ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

DPHYS
Department of Physics

Walking technicolor: testing infrared conformality with exact results in two dimensions

Oscar Åkerlund*
Philippe de Forcrand

LATTICE 2014
June 23, 2014

Conformal invariance. . .

- Conformal invariance is typically connected with an infrared (IR) fixed point.
- There the coupling constant stops running and thus becomes scale invariant.
- A general feature is that all correlation functions are given by power laws.
- Perturbatively, if you add enough fermions to SU(N) Yang-Mills theory, it becomes conformal.

. . . on the lattice.

- A finite lattice is not well suited to study a conformal theory.
- There are two intrinsic scales which you can never truly get rid of, the lattice spacing a and the lattice extent L.
- Nevertheless, lattice calculations are the method of choice for nonperturbative investigation of QFT's.
- It is important to control the deformations of the conformal theory induced by a lattice and by external masses.

A first example: zero momentum Green's functions

- Let some field have an anomalous scaling dimension Δ. The Green's functions in coordinate and Fourier space are related:

$$(x^2)^{-(d-2)/2-\Delta} \Leftrightarrow (p^2)^{-1+\Delta}$$

A first example: zero momentum Green's functions

- Let some field have an anomalous scaling dimension Δ. The Green's functions in coordinate and Fourier space are related:

$$(x^2)^{-(d-2)/2-\Delta} \Leftrightarrow (p^2)^{-1+\Delta}$$

- Now let us add a mass deformation to this, $\widetilde{G}(p) = (p^2 + m^2)^{-1+\Delta}$. A short calculation suggests that for large separations the zero spatial momentum temporal correlator is given by:

$$G(t, \vec{p} = 0) \propto \left(\frac{t}{m}\right)^{\frac{1}{2}-\Delta} K_{\frac{1}{2}-\Delta}(mt) \overset{mt \gg 1}{\propto} \frac{e^{-mt}}{t^{\Delta}}$$

- Turned into analysis method of lattice correlators (*Iwasaki et al.*) but does not apply on the lattice!

Can lattice simulations be used to study a conformal theory?

- Three important questions to answer:
  1. How do the two intrinsic scales, *a*, *L*, modify the conformal behavior?
  2. Can we observe power law decay or power law corrected exponential decay on the lattice?
  3. Can a mass deformation be used to obtain anomalous dimensions?

Can lattice simulations be used to study a conformal theory?

- Three important questions to answer:
  1. How do the two intrinsic scales, a, L, modify the conformal behavior?
  2. Can we observe power law decay or power law corrected exponential decay on the lattice?
  3. Can a mass deformation be used to obtain anomalous dimensions?

- Answers from exactly solvable models in $2d$:
  1. The critical Ising model (1 and 2).
  2. The Sommerfield model (3).

1. Critical $2d$ Ising model

- In an infinite volume the spin-spin correlation is given by:

$$\langle \sigma(0,0)\sigma(x,t)\rangle \propto \left(x^2 + t^2\right)^{-\Delta}$$

$\Delta = \frac{1}{8}$ is the scaling dimension of the spin field.

1. Critical 2d Ising model

- In an infinite volume the spin-spin correlation is given by:

$$\langle \sigma(0,0)\sigma(x,t)\rangle \propto \left(x^2+t^2\right)^{-\Delta}$$

$\Delta = \frac{1}{8}$ is the scaling dimension of the spin field.

- On a torus with size $L_s \times L_t \equiv L \times (\tau L)$, $\quad \tau = \frac{N_t}{N_s}$ we have:

$$\langle \sigma(0,0)\sigma(x,t)\rangle \propto |\vartheta_1(z,q)|^{-\frac{1}{4}} \sum_{\nu=1}^{4} \left|\vartheta_\nu(\tfrac{z}{2},q)\right|$$

$\vartheta_\nu(z,q)$ is the ν'th Jacobi theta function with $q = \exp(-\pi\tau)$ and $z = \frac{\pi}{L}(x+it)$.

We sum over x to obtain the zero spatial momentum correlator:

$$C(t,q,L) \propto \cosh\left(\frac{\pi}{4L}\left(t-\frac{\tau L}{2}\right)\right) + \sum_{n=1}^{\infty}\left(c_n q^{2n}\cosh\left(\frac{\pi(4n+\frac{1}{4})}{L}\left(t-\frac{\tau L}{2}\right)\right)\right)$$
$$+ q^{\frac{1}{4}}\sum_{n=1}^{\infty}\left(c_n^q q^n \cosh\left(\frac{\pi(2n-\frac{1}{4})}{L}\left(t-\frac{\tau L}{2}\right)\right)\right) + \mathcal{O}(q^2), \quad q = e^{-\pi\tau}$$

The coefficients c_n and c_n^q depends on n and L and can be exactly calculated.

We sum over x to obtain the zero spatial momentum correlator:

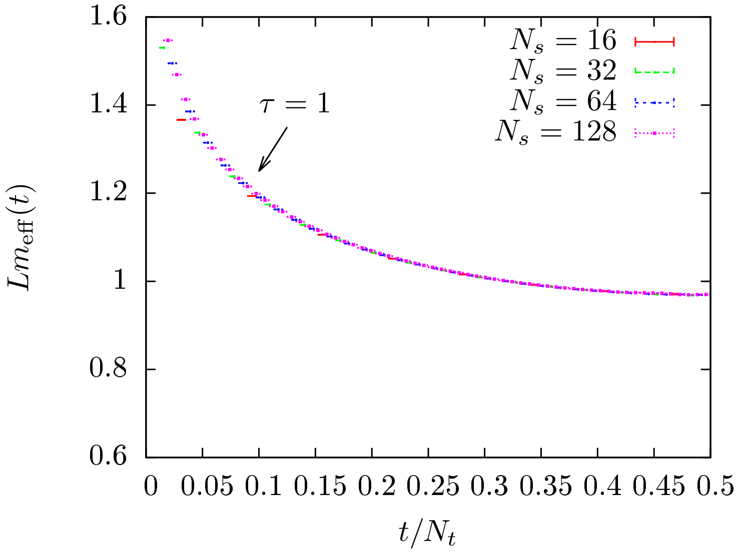
$$C(t,q,L) \propto \cosh\left(\frac{\pi}{4L}\left(t-\frac{\tau L}{2}\right)\right) + \sum_{n=1}^{\infty}\left(c_n q^{2n}\cosh\left(\frac{\pi(4n+\frac{1}{4})}{L}\left(t-\frac{\tau L}{2}\right)\right)\right)$$
$$+\, q^{\frac{1}{4}}\sum_{n=1}^{\infty}\left(c_n^q q^n \cosh\left(\frac{\pi(2n-\frac{1}{4})}{L}\left(t-\frac{\tau L}{2}\right)\right)\right) + \mathcal{O}(q^2), \quad q = e^{-\pi\tau}$$

The coefficients c_n and c_n^q depends on n and L and can be exactly calculated.

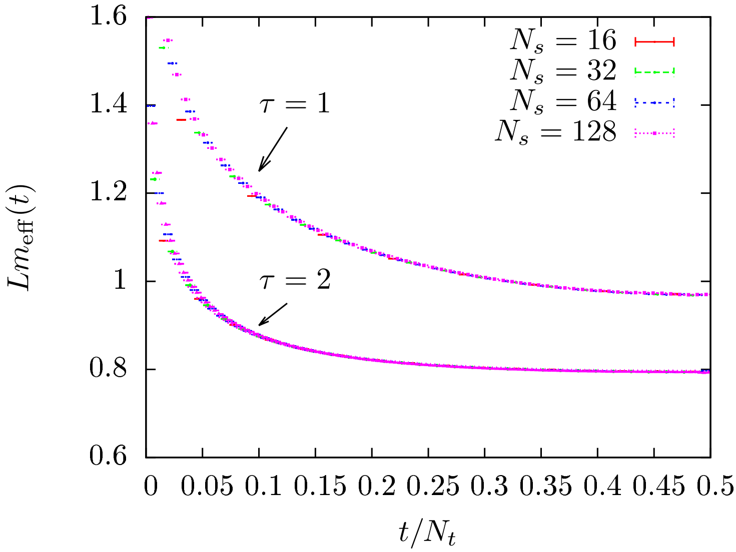
- $q = 0 = e^{-\pi\tau}$ corresponds to a cylindrical geometry:

$$C(t,0,L) \propto e^{-\frac{\pi}{4}\frac{t}{L}}\left(1+\sum_{n=1}^{\infty} c_n e^{-4n\pi\frac{t}{L}}\right)$$

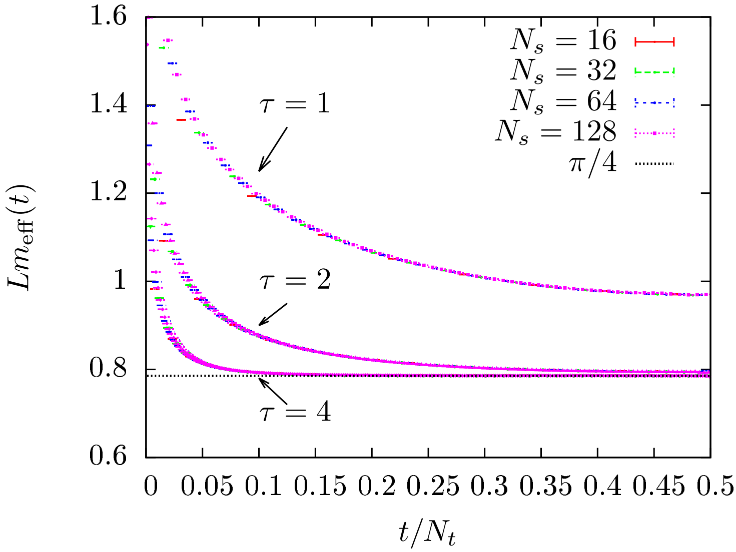
Numerical results



Numerical results

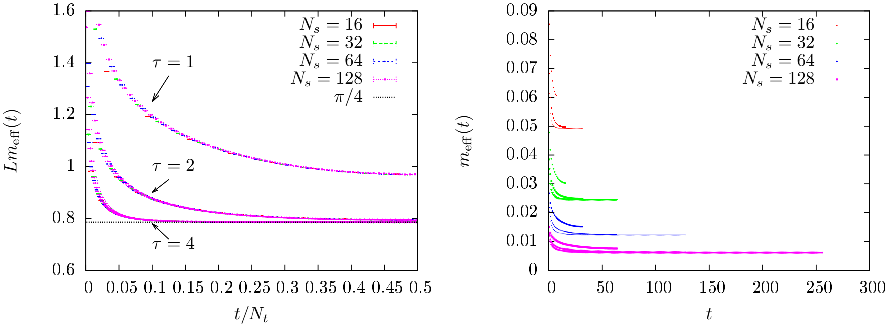


Numerical results



Numerical results

- Perfect matching of correlators after re-scaling the dimensions $\Leftrightarrow$ Conformal behavior.
- Effective mass plateau says nothing about conformal behavior



From exponentials to a power law

- The Euclidean time correlator is related to the spectral density:

$$C(t) \propto \int_0^\infty \mathrm{d}\omega \, \rho(\omega) e^{-\omega t}.$$

- Each exponential decay, $\omega_n = n\frac{\delta}{L}$ in $C(t)$ corresponds to a delta function in $\rho(\omega)$.

From exponentials to a power law

- The Euclidean time correlator is related to the spectral density:

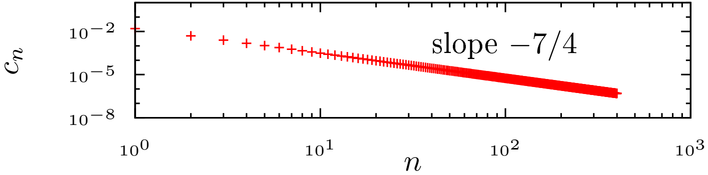
$$C(t) \propto \int_0^\infty \mathrm{d}\omega \, \rho(\omega) e^{-\omega t}.$$

- Each exponential decay, $\omega_n = n\frac{\delta}{L}$ in $C(t)$ corresponds to a delta function in $\rho(\omega)$.
- Taking $L \to \infty$, i.e. the level spacing to zero, leads to an integral over the delta functions and $\rho(\omega) = c\left(\frac{\omega L}{\delta}\right)$ where $c(x)$ is the continuous version of the coefficients c_n, (*cf. M. Stephanov arXiv:0705.3049*).
- Since $L \to \infty$ we will probe the large n asymptotic of c_n:

$$\boxed{c_n \propto n^{-\alpha} \Rightarrow \rho(\omega) \propto \omega^{-\alpha}}$$

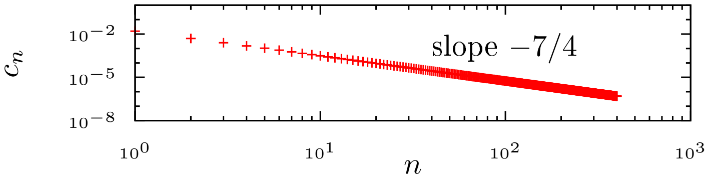
An explicit calculation

- For the Ising model in a cylinder we find $c_n = \dfrac{\Gamma\left(\frac{1}{8}+n\right)^2}{(n!)^2\Gamma\left(\frac{1}{8}\right)^2}$ for $L \to \infty$.
-



An explicit calculation

- For the Ising model in a cylinder we find $c_n = \frac{\Gamma\left(\frac{1}{8}+n\right)^2}{(n!)^2\Gamma\left(\frac{1}{8}\right)^2}$ for $L \to \infty$.

- 

- This implies $\rho(\omega) \propto \omega^{-\frac{7}{4}}$ which is what we expect:

$$(x^2 + t^2)^{-\frac{1}{8}} \Rightarrow (p^2 + \omega^2)^{-1+\frac{1}{8}} = \{p = 0\} = \omega^{-\frac{7}{4}} \propto \rho(\omega)$$

Consequences

- We find that there is a mass $\frac{\pi}{4L}$ even though the theory should be massless.
- There is no power law correction to the exponential decay. Instead, as $L \to \infty$, a continuum of excitations emerge to give a massless correlator.

Consequences

- We find that there is a mass $\frac{\pi}{4L}$ even though the theory should be massless.

- There is no power law correction to the exponential decay. Instead, as $L \to \infty$, a continuum of excitations emerge to give a massless correlator.

- A finite temporal extent introduces new corrections, $\sim e^{-\frac{\pi\tau}{4}} e^{-\delta E t}$, which are exponentially suppressed by the aspect ratio.

- These new states may conceal the true ground state mass $\frac{\pi}{4L}$, even at $L \to \infty$. Large τ essential.

2. The $2d$ Sommerfield model

C. M. Sommerfield, *Ann. Phys.* 26 (1964) 1, H. Georgi and Y. Kats, *JHEP* 1002 (2010) 065

The Sommerfield Lagrangian is given by:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{m_0^2}{2}A^\mu A_\mu$$

- It is the Schwinger model with a mass term for the vector boson, i.e. there is no gauge symmetry.
- In the infrared it becomes scale-invariant (cf. Thirring model) and has anomalous dimensions.

- Solved by introducing: $A_\mu = \partial_\mu \mathcal{V} + \epsilon_{\mu\nu}\partial^\nu \mathcal{A}$ and $\Psi = e^{ie(\mathcal{V}+\mathcal{A}\gamma^5)}\psi$.
- Fermion becomes free:

$$\mathcal{L} = i\bar{\Psi}\partial\!\!\!/\Psi + \frac{m_0^2}{2}\partial_\mu\mathcal{V}\partial^\mu\mathcal{V} + \frac{1}{2}\mathcal{A}(\partial_\mu\partial^\mu)^2\mathcal{A} - \frac{m^2}{2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \qquad m^2 = m_0^2 + \frac{e^2}{\pi}$$

- Solved by introducing: $A_\mu = \partial_\mu \mathcal{V} + \epsilon_{\mu\nu}\partial^\nu \mathcal{A}$ and $\Psi = e^{ie(\mathcal{V}+\mathcal{A}\gamma^5)}\psi$.
- Fermion becomes free:

$$\mathcal{L} = i\bar{\Psi}\partial\!\!\!/\Psi + \frac{m_0^2}{2}\partial_\mu\mathcal{V}\partial^\mu\mathcal{V} + \frac{1}{2}\mathcal{A}(\partial_\mu\partial^\mu)^2\mathcal{A} - \frac{m^2}{2}\partial_\mu\mathcal{A}\partial^\mu\mathcal{A} \qquad m^2 = m_0^2 + \frac{e^2}{\pi}$$

- The fermionic 2-point function is given by a product of a free fermionic Green's function and a bosonic correction:

$$\begin{aligned}\langle 0|\mathrm{T}\psi_\alpha(x)\psi^*_\beta(0)|0\rangle =& \langle 0|\mathrm{T}\Psi_\alpha(x)\Psi^*_\beta(0)|0\rangle \\ &\times \langle 0|\mathrm{T}e^{-ie(\mathcal{V}(x)+\mathcal{A}(x)\gamma^5)}e^{-ie(\mathcal{V}(0)+\mathcal{A}(0)\gamma^5)}|0\rangle\end{aligned}$$

- The bosonic correction can be calculated in a closed form and is given by an exponential of massless and massive bosonic Green's functions.

- Define the composite operator $\mathcal{O} \equiv \bar{\psi}\frac{1}{2}(1+\gamma^5)\psi = \psi_2^*\psi_1$ (cf. meson/pion). This operator will have an anomalous dimension at low energies.
- Using the Operator Product Expansion of a product of two ψ's one can show (Georgi & Kats) that:

$$\langle 0|\mathrm{T}\mathcal{O}(x)\mathcal{O}(0)|0\rangle = C(x)^4 \left|S_0(x)\right|^2,$$

where $S_0(x)$ is a free fermion propagator and $C(x)$ is due to the bosons.

- Define the composite operator $\mathcal{O} \equiv \bar{\psi}\frac{1}{2}(1+\gamma^5)\psi = \psi_2^*\psi_1$ (cf. meson/pion). This operator will have an anomalous dimension at low energies.
- Using the Operator Product Expansion of a product of two ψ's one can show (Georgi & Kats) that:

$$\langle 0|\mathrm{T}\mathcal{O}(x)\mathcal{O}(0)|0\rangle = C(x)^4 \left|S_0(x)\right|^2 ,$$

where $S_0(x)$ is a free fermion propagator and $C(x)$ is due to the bosons.
- Using the asymptotic behaviors of $C(x)$ and $S_0(x)$ one finds

$$\langle 0|\mathrm{T}\mathcal{O}(x)\mathcal{O}(0)|0\rangle \propto \begin{cases} (x^2)^{-1}, & xm \ll 1 \\ (x^2)^{-1-\gamma_{\mathcal{O}}}, & xm \gg 1 \end{cases}$$

where $\gamma_{\mathcal{O}} \equiv -\frac{e^2}{\pi m^2} = -\frac{1}{1+\frac{m_0^2\pi}{e^2}}$ is the anomalous dimension of $\mathcal{O}$.

On the lattice

- In principle straightforward to calculate the propagator on the lattice.
- We can also add a bare mass, m_q, to the fermion to measure the anomalous mass dimension.
- $N_f = 4$ (naive lattice fermions) is similar to $N_f = 1$, *except* when $m_0 \to 0$ (Schwinger model).

On the lattice

- In principle straightforward to calculate the propagator on the lattice.
- We can also add a bare mass, m_q, to the fermion to measure the anomalous mass dimension.
- $N_f = 4$ (naive lattice fermions) is similar to $N_f = 1$, *except* when $m_0 \to 0$ (Schwinger model).
- Note: if Lm_q is small, fermionic spatial boundary conditions start to play a role:
  - Periodic boundary conditions $\Rightarrow \vec{p}_x = \vec{0}$ fermion, power law corrections in $1/L$.
  - Anti-periodic boundary conditions $\Rightarrow |\vec{p}_x| \geq \frac{\pi}{L}$.

Results

- We consider both periodic and anti-periodic boundary conditions.
- With zero quark mass we obtain (for apbc) the composite mass

$$m_{\mathcal{O}} = \frac{2\pi}{L}(1 + \gamma_{\mathcal{O}}), \quad \gamma_{\mathcal{O}} = -\frac{e^2}{\pi m^2} \quad \text{(anomalous dimension of } \mathcal{O}\text{)}$$

Note how this is analogous to the Ising case.

Results

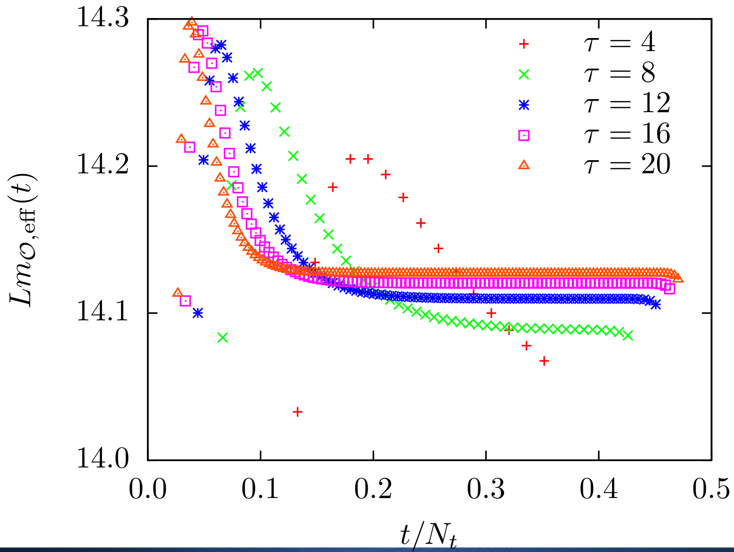
- We consider both periodic and anti-periodic boundary conditions.
- With zero quark mass we obtain (for apbc) the composite mass

$$m_{\mathcal{O}} = \frac{2\pi}{L}(1+\gamma_{\mathcal{O}}), \quad \gamma_{\mathcal{O}} = -\frac{e^2}{\pi m^2} \quad (\text{ anomalous dimension of } \mathcal{O})$$

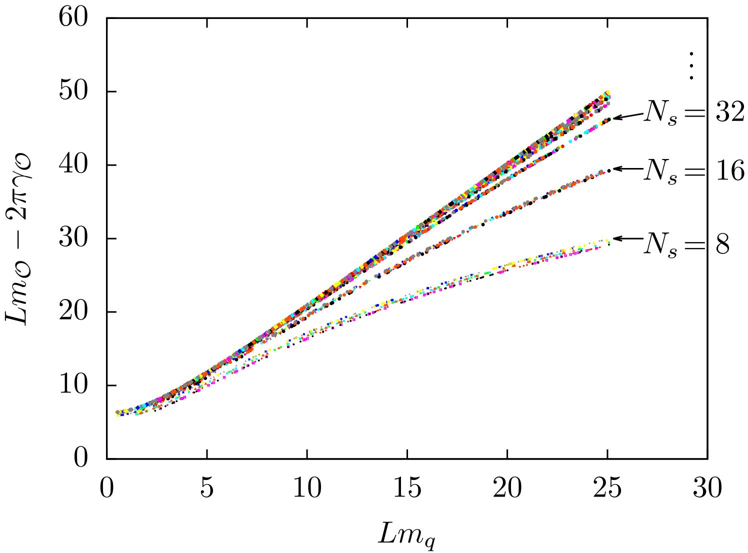
  Note how this is analogous to the Ising case.
- We then calculate $m_{\mathcal{O}}$ as a function of the mass deformation m_q.
- Lattice of size $N_s \times (16N_s)$ with $8 \leq N_s \leq 96$. Also three different values of $Lm \in \{3, 5, 10\}$ and $\gamma_{\mathcal{O}} \in \{-\frac{5}{6}, -\frac{1}{2}, -\frac{1}{6}\}$.

Numerical exact results

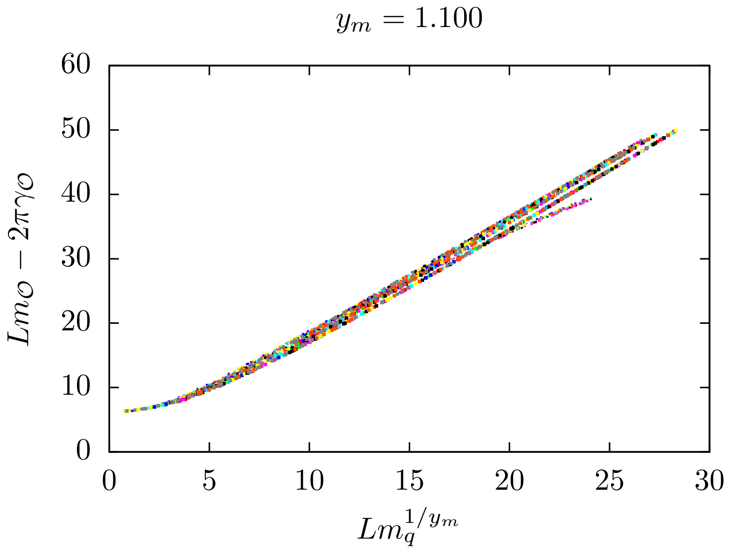
$$C(t) \sim e^{-m_{\mathcal{O}}\frac{t}{L}-\frac{m_2}{\tau}\left(\frac{t}{L}\right)^2}$$



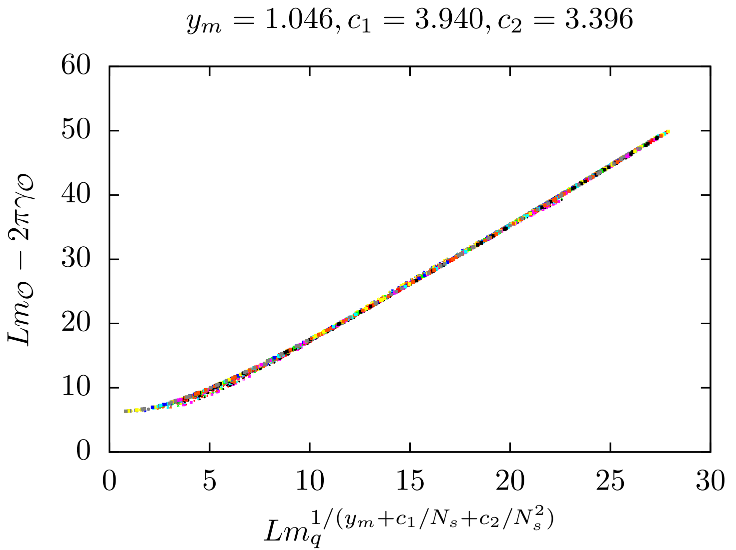
Anti-periodic boundary conditions with mass deformation m_q



Single anomalous dimension y_m, exact result: $y_m = 1$

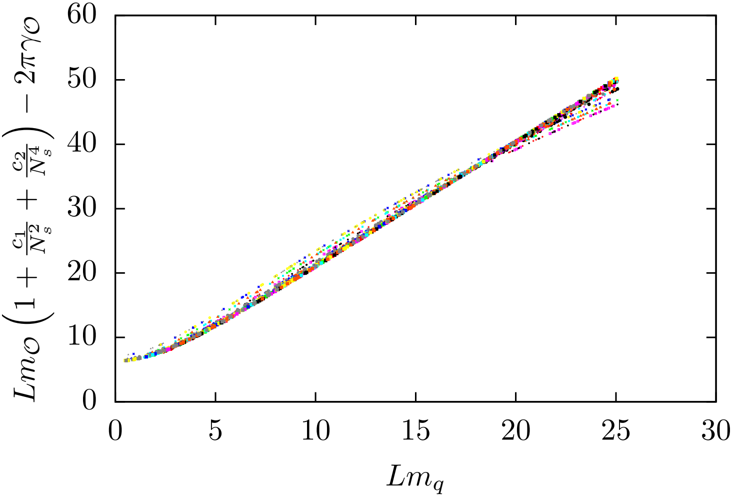


Effective $y_m(N_s)$

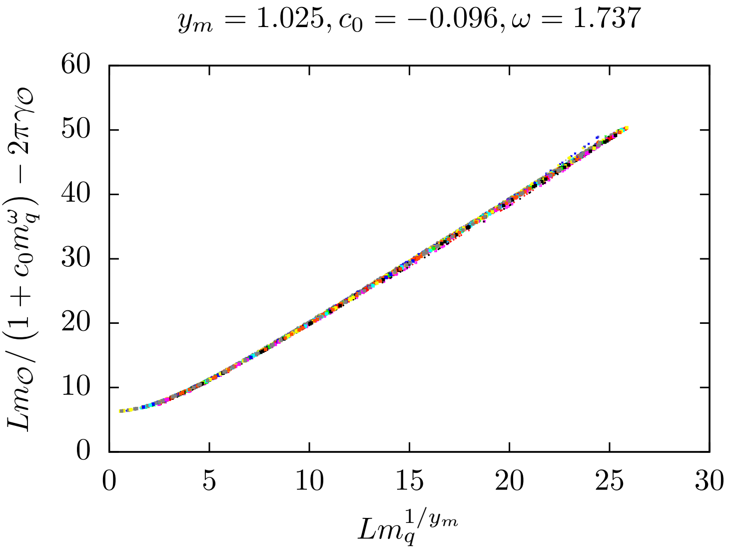


Discretization errors with $y_m = 1$ fixed to exact value

$$c_1 = 62.424, c_2 = -2507.945$$

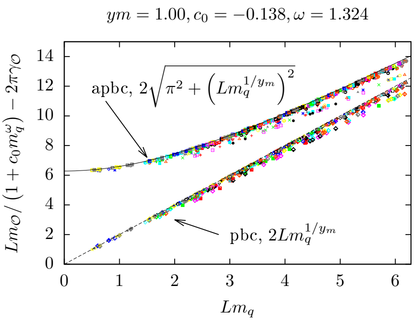


Corrections to scaling (*cf. Cheng et al. arXiv:1401.0195*)

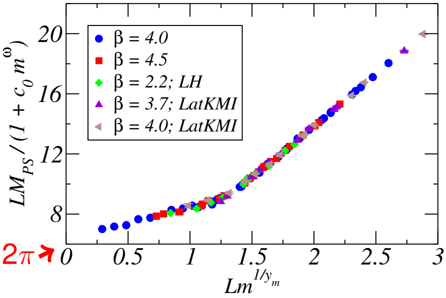


Comparison apbc and pbc

This work



arXiv:1401.0195



SU(3) with 12 fundamental flavors:

A. Cheng, A. Hasenfratz, Y. Liu, G. Petropoulos and D. Schaich, arXiv:1401.0195

Conclusions and lessons learned:
Points to keep in mind when studying conformality on the lattice

- Direct test of scale invariance: compare correlators $C(t)$ in lattice $L^3 \times L_t$ with $C(\lambda t)$ in lattice $(\lambda L)^3 \times \lambda L_t$.

Conclusions and lessons learned:

Points to keep in mind when studying conformality on the lattice

- Direct test of scale invariance: compare correlators $C(t)$ in lattice $L^3 \times L_t$ with $C(\lambda t)$ in lattice $(\lambda L)^3 \times \lambda L_t$.
- Anti-periodic boundary conditions allows zero external mass but there is still a mass plateau. Also, small external masses may be over-shadowed by the constant term.
- Periodic boundary conditions allows zero meson mass but may come with high computational costs.

Conclusions and lessons learned:

Points to keep in mind when studying conformality on the lattice

- Direct test of scale invariance: compare correlators $C(t)$ in lattice $L^3 \times L_t$ with $C(\lambda t)$ in lattice $(\lambda L)^3 \times \lambda L_t$.
- Anti-periodic boundary conditions allows zero external mass but there is still a mass plateau. Also, small external masses may be over-shadowed by the constant term.
- Periodic boundary conditions allows zero meson mass but may come with high computational costs.
- It is mandatory to use a large aspect ratio N_t/N_s because of excited states.

Conclusions and lessons learned:

Points to keep in mind when studying conformality on the lattice

- Direct test of scale invariance: compare correlators $C(t)$ in lattice $L^3 \times L_t$ with $C(\lambda t)$ in lattice $(\lambda L)^3 \times \lambda L_t$.
- Anti-periodic boundary conditions allows zero external mass but there is still a mass plateau. Also, small external masses may be over-shadowed by the constant term.
- Periodic boundary conditions allows zero meson mass but may come with high computational costs.
- It is mandatory to use a large aspect ratio N_t/N_s because of excited states.
- It is necessary to consider corrections to scaling and discretization errors to obtain correct anomalous dimensions.

Thank you for your attention!

Ising model $\Leftrightarrow$ Free fermions

- The $2d$ Ising model can be reformulated in terms of free staggered fermions with pbc and apbc in space and time (4 combinations).

- At criticality all bc's are equally important and are averaged over (cf. the four Jacobi theta functions).

- The apbc contributions are related to the $1/L$ mass.

The coefficients c_n and c_n^q

Recall the Ising propagator on a torus of size $L \times (\tau L)$:

$$C(t, x, L, \tau) \equiv \langle \sigma(0,0)\sigma(x,t) \rangle \propto |\vartheta_1(z,q)|^{-\frac{1}{4}} \sum_{\nu=1}^{4} \left|\vartheta_\nu(\tfrac{z}{2}, q)\right|$$

Defining $u = e^{-\frac{\pi t}{L}}$, to lowest order in q, the magnitude of the theta functions are given by:

$$\left|\vartheta_1\left(\frac{x+it}{\pi L}, q\right)\right| = q^{\frac{1}{4}} u^{-1} \left(1 + u^4 - 2u^2 \cos\left(\tfrac{2\pi x}{L} x\right)\right)^{\frac{1}{2}}$$

$$\left|\vartheta_2\left(\frac{x+it}{\pi L}, q\right)\right| = q^{\frac{1}{4}} u^{-1} \left(1 + u^4 + 2u^2 \cos\left(\tfrac{2\pi x}{L}\right)\right)^{\frac{1}{2}}$$

$$\left|\vartheta_3\left(\frac{x+it}{\pi L}, q\right)\right| = \left|\vartheta_4\left(\frac{x+it}{\pi L}, q\right)\right| = 1$$

Inserting this yields

$$C(t,x,L,q) = q^{-1/16} u^{\frac{1}{4}} \left(1 + u^4 - 2u^2 \cos\left(\tfrac{2\pi x}{L}\right)\right)^{-\frac{1}{8}}$$
$$\times \left(2 + q^{\frac{1}{4}} u^{-\frac{1}{2}} \left(\left(\left(1 + u^2 - 2u^1 \cos\left(\tfrac{\pi x}{L}\right)\right)^{\frac{1}{2}} + \right.\right.\right.$$
$$\left.\left.\left.\left(1 + u^2 + 2u^1 \cos\left(\tfrac{\pi x}{L}\right)\right)^{\frac{1}{2}}\right)\right)\right)$$

We discard the overall $q^{-\frac{1}{8}} u^{\frac{1}{4}}$ and expand the expressions containing the cosines in order to sum over x. Note that $u^{\frac{1}{4}} = e^{-\frac{\pi}{4L} t}$ is the leading order decay. We also treat the q-independent and q-dependent parts separately.

q-independent part

An expansion in powers of u gives

$$\left(1+u^4-2u^2\cos\left(\tfrac{2\pi x}{L}\right)\right)^{-\frac{1}{8}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{8}}{n}\left(u^4-2u^2\cos\left(\tfrac{2\pi x}{L}\right)\right)^n$$

$$= \sum_{n=0}^{\infty}\binom{-\frac{1}{8}}{n}\sum_{m=0}^{n}\binom{n}{m}(-2)^m u^{4n-2m}\cos\left(\tfrac{2\pi x}{L}\right)^m$$

We can now sum over x term by term using:

$$\frac{1}{L}\sum_{x=0}^{L-1}\cos\left(\tfrac{2\pi x}{L}\right)^m = 2^{-2r}\left(\binom{2r}{r}+2\sum_{n=1}^{\left\lfloor\frac{2r}{L}\right\rfloor}\binom{2r}{r+\frac{nL}{2}}\right)\delta_{m,2r}$$

This gives:

$$C(t, p_x = 0, L, 0) \propto \sum_{n=0}^{\infty} \binom{-\frac{1}{8}}{n} \sum_{r=0}^{n} \binom{n}{2r} u^{4(n-r)} \left(\binom{2r}{r} + 2 \sum_{m=1}^{\left\lfloor \frac{2r}{L} \right\rfloor} \binom{2r}{r + \frac{mL}{2}} \right)$$

$$= \sum_{n=0}^{\infty} u^{4n} \sum_{r=0}^{n} \binom{-\frac{1}{8}}{n+r} \binom{n+r}{2r} \left(\binom{2r}{r} + 2 \sum_{m=1}^{\left\lfloor \frac{2r}{L} \right\rfloor} \binom{2r}{r + \frac{mL}{2}} \right)$$

$$= \sum_{n=0}^{\infty} u^{4n} \underbrace{\left(\frac{\Gamma\left(\frac{1}{8} + n\right)^2}{(n!)^2 \Gamma\left(\frac{1}{8}\right)^2} + \sum_{m=1}^{\left\lfloor \frac{2n}{L} \right\rfloor} \frac{\Gamma\left(\frac{7}{8} - n + \frac{mL}{2}\right)^{-1} \Gamma\left(1 + n + \frac{mL}{2}\right)^{-1}}{\Gamma\left(\frac{7}{8} - n - \frac{mL}{2}\right) \Gamma\left(1 + n - \frac{mL}{2}\right)} \right)}_{c_n}$$

q-dependent part

For the q-dependent part we have to expand three roots before we can sum over x. After expanding we find that we need to calculate the following sums:

$$\frac{1}{L}\sum_{x=0}^{L-1}\cos\left(\frac{2\pi x}{L}\right)^{r}\cos\left(\frac{\pi x}{L}\right)^{2p}$$

$$= 2^{-r-2p}\sum_{n=0}^{r}\binom{r}{n}\left(\binom{2p}{p+r-2n}+2\sum_{q=1}^{\left\lfloor\frac{r+p}{L}\right\rfloor}\binom{2p}{p+r-2n+qL}\right)$$

Using this we find the coefficient

$$c_n^q = \sum_{m=0}^{n} \sum_{r=\max(2m-n,0)}^{m} \sum_{p=0}^{n+r-2m} (-1)^r \binom{\frac{1}{2}}{n+r+p-2m} \binom{n+r+p-2m}{2p}$$

$$\times \binom{-\frac{1}{8}}{m} \binom{m}{r} \left(\sum_{k=0}^{r} \binom{r}{k} \left(\binom{2p}{p+r-2k} + 2 \sum_{q=1}^{\left\lfloor \frac{r+p}{L} \right\rfloor} \binom{2p}{p+r-2k+qL} \right) \right)$$

The bosonic corrections

We need to calculate

$$B(x) = \langle 0|\mathrm{T} e^{-ie(\mathcal{V}(x)+\mathcal{A}(x)\gamma^5)} e^{-ie(\mathcal{V}(0)+\mathcal{A}(0)\gamma^5)}|0\rangle.$$

Perform Wick contractions with:

$$\int \mathrm{d}^2 x\, e^{ipx} \langle 0|\mathrm{T}\mathcal{V}(x)\mathcal{V}(0)|0\rangle = \frac{1}{m_0^2 p^2}$$

$$\int \mathrm{d}^2 x\, e^{ipx} \langle 0|\mathrm{T}\mathcal{A}(x)\mathcal{A}(0)|0\rangle = \frac{1}{(p^2)^2 + m^2 p^2} = \frac{1}{m^2}\left(\frac{1}{p^2+m^2} - \frac{1}{p^2}\right)$$

One finds

$$B(x) = \frac{C_0(x)}{C(x)^{\kappa_{\alpha\beta}}}, \quad \kappa_{\alpha\beta} = \begin{cases} 1, & \alpha \neq \beta \\ -1, & \alpha = \beta \end{cases}$$

where

$$C_0(x) = exp\left[\frac{e^2}{m_0^2}\left(D(x,0) - D(0,0)\right)\right]$$

$$C(x) = exp\left[\frac{e^2}{m^2}\left(\left(D(x,m) - D(0,m)\right) - \left(D(x,0) - D(0,0)\right)\right)\right]$$

$$D(x,m) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{e^{-ipx}}{p^2 + m^2}$$

which defines the $C(x)$ used in the $\mathcal{O}$ propagator.

The composite correlator

To derive the composite two-point correlator we start by considering the free fermion four-point correlator:

$$G^{(4)}_{\mathrm{free}} = \langle 0|\mathrm{T}\psi^*_{\alpha_2}(x_2)\psi_{\alpha_1}(x_1)\psi^*_{\beta_2}(y_2)\psi_{\beta_1}(y_1)|0\rangle.$$

which after Wick contractions becomes

$$-S_0^{\alpha_1\alpha_2}(x_1-x_2)S_0^{\beta_1\beta_2}(y_1-y_2)+S_0^{\alpha_1\beta_2}(x_1-y_2)S_0^{\beta_1\alpha_2}(y_1-x_2),$$

where the free fermion two-point function is given by

$$S_0^{\alpha_1\alpha_2}(x) = \int \frac{\mathrm{d}p^2}{(2\pi)^2}\frac{e^{ipx}}{p^2+m^2} \times \begin{cases} m, & \alpha_1 \neq \alpha_2 \\ (p_1 - ip_2), & \alpha_1 = \alpha_2 = 1 \\ (p_1 + ip_2) & \alpha_1 = \alpha_2 = 2 \end{cases}$$

The bosonic contribution is given by

$$\prod_{i<j} C_0(x_i - x_j)^{\eta_{ij}} C(x_i - x_j)^{\eta_{ij}\kappa_{ij}},$$

with the sign factors depending on which two fermionic fields are contracted via

$$\eta_{ij} = \begin{cases} +1, & \psi \text{ and } \psi^* \\ -1, & \text{otherwise} \end{cases}, \quad \kappa_{ij} = \begin{cases} +1, & \alpha = \beta \\ -1, & \alpha \neq \beta \end{cases}.$$

Since $\mathcal{O} = \psi_2^* \psi_1$ we will be interested in the case $\alpha_1 = \beta_2 \neq \alpha_2 = \beta_1$ and $x_1 \to x_2 \equiv x, \ y_1 \to y_2 \equiv y$. We thus get the full four point function

$$G^{(4)} = \frac{C_0(x_1 - x_2) C_0(y_1 - y_2)}{C(x_1 - x_2) C(y_1 - y_2)} C(x - y)^4 S_0^{11}(x - y) S_0^{22}(x - y).$$

Using a leading order operator product expansion of two fermionic fields we find

$$\mathrm{T}\psi_2^*(x_2)\psi_1(x_1) \approx c(x_2 - x_1)\psi_2^*\psi_1(x_2) \approx c(x_2 - x_1)\mathcal{O}(x_2),$$

which gives

$$G^{(4)} = c(x_2 - x_1)c(y_2 - y_1)\langle 0|\mathrm{T}\mathcal{O}(x)\mathcal{O}(y)|0\rangle.$$

Comparing to the previous slide we see that we should take $c(x) = C_0(x)/C(x)$ to arrive at

$$\langle 0|\mathrm{T}\mathcal{O}(x)\mathcal{O}(0)|0\rangle = C(x)^4 \left|S_0^{11}(x)\right|^2$$