Understanding localisation in QCD through an Ising-Anderson model

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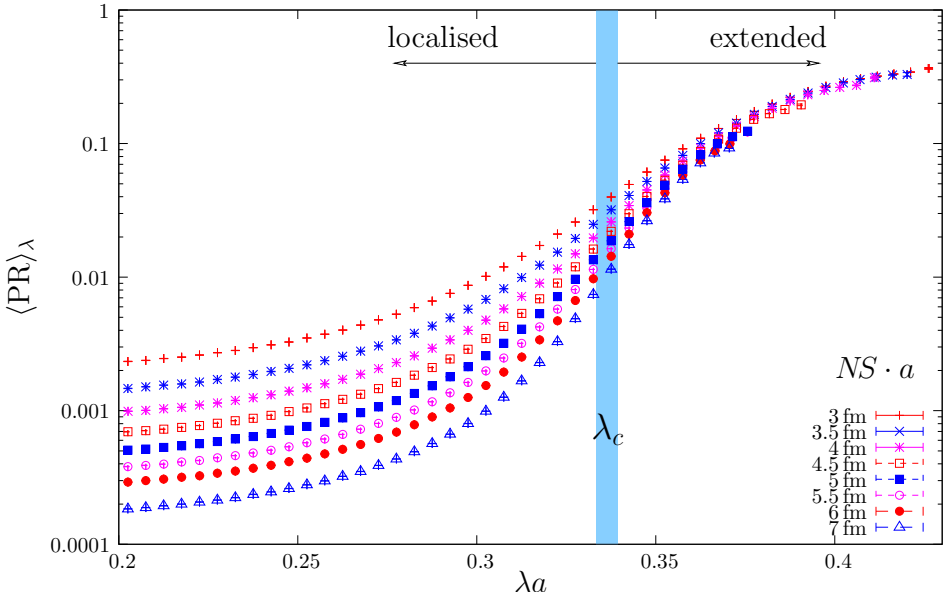
Localisation in the Dirac Spectrum

Low-lying modes of the Dirac operator are localised above T_c

[García-García, Osborn (2007), Kovács (2010), Kovács, Pittler (2010), Kovács, Pittler (2012)]



$$\mathrm{IPR} = \sum_x |\psi(x)|^4$$

$$\mathrm{PR} = \mathrm{IPR}^{-1}/V_4$$

$$T \simeq 2.6\, T_c$$

- Eigenmodes localised/delocalised for $\lambda < \lambda_c(T)$ / $\lambda > \lambda_c(T)$
- Anderson-type phase transition in the spectrum at λ_c, same universality class of the 3D unitary Anderson model

  [MG, Kovács, Pittler (2014)]

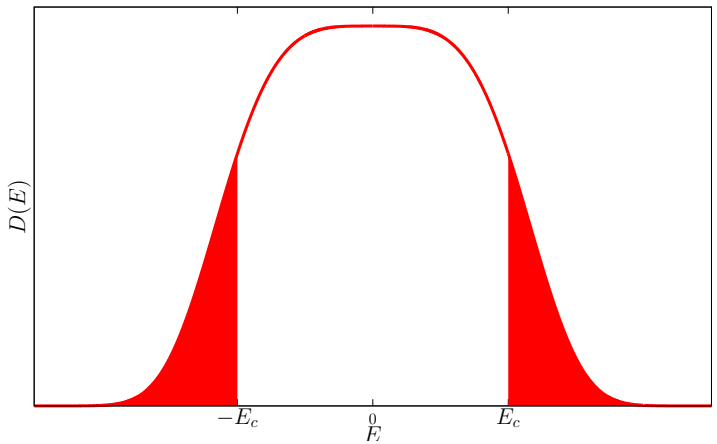
Anderson Transition in 3D

Tight-binding Hamiltonian for "dirty" conductors [Anderson (1958)]

$$H_{\vec{x}\vec{y}} = \varepsilon_{\vec{x}}\delta_{\vec{x}\vec{y}} + \sum_{\mu}(\delta_{\vec{x}+\hat{\mu}\,\vec{y}} + \delta_{\vec{x}-\hat{\mu}\,\vec{y}})$$

Random on-site potential $\varepsilon_{\vec{x}} \in \left[-\frac{W}{2}, \frac{W}{2}\right]$ ($W \sim$ amount of disorder) plus hopping

- eigenstates localised for $E > E_c(W)$ (mobility edge)
- second-order phase transition with divergent $\xi \sim |E - E_c|^{-\nu}$



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In a magnetic field $\rightarrow$ random phases, unitary class

- eigenstates localised for $E > E_c(W)$ (mobility edge)
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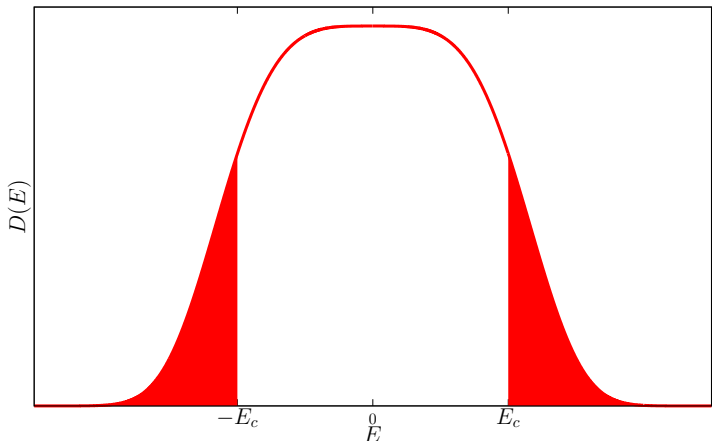
$$\nu_{\mathrm{UAM}} = 1.43(4)$$

[Slevin, Ohtsuki (1999)]



Anderson Model vs. High-Temperature QCD

	Anderson model	QCD at $T > T_c$
dimensionality	3D	4D
mobility edge	energy $E_c(W)$	Dirac eigenvalue $\lambda_c(T)$
amount of disorder	width W	temperature T
type of disorder	diagonal (sites) uncorrelated	off-diagonal (links) correlated (short range)
critical exponent	ν_{UAM}=1.43(4) [Slevin, Ohtsuki (1999)]	$\nu_{\mathrm{QCD}} = 1.43(6)$ [MG, Kovács, Pittler (2014)]

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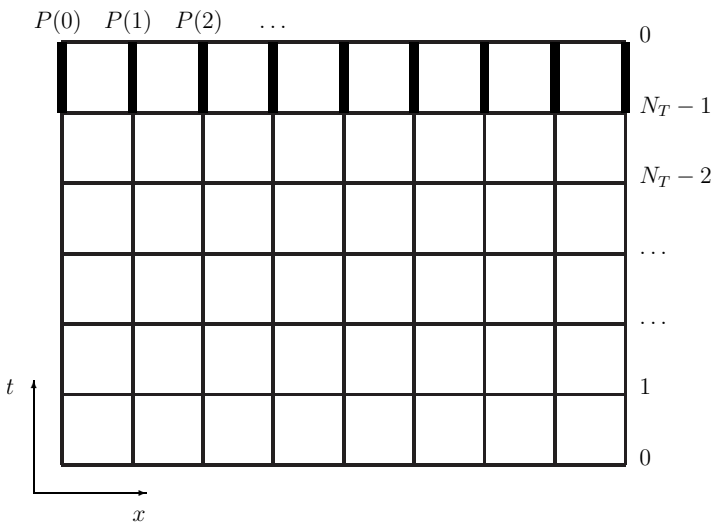
Dimensional Reduction

QCD above T_c is effectively a 3D model: time slices strongly correlated, quark eigenfunctions look qualitatively the same at all t

Temporal gauge: $U_4 = 1$ except at the temporal boundary $\Rightarrow$ wave functions obey effective boundary conditions involving the Polyakov loop

$$\psi(N_T, \vec{x}) = -P(\vec{x})\psi(0, \vec{x})$$



Correlated time slices $\Rightarrow$ effective, x-dependent boundary conditions will affect the behaviour at x for all t

$P(\vec{x})$ fluctuates in space, providing effective 3D diagonal disorder

Localisation and Polyakov Loops

Simplified setting: $SU(2)$, constant U_4, $U_j = 1$
Temporal diagonal gauge, $P = \mathrm{diag}(e^{i\varphi}, e^{-i\varphi})$

$$\not{D}\psi = i\lambda\psi \qquad \psi(t,\vec{x}) \propto e^{i\omega t + i\vec{p}\cdot\vec{x}} \qquad \lambda^2 = \sin^2\omega + \sum_{j=1}^{3}\sin^2 p_j$$

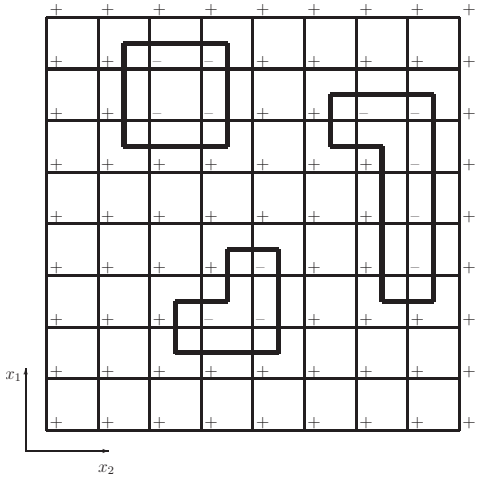
$\frac{L}{2\pi}p_j = 0, 1, \ldots, L-1$ to fulfill spatial bc

$$\omega(\varphi) = \frac{1}{N_T}(\pi \pm \varphi \mod 2\pi) = aT(\pi \pm \varphi \mod 2\pi)$$

to fulfill temporal bc (effective Matsubara frequencies)

Above T_c $P(\vec{x})$ gets ordered along 1 with "islands" of "wrong" $P(\vec{x}) \neq 1$

- $\omega(0) = \pi a T$ provides an effective gap in the spectrum
- "wrong" $P(\vec{x})$ allows for smaller $\lambda \Rightarrow$ localising "trap" for eigenmodes



[Bruckmann, Kovács, Schierenberg (2011)]

Effective 3D Model

It should be possible to understand the qualitative features of the Dirac spectrum and eigenfunctions in QCD using a genuinely 3D model

$$i\not{D}_{xy} = i\gamma^4 (D_4)_{xy} + i\vec{\gamma}\cdot\vec{D}_{xy} \longrightarrow$$
$$H_{\vec{x}\vec{y}} = \gamma^4 \mathcal{N}_{\vec{x}} \delta_{\vec{x}\vec{y}} + i\vec{\gamma}\cdot\vec{D}_{\vec{x}\vec{y}}$$

Off-diagonal disorder less effective in producing localisation: $\vec{D} \to \vec{\partial}$, colour components decouple (changes the symmetry class)

Diagonal disorder $\mathcal{N}_{\vec{x}}$:

- not uncorrelated, should be governed by Polyakov-loop-like dynamics $\sim$ spin models in the ordered phase
- phase of $P(\vec{x})$ enters the effective boundary conditions $\Rightarrow$ continuous spin
- $\mathcal{N}_{\vec{x}}$ should produce an effective gap in the spectrum

Effective 3D Model

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Ising-Anderson Model

Enough to have a spin model which displays an ordered phase: Ising model with continuous spin

$$H_{\vec{x}\vec{y}} = \gamma^4 \Lambda \frac{1+s_{\vec{x}}}{2} \delta_{\vec{x}\vec{y}} + i\vec{\gamma} \cdot \vec{\partial}_{\vec{x}\vec{y}} \qquad s_{\vec{x}} \in [-1, 1]$$

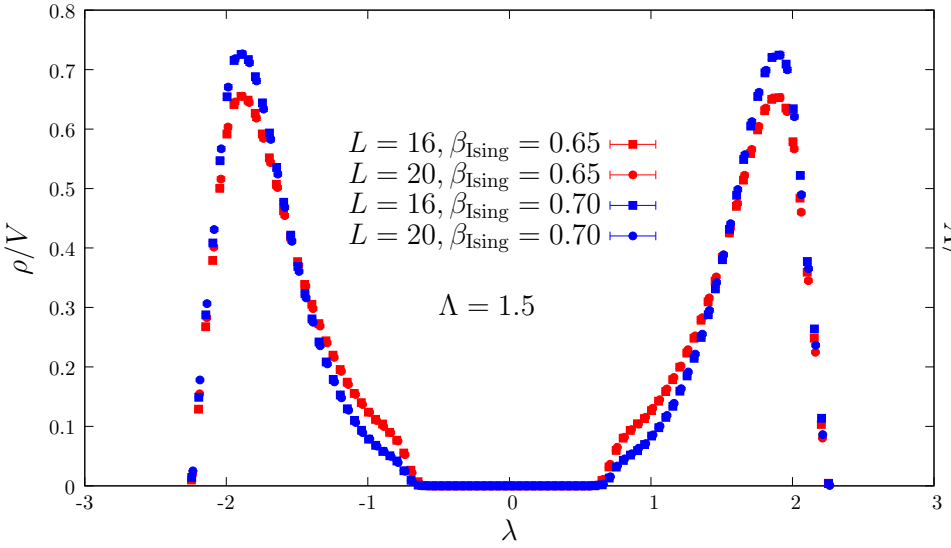
Ordered phase: "sea" of $s_{\vec{x}} = 1$ with "islands" of $s_{\vec{x}} \neq 1$ (magnetic field $= 0^+$)

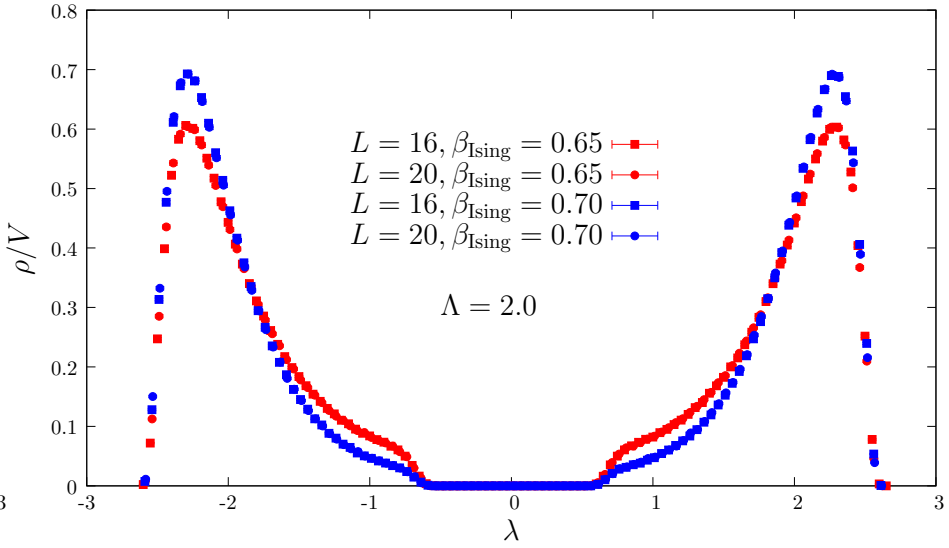
- $\frac{1+s_{\vec{x}}}{2}$: 1 for aligned spins, 0 for anti-aligned $\approx$ effective spectral gap
- Λ: spin-fermion coupling $\approx$ Matsubara frequency (size of the gap)
- ordering of the spin configuration governed by β_{Ising}

Symmetry class: orthogonal (L even) or symplectic (L odd)

Test of viability of the sea/islands explanation

Spectral Density



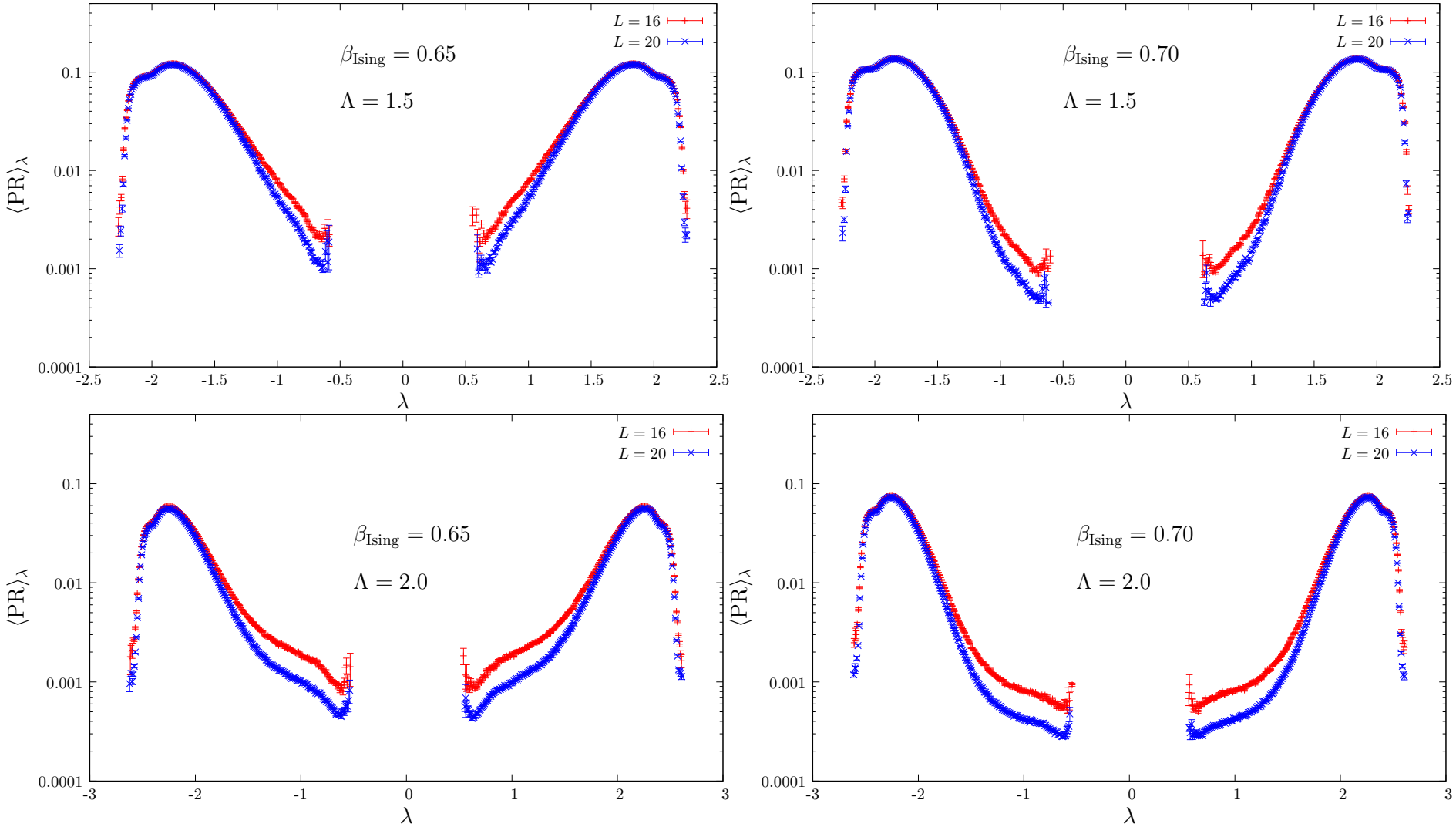


Low modes have small spectral density, which rapidly increases

Symmetry $\lambda \to -\lambda$ on average

Sharp gap in the spectrum

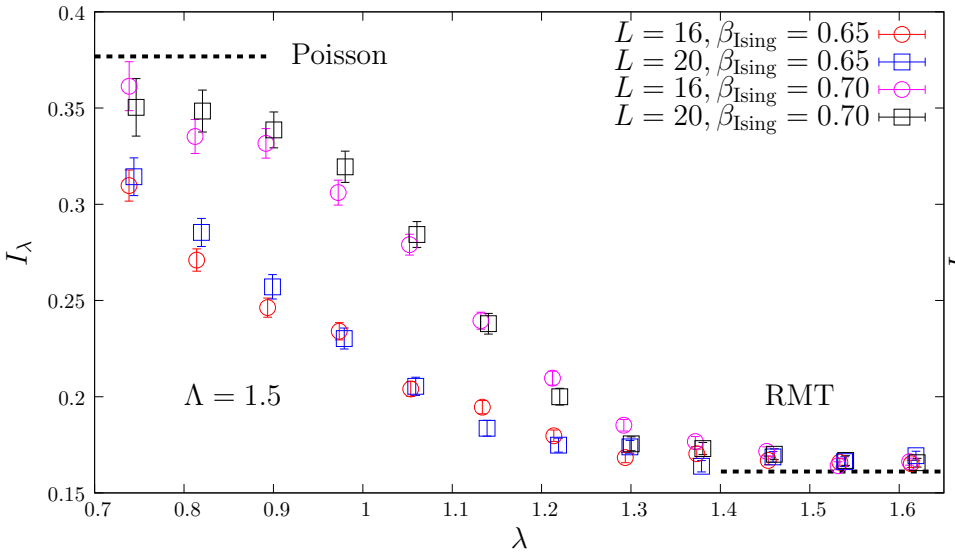
Participation Ratio

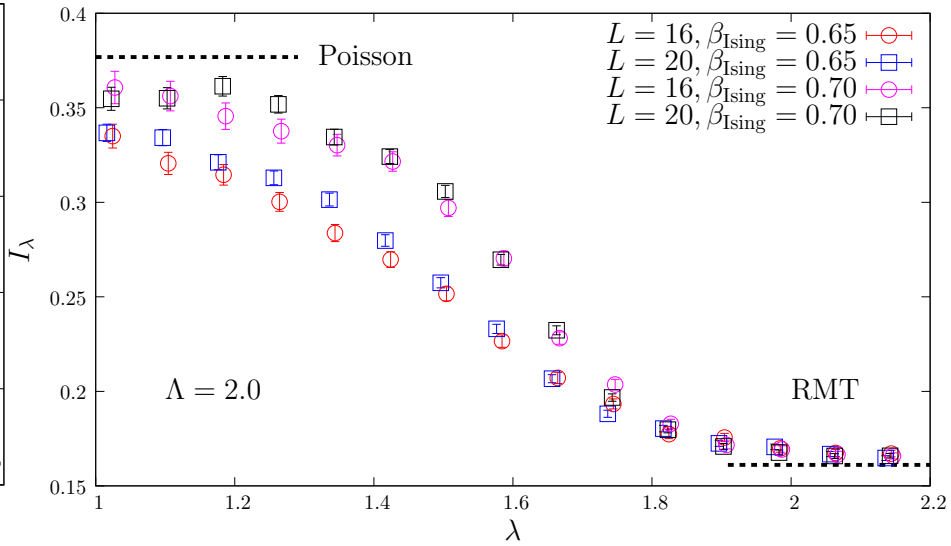


$$PR = \left(\sum_x |\psi(x)|^4\right)^{-1} / V$$

Modes change from localised to delocalised moving up in the spectrum

Spectral Statistics





Spectral statistics changes from Poisson to RMT (orthogonal)

$$I_\lambda = \int_0^{\bar{s}} ds\, p_\lambda(s) \qquad s_i = \frac{\lambda_{i+1}-\lambda_i}{\langle \lambda_{i+1}-\lambda_i \rangle} \qquad \bar{s} \simeq 0.5$$

Critical $\lambda_c \approx$ effective gap

$\beta_{\rm Ising}$	$\langle m \rangle$	$\langle \Lambda^{\frac{1+m}{2}} \rangle$
0.65	0.4854(2)	$1.114(1)\|_{\Lambda=1.5}$ — $1.485(2)\|_{\Lambda=2.0}$
0.70	0.5874(1)	$1.1905(7)\|_{\Lambda=1.5}$ — $1.587(1)\|_{\Lambda=2.0}$

Summary and Outlook

- Spatial fluctuations of the Polyakov loop provide a mechanism for localisation in QCD above T_c
- Preliminary results with the 3D Ising-Anderson model support the proposed mechanism

Open issues:

- Larger volumes, check if there is a true phase transition
- Tune the underlying spin model to study systems with larger correlation length

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