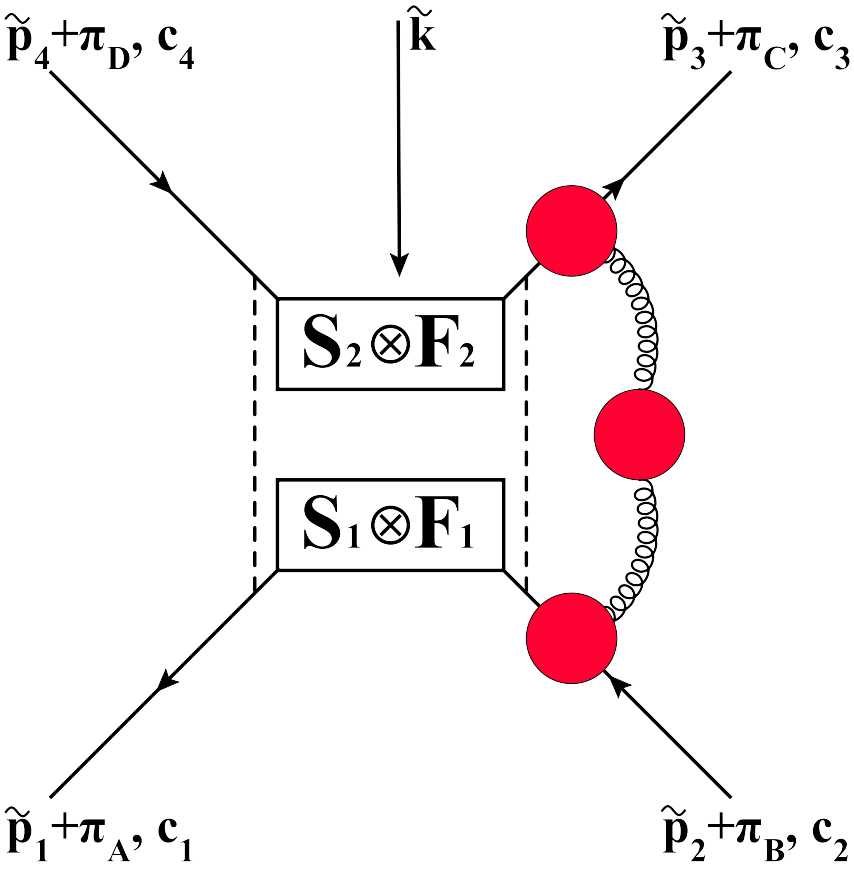
$$O_{\mathrm{II}}(x) = \left[\overline{\chi}_{c_1}(x_A)\overline{(\gamma_{S_1} \otimes \xi_{F_1})}_{AB}\chi_{c_2}(x_B)\right]\left[\overline{\chi}_{c_3}(x_C)\overline{(\gamma_{S_2} \otimes \xi_{F_2})}_{CD}\chi_{c_4}(x_D)\right] \times U_{AB;c_1c_2}(x)U_{CD;c_3c_4}(x) \quad (2)$$

- U : link variable.
- A, B, C, D: hypercube index
- c_1, c_2, c_3, c_4 : color index

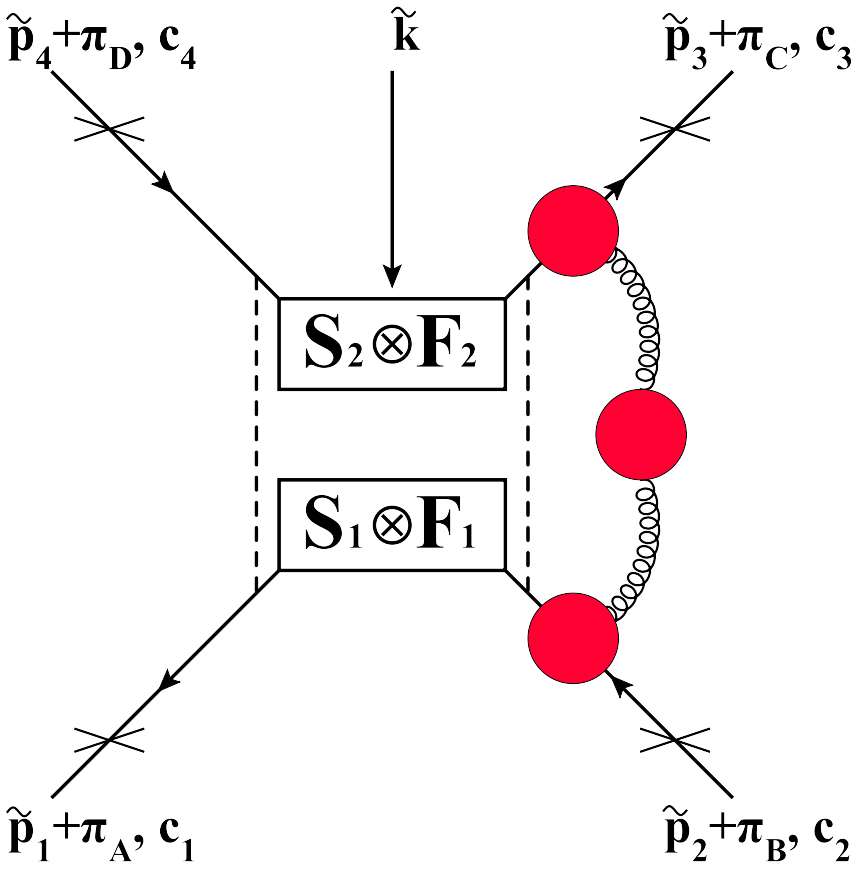


(a) one-color trace (b) two-color trace

Figure: One-color trace and two-color trace four-fermion operators

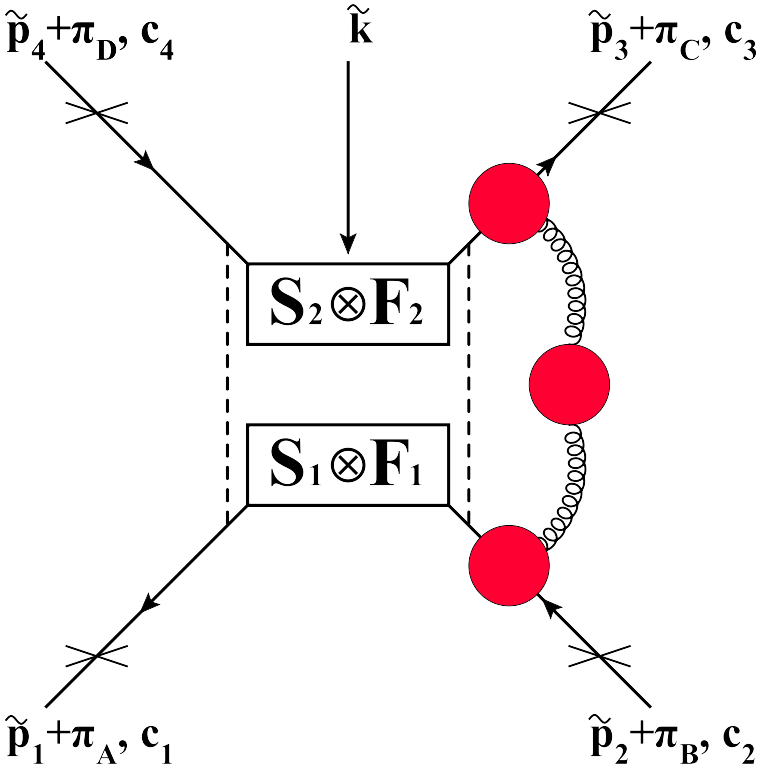


(a) Unamputated Green's Function

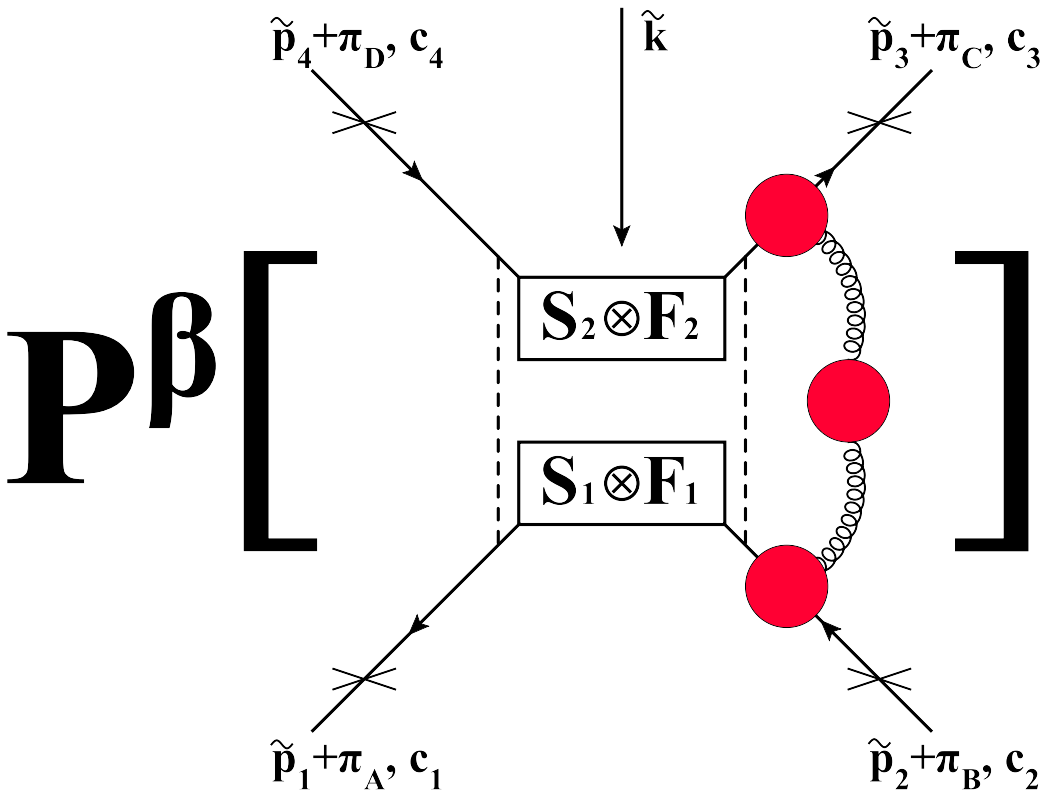


(b) Amputated Green's Function

Red Circle : 1PI Diagram.



(c) Amputated Green's Function : Λ^{α}



(d) Projected Amputated Green's Function : $\Gamma^{\alpha\beta}$

- α, β : the indices to represent different operators.
  ex) $\alpha = (V \otimes P)(V \otimes P)_{\mathrm{I}}$, $\beta = (A \otimes P)(A \otimes P)_{\mathrm{II}}$
- momentum conservation in reduced Brillouin zone. $\tilde{k} = \tilde{p}_1 - \tilde{p}_2 + \tilde{p}_3 - \tilde{p}_4$

The one-color trace projection operator is

$$\hat{P}^{\beta} = \overline{\overline{(\gamma^{\dagger}_{S'} \otimes \xi^{\dagger}_{F'})}}_{BA} \overline{\overline{(\gamma^{\dagger}_{S''} \otimes \xi^{\dagger}_{F''})}}_{DC} \delta_{c_4 c_1} \delta_{c_3 c_2}$$

The two-color trace projection operator is

$$\hat{P}^{\beta} = \overline{\overline{(\gamma^{\dagger}_{S'} \otimes \xi^{\dagger}_{F'})}}_{BA} \overline{\overline{(\gamma^{\dagger}_{S''} \otimes \xi^{\dagger}_{F''})}}_{DC} \delta_{c_2 c_1} \delta_{c_4 c_3}$$

- A, B, C, D: hypercube index
- c_1, c_2, c_3, c_4 : color index

For simplicity, we define the following notations.

$$
\begin{aligned}
O_{V1} &\equiv O_{[V\otimes P][V\otimes P],\mathrm{I}} \\
O_{V2} &\equiv O_{[V\otimes P][V\otimes P],\mathrm{II}} \\
O_{A1} &\equiv O_{[A\otimes P][A\otimes P],\mathrm{I}} \\
O_{A2} &\equiv O_{[A\otimes P][A\otimes P],\mathrm{II}}
\end{aligned} \tag{3}
$$

The tree level B_K operator is

$$
O_{B_K}^{\mathrm{tree}} = O_{V1}^{\mathrm{tree}} + O_{V2}^{\mathrm{tree}} + O_{A1}^{\mathrm{tree}} + O_{A2}^{\mathrm{tree}} \tag{4}
$$

The matching formula of B_K operator is

$$
O_{B_K}^{R} = z_1 O_{V1}^{B} + z_2 O_{V2}^{B} + z_3 O_{A1}^{B} + z_4 O_{A2}^{B} + \sum_{\alpha\in(D)} z_\alpha O_\alpha^{B} \tag{5}
$$

- The superscription R(B) denotes renormalized(bare) quantity.
- z_1, z_2, z_3, z_4 and z_α are the renormalization factors.
- We classify operators as follows.
  (C): {V1, V2, A1, A2} (diagonal operators)
  (D): remaining operators (off-diagonal operators)

The renormalization of quark field is

$$\chi_R = Z_q^{1/2} \chi_B \tag{6}$$

where z_q is wave function renormalization factor for quark field.
The renormalized amputated Green's function of B_K operator is

$$\Lambda^R_{B_K} = \frac{z_1}{z_q^2}\Lambda^B_{V1} + \frac{z_2}{z_q^2}\Lambda^B_{V2} + \frac{z_3}{z_q^2}\Lambda^B_{A1} + \frac{z_4}{z_q^2}\Lambda^B_{A2} + \sum_{\alpha\in(D)} \frac{z_\alpha}{z_q^2}\Lambda^B_\alpha \,, \tag{7}$$

For simplicity, the projection operators are defined as follows.

$$\begin{aligned}
\hat{P}_{V1} &= \overline{\overline{(V_\mu \otimes P)}}^{\dagger}_{BA} \overline{\overline{(V_\mu \otimes P)}}^{\dagger}_{DC} \delta_{c_4 c_1} \delta_{c_3 c_2} \\
\hat{P}_{V2} &= \overline{\overline{(V_\mu \otimes P)}}^{\dagger}_{BA} \overline{\overline{(V_\mu \otimes P)}}^{\dagger}_{DC} \delta_{c_2 c_1} \delta_{c_4 c_3} \\
\hat{P}_{A1} &= \overline{\overline{(A_\mu \otimes P)}}^{\dagger}_{BA} \overline{\overline{(A_\mu \otimes P)}}^{\dagger}_{DC} \delta_{c_4 c_1} \delta_{c_3 c_2} \\
\hat{P}_{A2} &= \overline{\overline{(A_\mu \otimes P)}}^{\dagger}_{BA} \overline{\overline{(A_\mu \otimes P)}}^{\dagger}_{DC} \delta_{c_2 c_1} \delta_{c_4 c_3}
\end{aligned} \tag{8}$$

We apply the projection operator to tree level amputated Green's function.

$$
\begin{aligned}
tr[\Lambda^{\text{tree}}_{B_K}\hat{P}_{V1}] &= tr[\Lambda^{\text{tree}}_{V1}\hat{P}_{V1}] + tr[\Lambda^{\text{tree}}_{V2}\hat{P}_{V1}] + \cancel{tr[\Lambda^{\text{tree}}_{A1}\hat{P}_{V1}]} + \cancel{tr[\Lambda^{\text{tree}}_{A2}\hat{P}_{V1}]} \\
&= (16\times16\times3\times3) + (16\times16\times3) = 3072 \equiv N \qquad (9)\\
tr[\Lambda^{\text{tree}}_{B_K}\hat{P}_{V2}] &= tr[\Lambda^{\text{tree}}_{V1}\hat{P}_{V2}] + tr[\Lambda^{\text{tree}}_{V2}\hat{P}_{V2}] + \cancel{tr[\Lambda^{\text{tree}}_{A1}\hat{P}_{V2}]} + \cancel{tr[\Lambda^{\text{tree}}_{A2}\hat{P}_{V2}]} \\
&= (16\times16\times3) + (16\times16\times3\times3) = 3072 \equiv N \qquad (10)\\
tr[\Lambda^{\text{tree}}_{B_K}\hat{P}_{A1}] &= \cancel{tr[\Lambda^{\text{tree}}_{V1}\hat{P}_{A1}]} + \cancel{tr[\Lambda^{\text{tree}}_{V2}\hat{P}_{A1}]} + tr[\Lambda^{\text{tree}}_{A1}\hat{P}_{A1}] + tr[\Lambda^{\text{tree}}_{A2}\hat{P}_{A1}] \\
&= (16\times16\times3\times3) + (16\times16\times3) = 3072 \equiv N \qquad (11)\\
tr[\Lambda^{\text{tree}}_{B_K}\hat{P}_{A2}] &= \cancel{tr[\Lambda^{\text{tree}}_{V1}\hat{P}_{A2}]} + \cancel{tr[\Lambda^{\text{tree}}_{V2}\hat{P}_{A2}]} + tr[\Lambda^{\text{tree}}_{A1}\hat{P}_{A2}] + tr[\Lambda^{\text{tree}}_{A2}\hat{P}_{A2}] \\
&= (16\times16\times3) + (16\times16\times3\times3) = 3072 \equiv N \qquad (12)\\
tr[\Lambda^{\text{tree}}_{B_K}\hat{P}_{(D)}] &= 0 \qquad (13)\\
tr[\Lambda^{\text{tree}}_{(D)}\hat{P}_{(C)}] &= 0\,, \qquad (14)
\end{aligned}
$$

Here,
(C): {V1, V2, A1, A2} (diagonal operators)
(D): remaining operators (off-diagonal operators)

The RI-MOM scheme prescription is

$$tr[\Lambda^R(\widetilde{p},\widetilde{p},\widetilde{p},\widetilde{p})\hat{P}] = tr[\Lambda^{tree}(\widetilde{p},\widetilde{p},\widetilde{p},\widetilde{p})\hat{P}] \tag{15}$$

Therefore,

$$N = tr[\Lambda^R_{B_K}\hat{P}_{V1}] \tag{16}$$
$$N = tr[\Lambda^R_{B_K}\hat{P}_{V2}] \tag{17}$$
$$N = tr[\Lambda^R_{B_K}\hat{P}_{A1}] \tag{18}$$
$$N = tr[\Lambda^R_{B_K}\hat{P}_{A2}] \tag{19}$$
$$0 = tr[\Lambda^R_{B_K}\hat{P}_{(D)}] \tag{20}$$
$$0 = tr[\Lambda^R_{(D)}\hat{P}_{(C)}] \tag{21}$$
$$\tag{22}$$

Here,
(C): {V1, V2, A1, A2} (diagonal operators)
(D): remaining operators (off-diagonal operators)

We define the projected amputated Green's function as follows.

$$\Gamma^B_{\alpha\beta} \equiv \frac{1}{N z_q^2} tr[\Lambda^B_\alpha \hat{P}_\beta] \tag{23}$$

where α, β are operator such as $V1, V2, A1, A2, \ldots$ and z_q is obtained from conserved vector current channel.

$$1 = (z_1 \Gamma^B_{V1V1} + z_2 \Gamma^B_{V2V1} + z_3 \Gamma^B_{A1V1} + z_4 \Gamma^B_{A2V1} + \sum_{\alpha \in (D)} z_\alpha \Gamma^B_{\alpha V1}) \tag{24}$$

$$1 = (z_1 \Gamma^B_{V1V2} + z_2 \Gamma^B_{V2V2} + z_3 \Gamma^B_{A1V2} + z_4 \Gamma^B_{A2V2} + \sum_{\alpha \in (D)} z_\alpha \Gamma^B_{\alpha V2}) \tag{25}$$

$$1 = (z_1 \Gamma^B_{V1A1} + z_2 \Gamma^B_{V2A1} + z_3 \Gamma^B_{A1A1} + z_4 \Gamma^B_{A2A1} + \sum_{\alpha \in (D)} z_\alpha \Gamma^B_{\alpha A1}) \tag{26}$$

$$1 = (z_1 \Gamma^B_{V1A2} + z_2 \Gamma^B_{V2A2} + z_3 \Gamma^B_{A1A2} + z_4 \Gamma^B_{A2A2} + \sum_{\alpha \in (D)} z_\alpha \Gamma^B_{\alpha A2}) \tag{27}$$

$$0 = (z_1 \Gamma^B_{V1\beta} + z_2 \Gamma^B_{V2\beta} + z_3 \Gamma^B_{A1\beta} + z_4 \Gamma^B_{A2\beta} + \sum_{\alpha \in (D)} z_\alpha \Gamma^B_{\alpha\beta}), \quad \beta \in (\mathrm{D}) \tag{28}$$

We can express these equations as a matrix equation.

$$\vec{z}_{\text{tree}} = \vec{z}\,\Gamma^B\,, \tag{29}$$

where

$$\vec{z}_{\text{tree}} = (1, 1, 1, 1, 0, \cdots, 0) \tag{30}$$

$$\vec{z} = (z_1, z_2, z_3, z_4, z_\alpha, z_\beta, \cdots) \tag{31}$$

$$\Gamma^B = \left(\begin{array}{cccc|ccc}
\Gamma^B_{V1V1} & \Gamma^B_{V1V2} & \Gamma^B_{V1A1} & \Gamma^B_{V1A2} & \Gamma^B_{V1\alpha} & \Gamma^B_{V1\beta} & \cdots \\
\Gamma^B_{V2V1} & \Gamma^B_{V2V2} & \Gamma^B_{V2A1} & \Gamma^B_{V2A2} & \Gamma^B_{V2\alpha} & \Gamma^B_{V2\beta} & \cdots \\
\Gamma^B_{A1V1} & \Gamma^B_{A1V2} & \Gamma^B_{A1A1} & \Gamma^B_{A1A2} & \Gamma^B_{A1\alpha} & \Gamma^B_{A1\eta} & \cdots \\
\Gamma^B_{A2V1} & \Gamma^B_{A2V2} & \Gamma^B_{A2A1} & \Gamma^B_{A2A2} & \Gamma^B_{A2\alpha} & \Gamma^B_{A2\beta} & \cdots \\
\hline
\Gamma^B_{\alpha V1} & \Gamma^B_{\alpha V2} & \Gamma^B_{\alpha A1} & \Gamma^B_{\alpha A2} & \Gamma^B_{\alpha\alpha} & \Gamma^B_{\alpha\beta} & \cdots \\
\Gamma^B_{\beta V1} & \Gamma^B_{\beta V2} & \Gamma^B_{\beta A1} & \Gamma^B_{\beta A2} & \Gamma^B_{\beta\alpha} & \Gamma^B_{\beta\beta} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)$$

Then, we can obtain the $\vec{z}$ from the following equation.

$$\vec{z} = \vec{z}_{\text{tree}}(\Gamma^B)^{-1} \tag{32}$$

We can make the matrix Γ^B as a block matrix.

$$\Gamma^B = \begin{pmatrix} X_{4\times 4} & Y_{4\times M} \\ Z_{M\times 4} & W_{M\times M} \end{pmatrix} \tag{33}$$

Here, X is 4×4 matrix.

$$X = \begin{pmatrix} \Gamma^B_{V1V1} & \Gamma^B_{V1V2} & \Gamma^B_{V1A1} & \Gamma^B_{V1A2} \\ \Gamma^B_{V2V1} & \Gamma^B_{V2V2} & \Gamma^B_{V2A1} & \Gamma^B_{V2A2} \\ \Gamma^B_{A1V1} & \Gamma^B_{A1V2} & \Gamma^B_{A1A1} & \Gamma^B_{A1A2} \\ \Gamma^B_{A2V1} & \Gamma^B_{A2V2} & \Gamma^B_{A2A1} & \Gamma^B_{A2A2} \end{pmatrix} \tag{34}$$

The matrix Y is $4 \times M$.

$$Y = \begin{pmatrix} \Gamma^B_{V1\alpha} & \Gamma^B_{V1\beta} & \cdots \\ \Gamma^B_{V2\alpha} & \Gamma^B_{V2\beta} & \cdots \\ \Gamma^B_{A1\alpha} & \Gamma^B_{A1\beta} & \cdots \\ \Gamma^B_{A2\alpha} & \Gamma^B_{A2\beta} & \cdots \end{pmatrix} \tag{35}$$

- $\{\alpha\ ,\ \beta, \cdots\} = \{(S\otimes V)(S\otimes V)_I, (S\otimes V)(S\otimes V)_{II}, (S\otimes A)(S\otimes A)_I, \cdots\}$.
- We calculate the complete set of off-diagonal mixing matrix Y.
- We assume that $Z_{ij} \simeq Y_{ji}$. It is an approximation within factor of 2.
- $W \simeq 1$.

$$\Gamma^B = \begin{pmatrix} X_{4\times 4} & Y_{4\times M} \\ Z_{M\times 4} & W_{M\times M} \end{pmatrix} \tag{36}$$

The inverse of block matrix is

$$(\Gamma^B)^{-1} = \begin{pmatrix} (X - YW^{-1}Z)^{-1} & -X^{-1}Y(W - ZX^{-1}Y)^{-1} \\ -W^{-1}Z(X - YW^{-1}Z)^{-1} & (W - ZX^{-1}Y)^{-1} \end{pmatrix} \tag{37}$$

$$\simeq \begin{pmatrix} X^{-1} + X^{-1}YW^{-1}ZX^{-1} & -X^{-1}Y(W^{-1} + W^{-1}ZX^{-1}YW^{-1}) \\ -W^{-1}Z(X^{-1} + X^{-1}YW^{-1}ZX^{-1}) & W^{-1} + W^{-1}ZX^{-1}YW^{-1} \end{pmatrix} \tag{38}$$

with $Z_{ij} \simeq Y_{ji}$ and $W \simeq 1$

$$\simeq \begin{pmatrix} X^{-1} + X^{-1}YY^TX^{-1} & -X^{-1}Y(1 + Y^TX^{-1}Y) \\ -Y^T(X^{-1} + X^{-1}YY^TX^{-1}) & 1 + Y^TX^{-1}Y \end{pmatrix} \tag{39}$$

Simulation Detail

- $20^3 \times 64$ MILC asqtad lattice ($a \approx 0.12fm,\ am_\ell/am_s = 0.01/0.05$).
- HYP smeared staggered fermions as valence quarks.
- The number of configurations is 30.
- 5 valence quark masses (0.01, 0.02, 0.03, 0.04, 0.05)
- 9 external momenta in the units of $(\frac{2\pi}{L_s}, \frac{2\pi}{L_s}, \frac{2\pi}{L_s}, \frac{2\pi}{L_t})$.
- We use the jackknife resampling method to estimate statistical errors.

n(x, y, z, t)	$\lvert a\widetilde{p}\rvert$	GeV
$(2,2,2,7)$	1.2871	2.1332
$(2,2,2,8)$	1.3421	2.2243
$(2,2,2,9)$	1.4018	2.3233
$(2,3,2,7)$	1.4663	2.4302
$(2,3,2,8)$	1.5148	2.5106
$(2,3,2,9)$	1.5680	2.5987
$(3,2,3,8)$	1.6698	2.7674
$(3,3,3,7)$	1.7712	2.9355
$(3,3,3,9)$	1.8562	3.0764

Matrix X analysis

$$\Gamma^B = \begin{pmatrix} X_{4\times 4} & Y_{4\times M} \\ Z_{M\times 4} & W_{M\times M} \end{pmatrix} \tag{40}$$

As an example, we consider the data analysis of Γ^B_{V1V1}.

$$X = \begin{pmatrix} \Gamma^B_{V1V1} & \Gamma^B_{V1V2} & \Gamma^B_{V1A1} & \Gamma^B_{V1A2} \\ \Gamma^B_{V2V1} & \Gamma^B_{V2V2} & \Gamma^B_{V2A1} & \Gamma^B_{V2A2} \\ \Gamma^B_{A1V1} & \Gamma^B_{A1V2} & \Gamma^B_{A1A1} & \Gamma^B_{A1A2} \\ \Gamma^B_{A2V1} & \Gamma^B_{A2V2} & \Gamma^B_{A2A1} & \Gamma^B_{A2A2} \end{pmatrix} \tag{41}$$

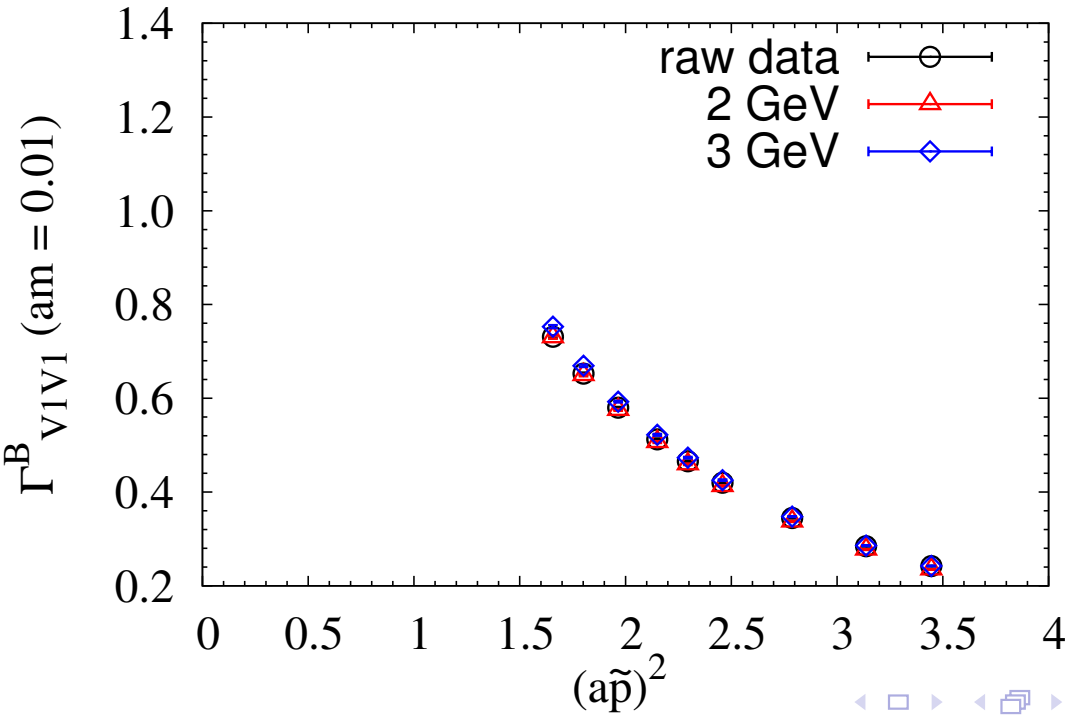
Γ^B_{V1V1} analysis

We convert the scale of raw data in the RI-MOM scheme from $\mu(=|\widetilde{p}|)$ to $\mu_0(=2\,\mathrm{GeV}$ or $3\,\mathrm{GeV})$ using two-loop RG evolution factor $U^{\text{RI-MOM}}_{B_K}(\mu_0,\mu)$.

$$\vec{z}^{\,\text{RI-MOM}}(\mu_0) = U^{\text{RI-MOM}}_{B_K}(\mu_0,\mu)\,\vec{z}^{\,\text{RI-MOM}}(\mu) \tag{42}$$

$$\vec{z}^{\,\text{RI-MOM}} = \vec{z}_{\text{tree}}(\Gamma^B)^{-1} \tag{43}$$

$$\Gamma^B_{V1V1}(\mu_0) = 1/U^{\text{RI-MOM}}_{B_K}(\mu_0,\mu)\Gamma^B_{V1V1}(\mu) \tag{44}$$



m-fit (fitting with respect to quark mass)

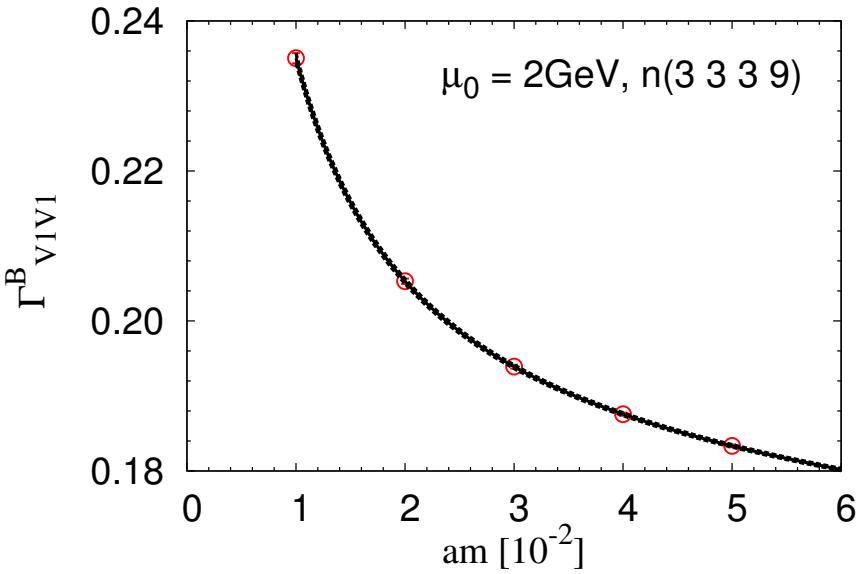
We fit the data with respect to quark mass for a fixed momentum to the following function f_{V1V1}. [RBC, PRD66, 2002]

$$f_{V1V1}(m, a, \tilde{p}) = a_1 + a_2 \cdot am + a_3 \cdot \frac{1}{(am)} + a_4 \cdot \frac{1}{(am)^2},$$

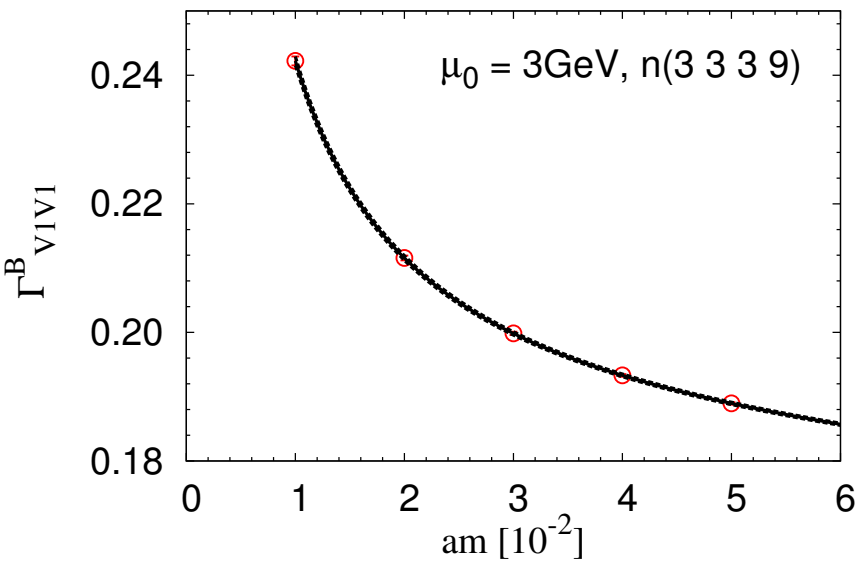
where a_i is a function of $\widetilde{p}$. We call this m-fit. After m-fit, we take the chiral limit values which corresponds to $a_1(a, \widetilde{p})$. Because of the sea quark determinant contributions($a_3, a_4 \propto m_\ell^2 m_s$), a_3 and a_4 term contribution vanishes in the chiral limit.

μ_0	a_1	a_2	a_3	a_4	χ^2/dof
2GeV	0.17457(19)	-0.0946(15)	0.0006907(83)	-0.000000766(35)	0.00194(40)
3GeV	0.17991(20)	-0.0975(15)	0.0007118(85)	-0.000000790(36)	0.00194(40)

m-fit plot



(e) $\mu_0 = 2\,\mathrm{GeV}$



(f) $\mu_0 = 3\,\mathrm{GeV}$

p-fit (fitting with respect to reduced momentum)

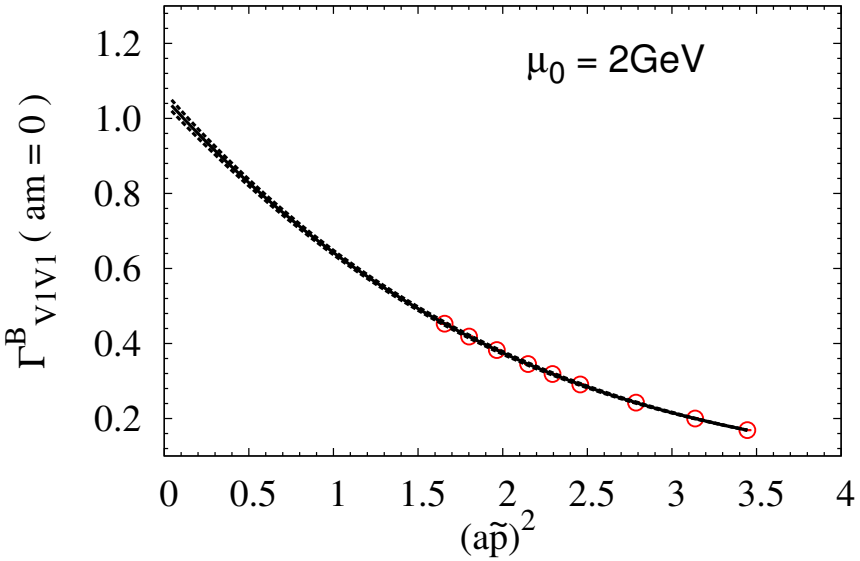
We fit $a_1(a, \widetilde{p})$ to the following fitting function.

$$f_{V1V1}(am = 0, a\tilde{p}) = b_1 + b_2(a\tilde{p})^2 + b_3((a\tilde{p})^2)^2 + b_4(a\tilde{p})^4 + b_5((a\tilde{p})^2)^3$$

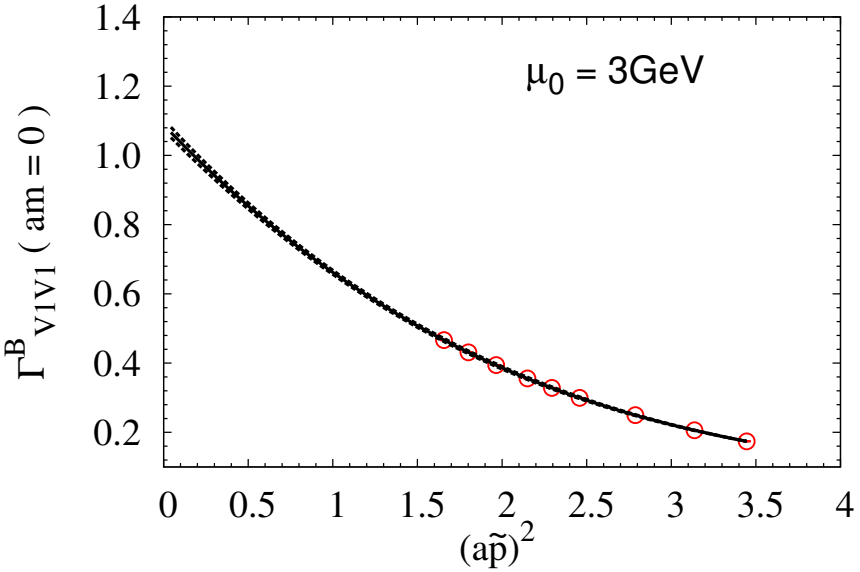
To avoid non-perturbative effects at small $(a\tilde{p})^2$, we choose the momentum window as $(a\tilde{p})^2 > 1$. Because we assume that those terms of $\mathcal{O}((a\widetilde{p})^2)$ and higher order are pure lattice artifacts, we take the b_1 as X_{11} value at $\mu = 2\,\mathrm{GeV}$ and $\mu = 3\,\mathrm{GeV}$ in the RI-MOM scheme.

μ_0	b_1	b_2	b_3	b_4	b_5	χ^2/dof
2GeV	1.056(15)	-0.500(18)	0.0925(72)	0.0019(63)	-0.00643(88)	0.08(17)
3GeV	1.088(16)	-0.515(18)	0.0953(74)	0.0020(65)	-0.00663(91)	0.08(17)

p-fit plot



(g) p-fit ($\mu_0 = 2\,\mathrm{GeV}$)



(h) p-fit ($\mu_0 = 3\,\mathrm{GeV}$)

Γ^{-1}(2GeV) matrix result

$$(\Gamma^B)^{-1} = \begin{pmatrix} X^{-1} + X^{-1}YY^TX^{-1} & -X^{-1}Y(1+Y^TX^{-1}Y) \\ -Y^T(X^{-1}+X^{-1}YY^TX^{-1}) & 1+Y^TX^{-1}Y \end{pmatrix}$$

$$X^{-1} = \begin{pmatrix} 1.374(33) & -0.818(47) & 0.347(21) & 0.007(32) \\ -0.748(45) & 1.999(43) & -0.009(31) & -0.049(34) \\ 0.352(31) & 0.033(47) & 1.260(33) & -0.623(39) \\ 0.019(46) & -0.086(54) & -0.656(40) & 1.590(37) \end{pmatrix}$$

$$X^{-1}YY^TX^{-1} = \begin{pmatrix} 0.0127(15) & -0.0079(11) & 0.0020(17) & -0.0001(10) \\ -0.00713(92) & 0.0064(11) & -0.0010(11) & -0.00045(73) \\ 0.0020(18) & 0.0000(15) & 0.0200(90) & -0.0103(48) \\ 0.0001(12) & -0.0012(11) & -0.0109(52) & 0.0059(28) \end{pmatrix} \tag{45}$$

$$-X^{-1}Y \approx 1\% \rightarrow 0.07\% \text{ by wrong taste suppression} \tag{46}$$

Result of renormalization factors of diagonal operators

We obtain $\vec{z}$ in RI-MOM scheme at 2GeV and 3GeV.

$$\vec{z} = \vec{z}_{\text{tree}}(\Gamma^B)^{-1}\,, \tag{47}$$

where

$$\vec{z}_{\text{tree}} = (1, 1, 1, 1, 0, \cdots, 0) \tag{48}$$

$$\vec{z} = (z_1, z_2, z_3, z_4, z_\alpha, z_\beta, \cdots) \tag{49}$$

We convert the scheme from RI-MOM $\rightarrow \overline{\text{MS}}$ using two-loop matching factor.

	RI-MOM(2GeV)	$\overline{\text{MS}}$(2GeV)	RI-MOM(3GeV)	$\overline{\text{MS}}$(3GeV)
z_1	0.9962(80)	1.0139(82)	0.9666(78)	0.9812(79)
z_2	1.128(31)	1.148(32)	1.095(30)	1.111(31)
z_3	0.9418(75)	0.9586(77)	0.9139(73)	0.9277(74)
z_4	0.926(29)	0.942(30)	0.898(28)	0.912(29)

Systematic Error

We estimate two different systematic errors.

- The size of $X^{-1}YY^TX^{-1}$.

$$(\Gamma^B)^{-1} = \begin{pmatrix} X^{-1} + X^{-1}YY^TX^{-1} & -X^{-1}Y(1+Y^TX^{-1}Y) \\ -Y^T(X^{-1}+X^{-1}YY^TX^{-1}) & 1+Y^TX^{-1}Y \end{pmatrix} \quad (50)$$

- The uncertainty comes from truncated higher order of the two-loop matching factor (RI-MOM$\rightarrow \overline{\mathrm{MS}}$):$O(\alpha_s^3)$

$$(\text{sys. err. of } z_i) = z_i \cdot \alpha_s^3 \quad (51)$$

We add these systematic errors in quadrature.

	$\overline{\mathrm{MS}}$(2GeV)	$\overline{\mathrm{MS}}$(3GeV)
z_1	1.0139(82)(274)	0.9812(79)(163)
z_2	1.148(32)(30)	1.111(31)(17)
z_3	0.9586(77)(269)	0.9277(74)(171)
z_4	0.942(30)(25)	0.912(29)(14)

Off-diagonal renormalization factors

$$(\Gamma^B)^{-1} = \begin{pmatrix} X^{-1} + X^{-1}YY^TX^{-1} & -X^{-1}Y(1 + Y^TX^{-1}Y) \\ -Y^T(X^{-1} + X^{-1}YY^TX^{-1}) & 1 + Y^TX^{-1}Y \end{pmatrix} \tag{52}$$

$$-X^{-1}Y(2\,\mathrm{GeV}) = \begin{pmatrix} -0.013(12) & 0.0075(64) & 0.013(12) & \cdots \\ 0.0155(38) & 0.0068(32) & -0.0008(31) & \cdots \\ 0.083(35) & 0.078(22) & -0.0162(45) & \cdots \\ -0.048(19) & -0.046(12) & 0.0104(13) & \cdots \end{pmatrix} \tag{53}$$

Off-diagonal renormalization factors

$$O^R_{B_K} = z_1 O^B_{V1} + z_2 O^B_{V2} + z_3 O^B_{A1} + z_4 O^B_{A2} + \sum_{\alpha \in (D)} z_\alpha O^B_\alpha \tag{54}$$

z_α	RI-MOM(2GeV)	$\overline{\mathrm{MS}}$(2GeV)	RI-MOM(3GeV)	$\overline{\mathrm{MS}}$(3GeV)
$z_{(S\otimes V)(S\otimes V)I}$	0.037(28)	0.038(28)	0.037(28)	0.038(28)
$z_{(S\otimes V)(S\otimes V)II}$	0.047(16)	0.047(17)	0.047(16)	0.047(17)
$z_{(V\otimes S)(V\otimes S)I}$	-0.00252(71)	-0.00257(73)	-0.00252(71)	-0.00256(72)
$z_{(V\otimes S)(V\otimes S)II}$	0.00038(10)	0.00039(11)	0.00038(10)	0.00039(10)
$z_{(T\otimes V)(T\otimes V)I}$	-0.0029(13)	-0.0030(14)	-0.0029(13)	-0.0029(14)
$z_{(T\otimes V)(T\otimes V)II}$	0.0094(42)	0.0096(42)	0.0094(42)	0.0096(42)
$z_{(A\otimes S)(A\otimes S)I}$	-0.0007(43)	-0.0007(44)	-0.0007(43)	-0.0007(44)
$z_{(A\otimes S)(A\otimes S)II}$	0.0027(13)	0.0027(14)	0.0027(13)	0.0027(14)
$z_{(P\otimes V)(P\otimes V)I}$	-0.0673(17)	-0.0685(17)	-0.0673(17)	-0.0683(17)
$z_{(P\otimes V)(P\otimes V)II}$	-0.0311(22)	-0.0317(23)	-0.0311(22)	-0.0316(23)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The effect of off-diagonal operators

The matrix element of the wrong taste operator with external K meson $(P \otimes P)$ is suppressed. Ref.[Thesis, Weonjong Lee]
The wrong taste channel with same normalization with B_K is.

$$\frac{\langle \overline{K}_0 | O_{(P\otimes V)(P\otimes V)_I} | K_0 \rangle}{\frac{8}{3}\langle \overline{K}_0 | A_\mu | 0 \rangle \langle 0 | A_\mu | K_0 \rangle} \leq 1\% \tag{55}$$

The size of wrong taste channel is smaller than 1% of B_K operator.
For example, the effect of $(P \otimes V)(P \otimes V)_I$ channel to B_K is

$$z_{(P\otimes V)(P\otimes V)I} \times 1\% \leq 0.069\% \tag{56}$$

Hence, this effect of wrong taste channel is neglected.

Compare with one-loop matching factor

We compare the NPR result($\overline{\mathrm{MS}}$ [NDR]) with those of one-loop perturbative matching.

	NPR(2GeV)	one-loop(2GeV)	NPR(3GeV)	one-loop(3GeV)
z_1	1.0139(82)(274)	1.080(95)	0.9812(79)(163)	1.035(62)
z_2	1.148(32)(30)	1.168(102)	1.111(31)(17)	1.120(67)
z_3	0.9586(77)(269)	1.088(95)	0.9277(74)(171)	1.043(63)
z_4	0.942(30)(25)	0.994(87)	0.912(29)(14)	0.953(57)

We estimate the systematic error of one-loop result as two-loop uncertainty(α_s^2).

$$(\text{sys. err. of } z_i) = z_i \cdot \alpha_s^2 \tag{57}$$

Conclusion

- We have done the first round data analysis for the B_K operator.
- We plan to analyse the BSM operators in near future.