$\Delta N \gamma^*$ transition form factor on the lattice

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arXiv:1405.3476

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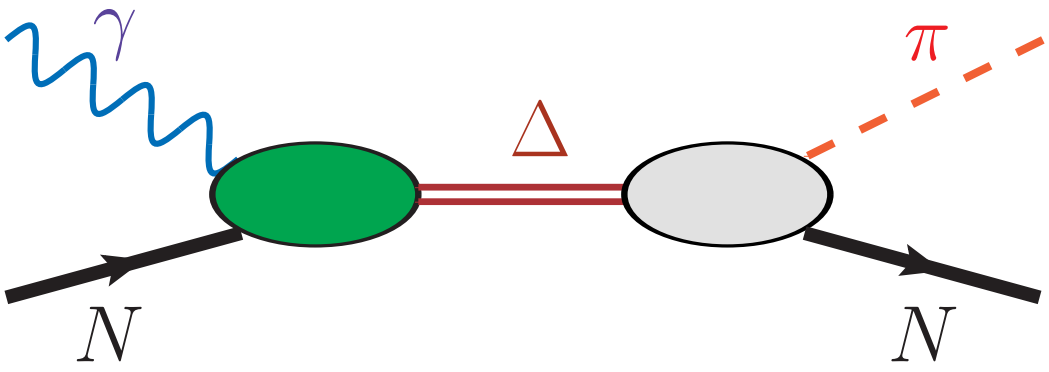
Plan

- Introduction: Resonance matrix elements in the continuum
- Kinematics, projecting out the formfactors
- Lüscher-Lellouch relation for the pion photoproduction amplitude
- Analytic continuation to the resonance pole
- Conclusions, outlook

Resonance matrix elements in the continuum



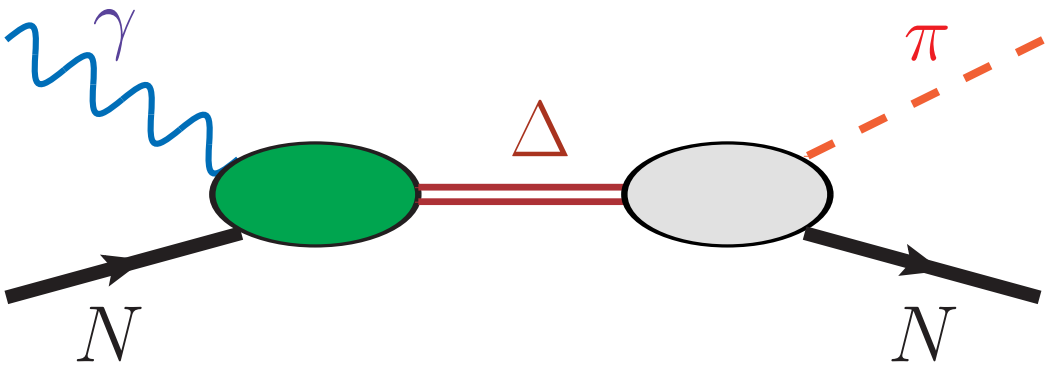
In the vicinity of the resonance pole. . .

$$\Longrightarrow \quad i \int d^4x\, e^{iPx}\, \langle 0|T\Delta(x)\bar{\Delta}(0)|0\rangle \to \frac{Z_R}{s_R - P^2} + \cdots$$

$$\Longrightarrow \quad i^2 \int d^4x d^4y\, e^{iPx - iQy}\, \langle 0|T\Delta(x)J(0)\bar{N}(y)|0\rangle$$

$$\to \frac{Z_R^{1/2}}{s_R - P^2}\, \langle \Delta|J(0)|N\rangle\, \frac{Z_N^{1/2}}{m_N^2 - Q^2} + \cdots$$

Photoproduction amplitude



In the narrow width approximation...

$$|\mathrm{Im}\,\mathcal{A}(\gamma^* N \to \pi N)| = \sqrt{\frac{8\pi}{p_A \Gamma}}\,|\langle \Delta | J(0) | N \rangle|\,, \qquad \delta(p_A) = 90^{\circ}$$

- The form factor defined on the real axis is process dependent due to the background contribution
- Analytic continuation to the resonance pole is needed!

I.G. Aznaurian *et al.,* arXiv:0810.0997; D. Drechsel *et al.,* NPA 645 (1999) 145;
R.L. Workman *et al.,* PRC 87 (2013) 068201

Measuring the form factor on the lattice: problems

Assuming Δ to be a stable particle...

C. Alexandrou *et al.,* PRD 79 (2009) 14507; arXiv:1108.4112; PRD 83 (2011) 014501

How the formalism is generalized in case of an unstable Δ?

- Which quantities should be measured on the lattice?
- How does one perform the infinite-volume limit in the form factors?
- How does one calculate the photoproduction amplitude?
- How does one perform the analytic continuation to the resonance pole?
- How does one project out different form factors in case of the unstable Δ?

$\hookrightarrow$ Use EFT approach in a finite volume

Kinematics

$$\Delta(t) = \sum_{\mathbf{x}} \Delta(t, \mathbf{x}) \quad \text{(CM frame)} \ , \qquad N(t) = \sum_{\mathbf{x}} e^{-i\mathbf{Q}\mathbf{x}} N(t, \mathbf{x})$$

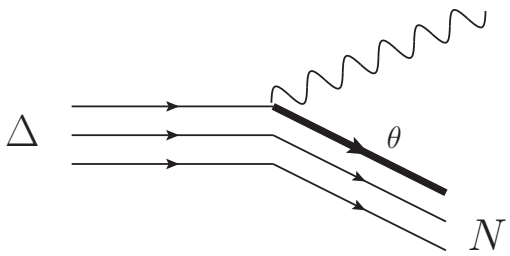
Measuring three-point functions:

$$R(t', t) = \langle 0 | \Delta(t') J(0) \bar{N}(0) | 0 \rangle \, , \qquad S_\Delta(t) \, , \ S_N(t) \ : \quad \text{propagators}$$

$$F = \lim_{t' \to \infty, t \to -\infty} \mathcal{N} \, \frac{R(t', t)}{S_\Delta(t' - t)} \left(\frac{S_N(t') S_\Delta(-t) S_\Delta(t' - t)}{S_\Delta(t') S_N(-t) S_N(t' - t)} \right)^{1/2}$$

Scanning the energy of Δ while keeping $\mathbf{Q}$ fixed:

- Choose $\mathbf{Q}$ along the third axis, use asymmetric boxes $L \times L \times L'$
- . . . or, use (partial) twisting in the nucleon



Projecting out the form factors

$$
\begin{aligned}
G_2: \quad \Delta_{3/2} &= \frac{1}{2}(1+\Sigma_3)\frac{1}{2}(1+\gamma_4)\frac{1}{\sqrt{2}}(\Delta^1 - i\Sigma_3\Delta^2) \\
G_1: \quad \Delta_{1/2} &= \frac{1}{2}(1-\Sigma_3)\frac{1}{2}(1+\gamma_4)\frac{1}{\sqrt{2}}(\Delta^1 + i\Sigma_3\Delta^2) \\
G_1: \quad \tilde{\Delta}_{1/2} &= \frac{1}{2}(1+\Sigma_3)\frac{1}{2}(1+\gamma_4)\Delta^3 \\
N_{\pm 1/2} &= \frac{1}{2}(1\pm\Sigma_3)\frac{1}{2}(1+\gamma_4)N \\
J^{\pm} &= \frac{1}{2}(J^1 \pm iJ^2)
\end{aligned}
$$

$$
\begin{aligned}
\langle\tilde{\Delta}(1/2)|J^3(0)|N(1/2)\rangle &\rightarrow A\frac{E_R - Q^0}{E_R}G_C(t) \\
\langle\Delta(1/2)|J^+(0)|N(-1/2)\rangle &\rightarrow A\frac{1}{\sqrt{2}}(G_M(t) - 3G_E(t)) \\
\langle\Delta(3/2)|J^+(0)|N(1/2)\rangle &\rightarrow A\sqrt{\frac{3}{2}}(G_M(t) + G_E(t))
\end{aligned}
$$

EFT: from a finite to the infinite volume

Strong Lagrangian:

$$\mathcal{L}_{NR} = N^\dagger\, 2w_N(i\partial_0 - w_N)N + \pi^\dagger\, 2w_\pi(i\partial_0 - w_\pi)\pi$$

$$+ \; C_0\, N^\dagger N \pi^\dagger \pi + X_i(\mathcal{O}_i^\dagger N\pi + \text{h.c.}) + \text{terms with derivatives}$$

$$w_N = \sqrt{m_N^2 - \triangle}\,, \quad w_\pi = \sqrt{M_\pi^2 - \triangle}$$

- LECs $C_0, \ldots$ are of unnatural size due to the presence of the Δ
- Electromagnetic interactions: $\partial_\mu \to \partial_\mu - ieA_\mu$

$\hookrightarrow$ Calculate matrix element in EFT in a finite and in the infinite volume

$\hookrightarrow$ Establish the relation between these two quantities

Two-point function

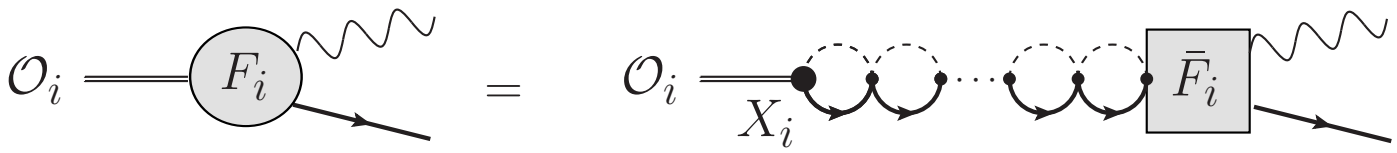
- $$\langle 0|\mathcal{O}_i(x_0)\bar{\mathcal{O}}_i(y_0)|0\rangle = \sum_n \frac{e^{-E_n(x_0-y_0)}}{4w_{1n}w_{2n}} \langle 0|\mathcal{O}_i(0)|n\rangle\langle n|\bar{\mathcal{O}}_i(0)|0\rangle$$
- $\langle 0|\mathcal{O}_i(x_0)\bar{\mathcal{O}}_i(y_0)|0\rangle =$ sum of bubble diagrams

Using Lüscher equation:

$$\text{bubble} = \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{4w_1(\mathbf{k})w_2(\mathbf{k})} \frac{1}{w_1(\mathbf{k}) + w_2(\mathbf{k}) - E_n} = \frac{p_n \cot\delta(p_n)}{8\pi E_n}$$

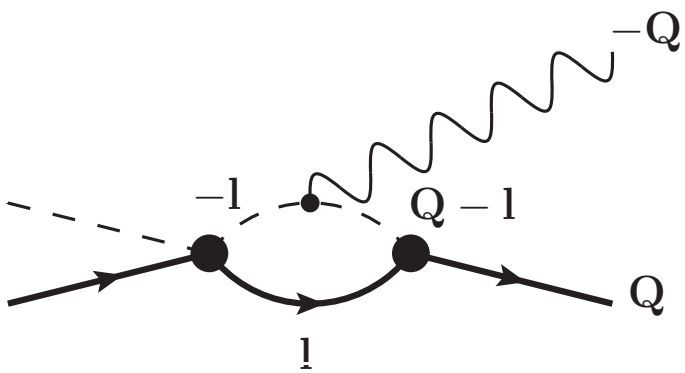
$$\hookrightarrow \quad |\langle 0|\mathcal{O}_i(0)|n\rangle| = \underbrace{U_i}_{\text{free spinor}} \; X_i V^{1/2} \left(\frac{\cos^2\delta(p_n)}{|\delta'(p_n) + L/2\pi\,\phi'(q_n)|} \frac{p_n^2}{2\pi} \right)^{1/2}$$

Three-point function



$$|F_i(p_n, |\mathbf{Q}|)| = V^{-1/2} \left(\frac{\cos^2 \delta(p_n)}{|\delta'(p_n) + L/2\pi\, \phi'(q_n)|} \frac{p_n^2}{2\pi} \right)^{1/2} |\bar{F}_i(p_n, |\mathbf{Q}|)|$$

- The irreducible amplitude $\bar{F}_i(p_n, |\mathbf{Q}|)$ contains only exponentially suppressed contributions in a finite volume



The nucleon is stable!

Extraction of the photoproduction amplitude

Watson's theorem, infinite volume:

$$\mathcal{A}_i(p, |\mathbf{Q}|) = e^{i\delta(p)} \cos\delta(p) \bar{F}_i(p, |\mathbf{Q}|)$$

$\hookrightarrow$ Lüscher-Lellouch formula for the photoproduction amplitude:

$$\mathcal{A}_i(p_n, |\mathbf{Q}|) = e^{i\delta(p_n)} V^{1/2} \left(\frac{1}{|\delta'(p_n) + L/2\pi\, \phi'(q_n)|} \frac{p_n^2}{2\pi} \right)^{-1/2} |F_i(p_n, |\mathbf{Q}|)|$$

Form factor, extracted from the photoproduction amplitude:

$$|\mathrm{Im}\, \mathcal{A}_i(p_A, |\mathbf{Q}|)| = \sqrt{\frac{8\pi}{p_A \Gamma}}\, |F_i^A(p_A, |\mathbf{Q}|)|\,, \qquad \delta(p_A) = 90^\circ$$

Continuation to the resonance pole

Effective-range expansion:

$$h(p) \doteq p^3 \cot\delta(p) = -\frac{1}{a} + \frac{1}{2}\, r\, p^2 + \cdots$$

$\hookrightarrow$ Position of the pole on the second Riemann sheet

$$-\frac{1}{a} + \frac{1}{2}\, r p_R^2 + \cdots = -i p_R^3$$

Form factor, extracted at the pole:

$$F_i^R(p_R, |\mathbf{Q}|) = Z_R^{1/2}\, \bar{F}_i(p_R, |\mathbf{Q}|)$$

$$Z_R = \left(\frac{p_R}{8\pi E_R}\right)^2 \left(\frac{16\pi p_R^3 E_R^3}{w_{1R} w_{2R}(2 p_R h'(p_R^2) + 3 i p_R^3)}\right)$$

Analytic continuation to the pole

- Extract irreducible amplitude $\bar{F}_i(p, |\mathbf{Q}|)$ from the measured amplitude $F_i(p, |\mathbf{Q}|)$, mutliplying by Lüscher-Lellouch factor
- The effective-range expansion reads:

$$p^3 \cot\delta(p)\, |\bar{F}_i(p, |\mathbf{Q}|)| = A_i(|\mathbf{Q}|) + p^2 B_i(|\mathbf{Q}|) + \cdots$$

  Fit the real coefficients $A_i(|\mathbf{Q}|), B_i(|\mathbf{Q}|), \ldots$ from the data
- Find the form factor at the pole by substituting $p \to p_R$ in the effective range expansion

$$F_i^R(p_R, |\mathbf{Q}|) = i p_R^{-3} Z_R^{1/2} (A_i(|\mathbf{Q}|) + p_R^2\, B_i(|\mathbf{Q}|) + \cdots)$$

$F_i^R(p_R, |\mathbf{Q}|)$ and $F_i^A(p_A, |\mathbf{Q}|)$ coinside in narrow width approximation!

Summary: prescription for the extraction of the form factor

Measure matrix elements $F_i(p, |\mathbf{Q}|)$ on the lattice: varying p, fixed $\mathbf{Q}$

Photoproduction multipoles are given by Lüscher-Lellouch formula:

$$\mathcal{A}_i(p_n, |\mathbf{Q}|) = e^{i\delta(p_n)} V^{1/2} \left(\frac{1}{|\delta'(p_n) + L/2\pi\, \phi'(q_n)|} \frac{p_n^2}{2\pi} \right)^{-1/2} |F_i(p_n, |\mathbf{Q}|)|$$

Form factor, extracted at real energies:

$$|\mathrm{Im}\, \mathcal{A}_i(p_A, |\mathbf{Q}|)| = \sqrt{\frac{8\pi}{p_A \Gamma}}\, |F_i^A(p_A, |\mathbf{Q}|)|\,, \qquad \delta(p_A) = 90^\circ$$

Analytic continuation through the fit to the effective-range formula:

The quantity $p^3 \cot\delta(p)\, |\bar{F}_i(p, |\mathbf{Q}|)|$ is a low-energy polynomial in p^2.

Fit the coefficients of this polynomial for real p^2 from the data, then continue to the pole by substituting $p \to p_R$

Conclusions, outlook

- Using effective field theory in a finite volume, we define the procedure of extraction of the $\Delta N \gamma^*$ form factor on the lattice in case of unstable Δ
- Analog of the Lüscher-Lellouch formula for the photoproduction amplitude is derived $\Rightarrow$ form factor at $\delta(p_A) = 90^{\circ}$

  Related work: R. Briceno, M. Hansen and A. Walker-loud, arXiv:1406.5965
- The value of the form factor at the resonance pole is determined by using analytic continuation trough the fit to the effective range expansion.
- For the infinitely narrow resonance, both definitions of the form factor coinside