On Curing the Divergence in Quark Number Susceptibility*

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Introduction

♠ Introducing chemical potential by adding μN (point-split) amounts to weights $f(a\mu) = 1 + a\mu$ & $g(a\mu) = 1 - a\mu$ to forward and backward time links respectively.

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♡ Important to note that analytical proof was for *free* quarks in these cases; Numerical computations showed it to work for the interacting case (Gavai-Gupta PRD 67, 034501 (2003).)

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♡ Using the definition of the chiral projectors for overlap fermions, we (Gavai-Sharma, PLB 2012) proposed a chirally invariant Overlap action for nonzero μ :

$$\begin{aligned} S^F &= \sum_n [\bar\psi_{n,L}(aD_{ov} + a\mu\gamma^4)\psi_{n,L} + \bar\psi_{n,R}(aD_{ov} + a\mu\gamma^4)\psi_{n,R}] \\ &= \sum_n \bar\psi_n[aD_{ov} + a\mu\gamma^4(1 - aD_{ov}/2)]\psi_n \ . \end{aligned}$$

- Easy to check that under the chiral transformations, $\delta\psi = i\alpha\gamma_5(1 - aD_{ov})\psi$ and $\delta\bar{\psi} = i\alpha\bar{\psi}\gamma_5$, it is invariant or all values of $a\mu$ and a.

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- Order parameter exists for all μ and T. It is $\langle\bar{\psi}\psi\rangle = \lim_{am\to 0}\lim_{V\to\infty}\langle \mathrm{Tr}\,\frac{(1-aD_{ov}/2)}{[aD_{ov}+(am+a\mu\gamma^4)(1-aD_{ov}/2)]}\rangle$.

- It, however, has a^{-2} divergences which cannot be removed by exponentiation of the μ-term (Narayanan-Sharma, JHEP 2011).

- The Overlap fermion dilemma : Either exact chiral invariance on lattice or divergences in $a \to 0$ limit.

Tackling the Divergences

- Opt for exact chiral invariance & learn to tackle the divergences.
- Note that contrary to common belief, divergences are **NOT** a lattice artifact. Indeed lattice regulator simply makes it easy to spot them. Using a Pauli-Villars cut-off Λ in the continuum theory, one can show the presence of $\mu\Lambda^2$ terms in number density easily (Gavai-Sharma, 1406.0474).

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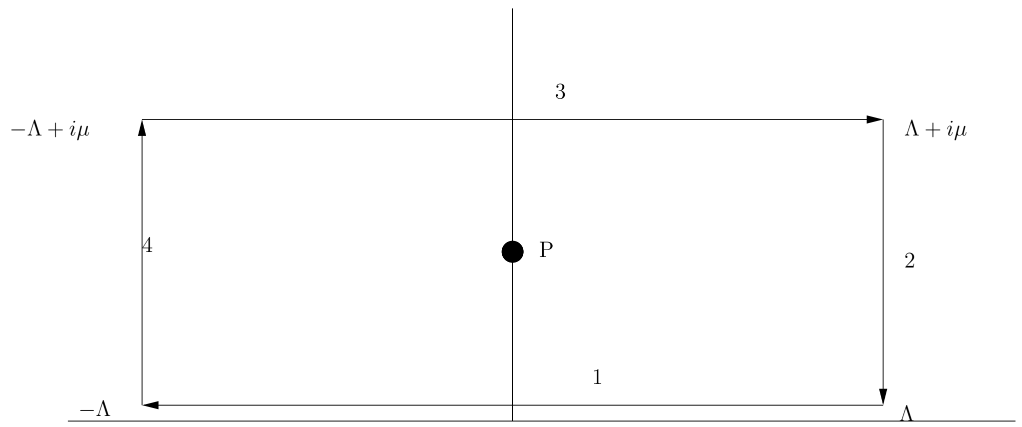
- The expression for the number density is

$$n = \frac{2iT}{V} \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{(\omega_n - i\mu)}{p^2 + (\omega_n - i\mu)^2} \equiv \frac{2iT}{V} \sum_n \int \frac{d^3p}{(2\pi)^3} \sum_{\omega_n} F(\omega_n, \mu, \vec{p}), \tag{1}$$

where $p^2 = p_1^2 + p_2^2 + p_3^2$. Here we take the gamma matrices as all Hermitian.

- In the usual contour method, but with a cut-off Λ, one has in the $T \to 0$ limit but $\mu \neq 0$ the following :



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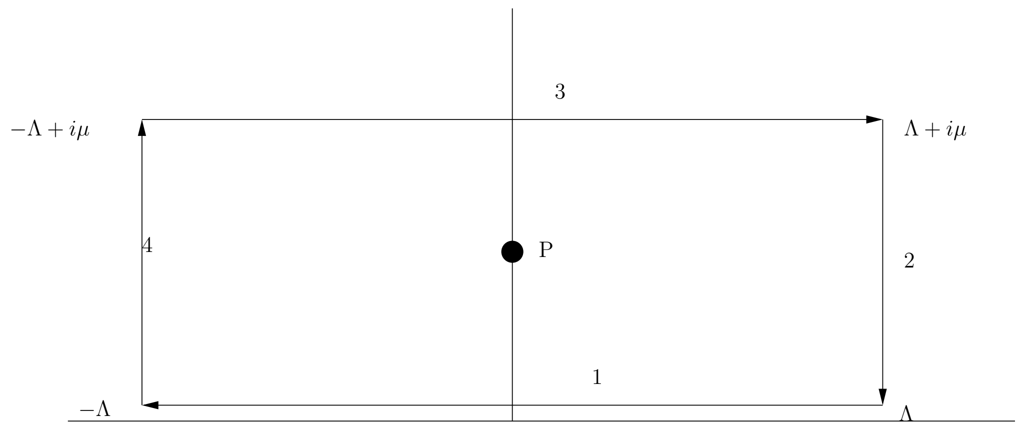

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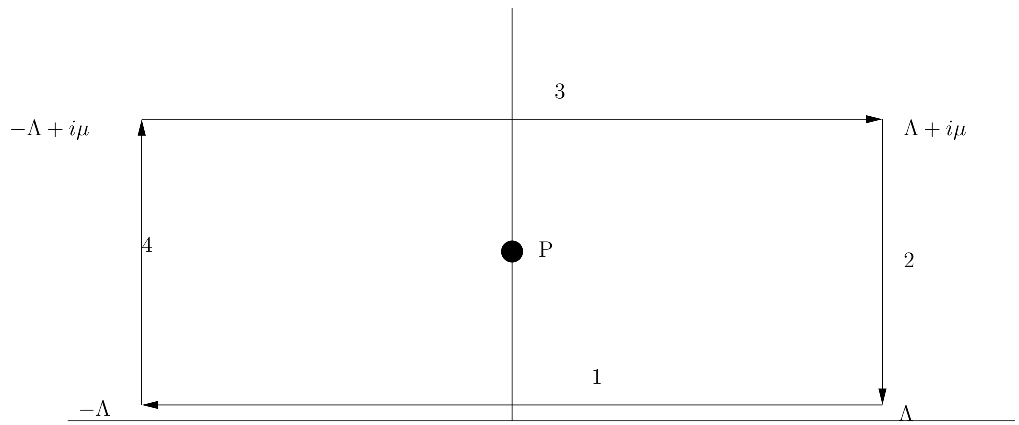

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- One may thus follow the prescription of subtracting the free theory divergence by hand. If it works, one can have several computational advantages in computing the higher order susceptibilities needed in critical point search.

- Indeed, for any fermion it leads to
  $M' = \sum_{x,y} N(x,y)$, and $M'' = M''' = M''''... = 0$,
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- Lot fewer terms in the Taylor coefficients, especially as the order increases. E.g., in the 4th order susceptibility, $\mathcal{O}_4 = -6 \text{ Tr } (M^{-1}M')^4$ in the linear case, compared to $\mathcal{O}_4 = -6 \text{ Tr } (M^{-1}M')^4 + 12 \text{ Tr } (M^{-1}M')^2 M^{-1}M'' - 3 \text{ Tr } (M^{-1}M'')^2 - 3 \text{ Tr } M^{-1}M'M^{-1}M''' + \text{Tr } M^{-1}M''''$.

- $\mathcal{O}_8$ has one term in contrast to 18 in the usual case. $\Longrightarrow$ Less Cancellations & Number of M^{-1} computations needed are lesser too.

Testing the idea

- On our $N_t = 6$ configurations (Gavai-Gupta PRD 2008), where we computed and published all the coefficients, the proposal of linear μ with simple subtraction was tested (Gavai-Sharma PRD 2012).

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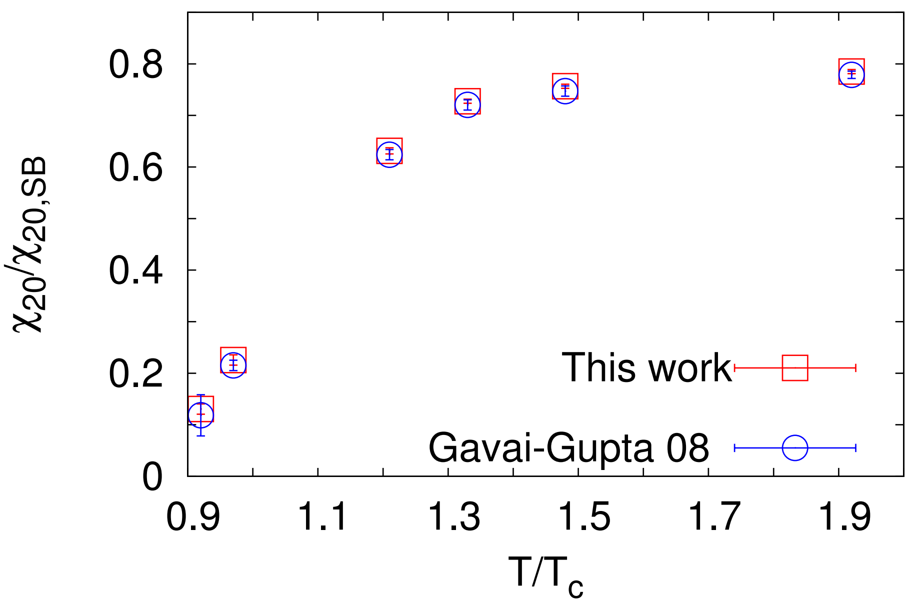
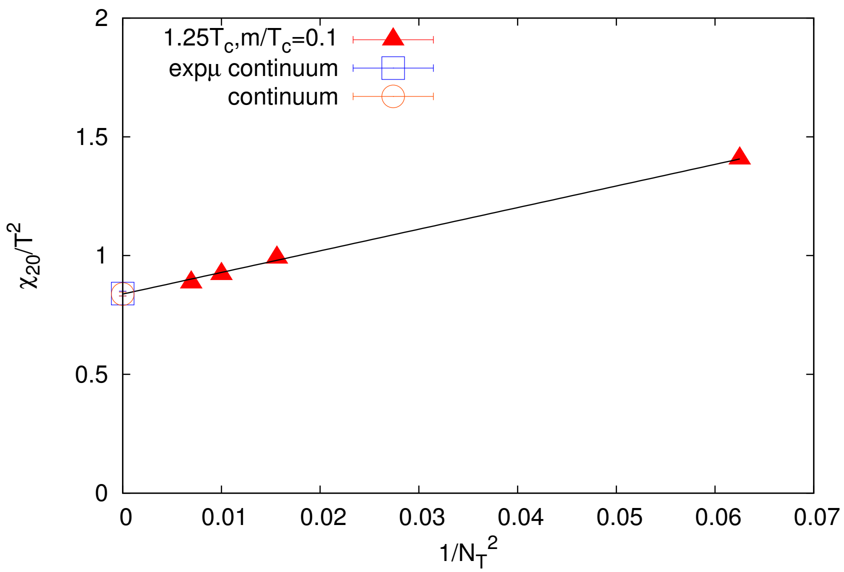
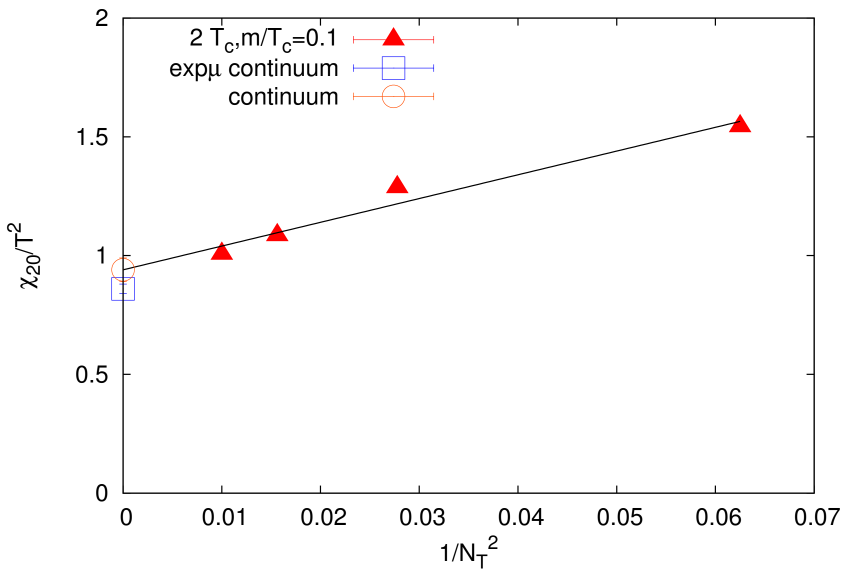


- Since the corresponding free fermion results approach the continuum limit differently, the $N_t = 6$ free results were divided out above.

- In order to test whether the divergence is truely absent, one needs to take the continuum limit $a \to 0$ or equivalently $N_t \to \infty$.

- We tested it for quenched QCD. For $m/T_c = 0.1$, we employed $N_t = 4$, 6, 8, 10 and 12 lattices and 50-100 independent configurations. At $T/T_c = 1.25$, 2 we obtained





- Absence of any divergent term is evident in the positive slope of the data.

- Moreover, our extrapolated continuum result coincides with the earlier result obtained with the $\exp(\pm a\mu)$ action (Swagato Mukherjee PRD 2006).

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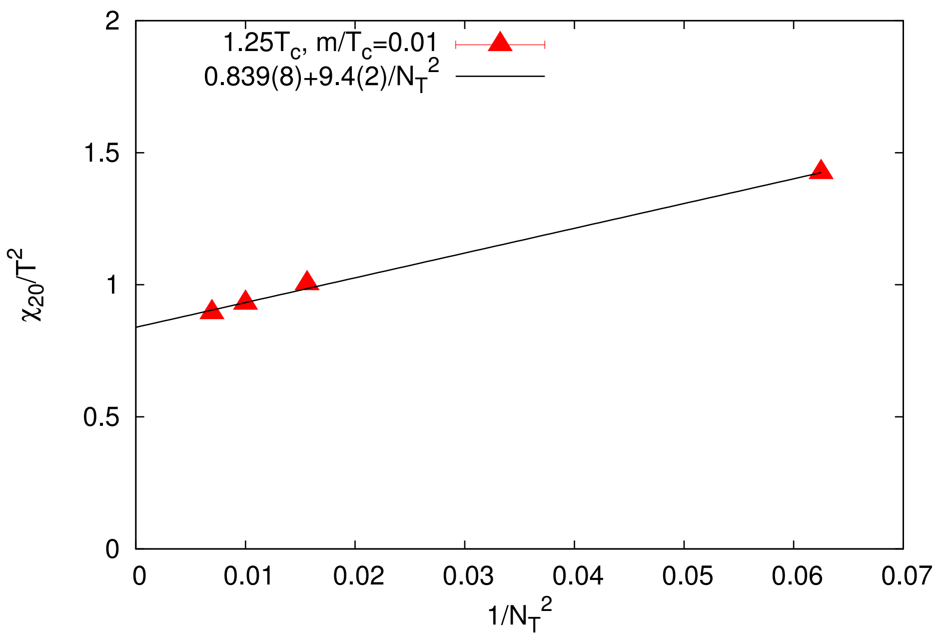
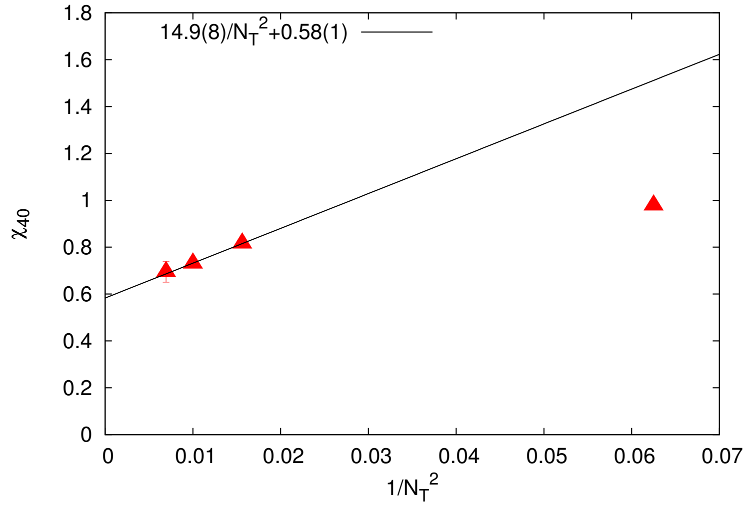
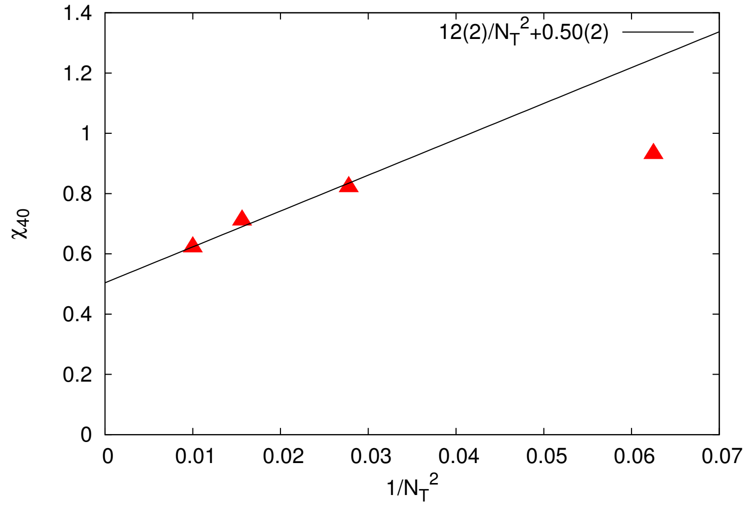
- We lowered the mass by a factor on 10 to $m/T_c = 0.01$ & repeated the exercise at a lower temperature on $T/T_c = 1.25$.



- Again no divergent term is evidently present in the slope of the data.

- Higher order susceptibility show similar finite result in continuum limit:





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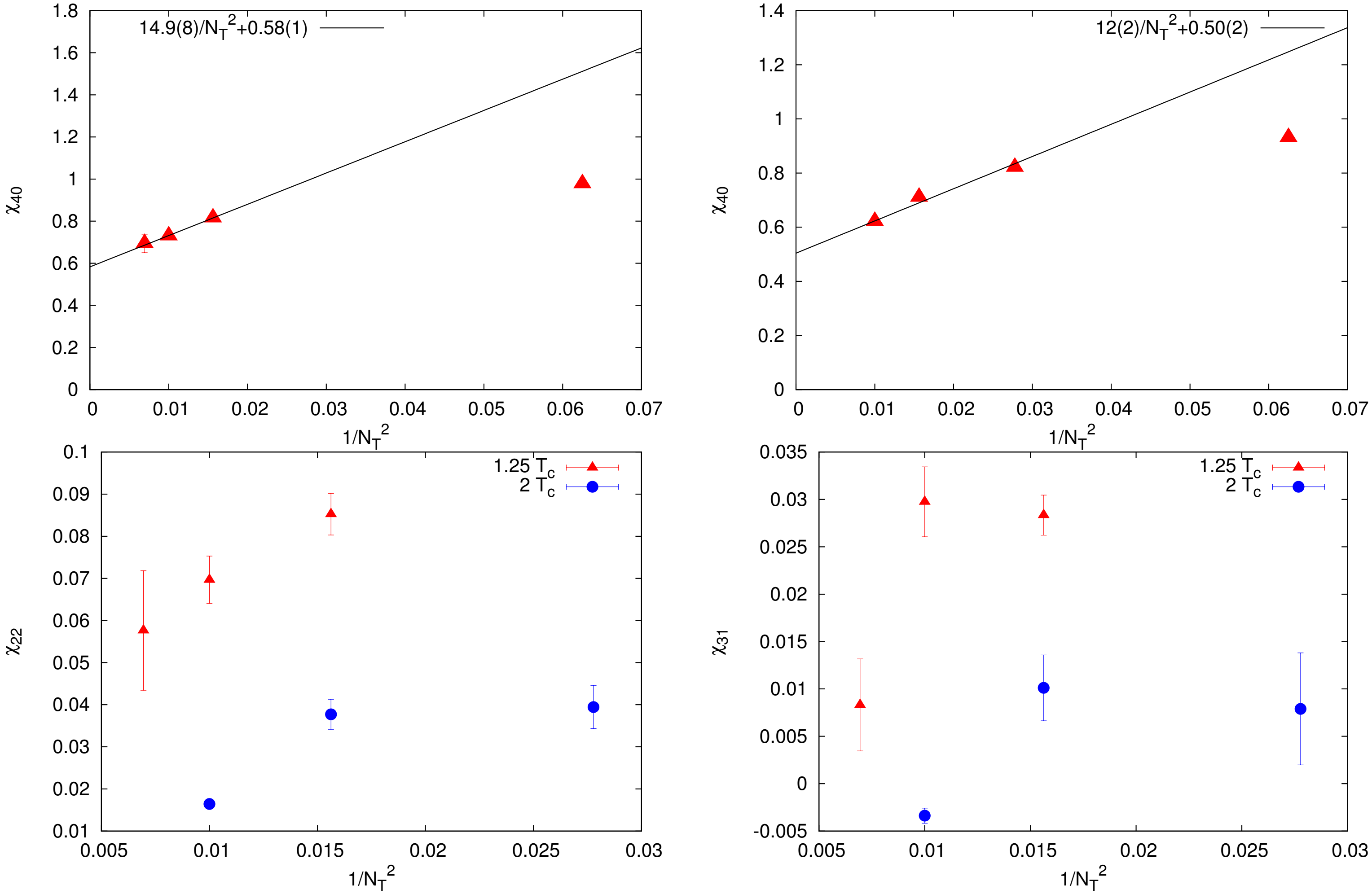

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- Divergence in the quark number susceptibility can be subtracted off by the corresponding free theory result. Continuum extrapolation yields the same result for both the linear and the exponential form, as it must.

- Interactions do not induce any additional divergence at finite T or μ once the zero temperature divergence is removed. This has been well known perturbatively but seems to hold non-perturbatively as well.

- Conserved charge N should not get renormalized.