$\Delta N \gamma^*$ transition form factor on the lattice

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Plan

- Introduction: Resonance matrix elements in the continuum
- Kinematics, projecting out the formfactors
- Lüscher-Lellouch relation for the pion photoproduction amplitude
- Analytic continuation to the resonance pole
- Conclusions, outlook
Resonance matrix elements in the continuum

In the vicinity of the resonance pole...

\[ i \int d^4x \ e^{iPx} \langle 0 | T \Delta(x) \bar{\Delta}(0) | 0 \rangle \rightarrow \frac{Z_R}{s_R - P^2} + \cdots \]

\[ i^2 \int d^4x d^4y \ e^{iPx - iQy} \langle 0 | T \Delta(x) J(0) \bar{N}(y) | 0 \rangle \]

\[ \rightarrow \frac{Z_R^{1/2}}{s_R - P^2} \langle \Delta | J(0) | N \rangle \frac{Z_N^{1/2}}{m_N^2 - Q^2} + \cdots \]
Photoproduction amplitude

In the narrow width approximation...

\[ |\text{Im} \mathcal{A}(\gamma^* N \rightarrow \pi N)| = \sqrt{\frac{8\pi}{p_A \Gamma}} |\langle \Delta |J(0)|N\rangle|, \quad \delta(p_A) = 90^\circ \]

- The form factor defined on the real axis is process dependent due to the background contribution
- Analytic continuation to the resonance pole is needed!

I.G. Aznaurian et al., arXiv:0810.0997; D. Drechsel et al., NPA 645 (1999) 145;
R.L. Workman et al., PRC 87 (2013) 068201
Assuming $\Delta$ to be a stable particle...  

How the formalism is generalized in case of an unstable $\Delta$?

- Which quantities should be measured on the lattice?
- How does one perform the infinite-volume limit in the form factors?
- How does one calculate the photoproduction amplitude?
- How does one perform the analytic continuation to the resonance pole?
- How does one project out different form factors in case of the unstable $\Delta$?

→ Use EFT approach in a finite volume
Kinematics

\[ \Delta(t) = \sum_x \Delta(t, x) \quad \text{(CM frame)} , \quad N(t) = \sum_x e^{-iQx} N(t, x) \]

Measuring three-point functions:

\[ R(t', t) = \langle 0|\Delta(t') J(0) \bar{N}(0)|0 \rangle , \quad S_\Delta(t) , S_N(t) : \text{ propagators} \]

\[ F = \lim_{t' \to \infty, t \to -\infty} N \frac{R(t', t)}{S_\Delta(t' - t)} \left( \frac{S_N(t') S_\Delta(-t) S_\Delta(t' - t)}{S_\Delta(t') S_N(-t) S_N(t' - t)} \right)^{1/2} \]

Scanning the energy of $\Delta$ while keeping $Q$ fixed:

- Choose $Q$ along the third axis, use asymmetric boxes $L \times L \times L'$
- ... or, use (partial) twisting in the nucleon
Projecting out the form factors

\[ G_2 : \quad \Delta_{3/2} = \frac{1}{2} (1 + \Sigma_3) \frac{1}{2} (1 + \gamma_4) \frac{1}{\sqrt{2}} (\Delta^1 - i\Sigma_3 \Delta^2) \]

\[ G_1 : \quad \Delta_{1/2} = \frac{1}{2} (1 - \Sigma_3) \frac{1}{2} (1 + \gamma_4) \frac{1}{\sqrt{2}} (\Delta^1 + i\Sigma_3 \Delta^2) \]

\[ G_1 : \quad \tilde{\Delta}_{1/2} = \frac{1}{2} (1 + \Sigma_3) \frac{1}{2} (1 + \gamma_4) \Delta^3 \]

\[ N_{\pm 1/2} = \frac{1}{2} (1 \pm \Sigma_3) \frac{1}{2} (1 + \gamma_4) N \]

\[ J^\pm = \frac{1}{2} (J^1 \pm iJ^2) \]

\[ \langle \tilde{\Delta}(1/2) | J^3(0) | N(1/2) \rangle \rightarrow A \frac{E_R - Q^0}{E_R} G_C(t) \]

\[ \langle \Delta(1/2) | J^+(0) | N(-1/2) \rangle \rightarrow A \frac{1}{\sqrt{2}} (G_M(t) - 3G_E(t)) \]

\[ \langle \Delta(3/2) | J^+(0) | N(1/2) \rangle \rightarrow A \sqrt{\frac{3}{2}} (G_M(t) + G_E(t)) \]
EFT: from a finite to the infinite volume

Strong Lagrangian:

\[ \mathcal{L}_{NR} = N^\dagger 2w_N(i\partial_0 - w_N)N + \pi^\dagger 2w_\pi(i\partial_0 - w_\pi)\pi \]

\[ + C_0 N^\dagger N \pi^\dagger \pi + X_i (\mathcal{O}_i^\dagger N\pi + \text{h.c.}) + \text{terms with derivatives} \]

\[ w_N = \sqrt{m_N^2 - \Delta}, \quad w_\pi = \sqrt{M_\pi^2 - \Delta} \]

- LECs \( C_0, \ldots \) are of unnatural size due to the presence of the \( \Delta \)
- Electromagnetic interactions: \( \partial_\mu \rightarrow \partial_\mu - ieA_\mu \)

→ Calculate matrix element in EFT in a finite and in the infinite volume

→ Establish the relation between these two quantities
Two-point function

\[
\langle 0| \mathcal{O}_i(x_0) \mathcal{O}_i(y_0)|0 \rangle = \sum_n \frac{e^{-E_n(x_0-y_0)}}{4w_1n w_2n} \langle 0| \mathcal{O}_i(0)|n \rangle \langle n| \mathcal{O}_i(0)|0 \rangle
\]

\[
\langle 0| \mathcal{O}_i(x_0) \mathcal{O}_i(y_0)|0 \rangle = \text{sum of bubble diagrams}
\]

Using Lüscher equation:

\[
\text{bubble} = \frac{1}{V} \sum_k \frac{1}{4w_1(k) w_2(k)} \frac{1}{w_1(k) + w_2(k) - E_n} = \frac{p_n \cot \delta(p_n)}{8\pi E_n}
\]

\[
\langle 0| \mathcal{O}_i(0)|n \rangle = U_i X_i V^{1/2} \left( \frac{\cos^2 \delta(p_n)}{\delta'(p_n) + L/2\pi \phi'(q_n)} \frac{p_n^2}{2\pi} \right)^{1/2}
\]
Three-point function

\[ \mathcal{O}_i \rightarrow F_i \quad \Rightarrow \quad \mathcal{O}_i \rightarrow \mathcal{O}_i \quad X_i \rightarrow \bar{F}_i \]

\[ |F_i(p_n, |Q|)| = V^{-1/2} \left( \frac{\cos^2 \delta(p_n)}{|\delta'(p_n) + L/2\pi \phi'(q_n)|} \frac{p_n^2}{2\pi} \right)^{1/2} |\bar{F}_i(p_n, |Q|)| \]

- The irreducible amplitude \( \bar{F}_i(p_n, |Q|) \) contains only exponentially suppressed contributions in a finite volume

The nucleon is stable!

A. Rusetsky, LATTICE 2014, New York, 27 June 2014 – p.10
Extraction of the photoproduction amplitude

Watson’s theorem, infinite volume:

\[ A_i(p, |Q|) = e^{i\delta(p)} \cos \delta(p) \tilde{F}_i(p, |Q|) \]

→ Lüscher-Lellouch formula for the photoproduction amplitude:

\[ A_i(p_n, |Q|) = e^{i\delta(p_n)} V^{1/2} \left( \frac{1}{|\delta'(p_n) + L/2\pi \phi'(q_n)|} \frac{p_n^2}{2\pi} \right)^{-1/2} |F_i(p_n, |Q|)| \]

Form factor, extracted from the photoproduction amplitude:

\[ |\text{Im} A_i(p_A, |Q|)| = \sqrt{\frac{8\pi}{p_{AT}}} |F_i^A(p_A, |Q|)|, \quad \delta(p_A) = 90^\circ \]
Continuation to the resonance pole

Effective-range expansion:

\[ h(p) \equiv p^3 \cot \delta(p) = -\frac{1}{a} + \frac{1}{2} r p^2 + \cdots \]

\[ \rightarrow \text{Position of the pole on the second Riemann sheet} \]

\[ -\frac{1}{a} + \frac{1}{2} r p_R^2 + \cdots = -i p_R^3 \]

Form factor, extracted at the pole:

\[ F_i^R(p_R, |Q|) = Z_R^{1/2} \bar{F}_i(p_R, |Q|) \]

\[ Z_R = \left( \frac{p_R}{8\pi E_R} \right)^2 \left( \frac{16\pi p_R^3 E_R^3}{w_1 R w_2 R (2p_R h'(p_R^2) + 3ip_R^3)} \right) \]
Analytic continuation to the pole

- Extract irreducible amplitude $\bar{F}_i(p, |Q|)$ from the measured amplitude $F_i(p, |Q|)$, multiplying by Lüscher-Lellouch factor.
- The effective-range expansion reads:

$$p^3 \cot \delta(p) |\bar{F}_i(p, |Q|)| = A_i(|Q|) + p^2 B_i(|Q|) + \cdots$$

Fit the real coefficients $A_i(|Q|), B_i(|Q|), \ldots$ from the data.
- Find the form factor at the pole by substituting $p \to p_R$ in the effective range expansion:

$$F^R_i(p_R, |Q|) = ip_R^{-3} Z_R^{1/2} (A_i(|Q|) + p_R^2 B_i(|Q|) + \cdots)$$

$F^R_i(p_R, |Q|)$ and $F^A_i(p_A, |Q|)$ coinside in narrow width approximation!
Summary: prescription for the extraction of the form factor

Measure matrix elements $F_i(p, |Q|)$ on the lattice: varying $p$, fixed $Q$

Photoproduction multipoles are given by Lüscher-Lellouch formula:

$$A_i(p_n, |Q|) = e^{i\delta(p_n)} V^{1/2} \left( \frac{1}{\delta'(p_n) + L/2\pi \phi'(q_n)} \frac{p_n^2}{2\pi} \right)^{-1/2} |F_i(p_n, |Q|)|$$

Form factor, extracted at real energies:

$$|\text{Im} A_i(p_A, |Q|)| = \sqrt{\frac{8\pi}{p_A \Gamma}} |F_i^A(p_A, |Q|)|, \quad \delta(p_A) = 90^\circ$$

Analytic continuation through the fit to the effective-range formula:

The quantity $p^3 \cot \delta(p) |\bar{F}_i(p, |Q|)|$ is a low-energy polynomial in $p^2$.

Fit the coefficients of this polynomial for real $p^2$ from the data, then continue to the pole by substituting $p \rightarrow p_R$
Conclusions, outlook

- Using effective field theory in a finite volume, we define the procedure of extraction of the $\Delta N \gamma^*$ form factor on the lattice in case of unstable $\Delta$
- Analog of the Lüscher-Lellouch formula for the photoproduction amplitude is derived $\Rightarrow$ form factor at $\delta(p_A) = 90^\circ$
  

- The value of the form factor at the resonance pole is determined by using analytic continuation through the fit to the effective range expansion.

- For the infinitely narrow resonance, both definitions of the form factor coincide