Understanding localisation in QCD through an Ising-Anderson model

Matteo Giordano, Tamás G. Kovács and Ferenc Pittler

Institute for Nuclear Research (ATOMKI), Debrecen

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Localisation in the Dirac Spectrum

Low-lying modes of the Dirac operator are localised above $T_c$


\[
\langle PR \rangle_{\lambda} = IPR - \frac{1}{V^4}
\]

\[
PR = IPR^{-1} / V^4
\]

\[
T \approx 2.6 T_c
\]

- Eigenmodes localised/delocalised for $\lambda < \lambda_c(T) / \lambda > \lambda_c(T)$
- Anderson-type phase transition in the spectrum at $\lambda_c$, same universality class of the 3D unitary Anderson model

[MG, Kovács, Pittler (2014)]
Anderson Transition in 3D

Tight-binding Hamiltonian for “dirty” conductors [Anderson (1958)]

\[ H_{\vec{x}\vec{y}} = \varepsilon_{\vec{x}} \delta_{\vec{x}\vec{y}} + \sum_{\mu} (\delta_{\vec{x}+\hat{\mu} \vec{y}} + \delta_{\vec{x}-\hat{\mu} \vec{y}}) \]

Random on-site potential \( \varepsilon_{\vec{x}} \in \left[-\frac{W}{2}, \frac{W}{2}\right] \) (\( W \sim \) amount of disorder) plus hopping

- eigenstates localised for \( E > E_c(W) \) (mobility edge)
- second-order phase transition with divergent \( \xi \sim |E - E_c|^{-\nu} \)
Anderson Transition in 3D

Tight-binding Hamiltonian for “dirty” conductors [Anderson (1958)]

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\[ \phi_{\vec{y}\vec{x}} = -\phi_{\vec{x}\vec{y}} \]

Random on-site potential \( \varepsilon_{\vec{x}} \in \left[ -\frac{W}{2}, \frac{W}{2} \right] \) (\( W \sim \) amount of disorder) plus hopping

In a magnetic field \( \rightarrow \) random phases, unitary class

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\[ \nu_{\text{UAM}} = 1.43(4) \]

[Slevin, Ohtsuki (1999)]
## Anderson Model vs. High-Temperature QCD

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QCD above $T_c$ is effectively a 3D model: time slices strongly correlated, quark eigenfunctions look qualitatively the same at all $t$

Temporal gauge: $U_4 = 1$ except at the temporal boundary $\Rightarrow$ wave functions obey effective boundary conditions involving the Polyakov loop

$$\psi(N_T, \vec{x}) = -P(\vec{x})\psi(0, \vec{x})$$

Correlated time slices $\Rightarrow$ effective, $x$-dependent boundary conditions will affect the behaviour at $x$ for all $t$

$P(\vec{x})$ fluctuates in space, providing effective 3D diagonal disorder
Localisation and Polyakov Loops

Simplified setting: \( SU(2) \), constant \( U_4, U_j = 1 \)
Temporal diagonal gauge, \( P = \text{diag}(e^{i\varphi}, e^{-i\varphi}) \)

\[
\mathcal{D}_\psi = i\lambda \psi \quad \psi(t, \vec{x}) \propto e^{i\omega t + i\vec{p} \cdot \vec{x}} \quad \lambda^2 = \sin^2 \omega + \sum_{j=1}^{3} \sin^2 p_j
\]

\[
\frac{L}{2\pi} p_j = 0, 1, \ldots, L - 1 \text{ to fulfill spatial bc}
\]

\[
\omega(\varphi) = \frac{1}{N_T} (\pi \pm \varphi \mod 2\pi) = aT(\pi \pm \varphi \mod 2\pi)
\]
to fulfill temporal bc (effective Matsubara frequencies)

Above \( T_c \) \( P(\vec{x}) \) gets ordered along 1 with “islands” of “wrong” \( P(\vec{x}) \neq 1 \)

- \( \omega(0) = \pi aT \) provides an effective gap in the spectrum
- “wrong” \( P(\vec{x}) \) allows for smaller \( \lambda \Rightarrow \) localising “trap” for eigenmodes

[Bruckmann, Kovács, Schierenberg (2011)]
Effective 3D Model

It should be possible to understand the qualitative features of the Dirac spectrum and eigenfunctions in QCD using a genuinely 3D model

\[ i \Phi_{xy} = i \gamma^4 (D_4)_{xy} + i \vec{\gamma} \cdot \vec{D}_{xy} \rightarrow \]
\[ H_{\vec{x}\vec{y}} = \gamma^4 \mathcal{N}_{\vec{x}} \delta_{\vec{x}\vec{y}} + i \vec{\gamma} \cdot \vec{D}_{\vec{x}\vec{y}} \]

Off-diagonal disorder less effective in producing localisation: \( \vec{D} \rightarrow \vec{\partial} \), colour components decouple (changes the symmetry class)

Diagonal disorder \( \mathcal{N}_{\vec{x}} \):

- not uncorrelated, should be governed by Polyakov-loop-like dynamics \( \sim \) spin models in the ordered phase

- phase of \( P(\vec{x}) \) enters the effective boundary conditions \( \Rightarrow \) continuous spin

- \( \mathcal{N}_{\vec{x}} \) should produce an effective gap in the spectrum
Effective 3D Model

It should be possible to understand the qualitative features of the Dirac spectrum and eigenfunctions in QCD using a genuinely 3D model

\[ i \Phi_{xy} = i \gamma^4 (D_4)_{xy} + i \vec{\gamma} \cdot \vec{D}_{xy} \rightarrow \]

\[ H_{xy} = \gamma^4 N_{\vec{x}} \delta_{\vec{x}\vec{y}} + i \vec{\gamma} \cdot \vec{\partial}_{\vec{x}\vec{y}} \]

Off-diagonal disorder less effective in producing localisation: \( \vec{D} \rightarrow \vec{\partial} \), colour components decouple (changes the symmetry class)

Diagonal disorder \( N_{\vec{x}} \):

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- \( N_{\vec{x}} \) should produce an effective gap in the spectrum
Ising-Anderson Model

Enough to have a spin model which displays an ordered phase: Ising model with continuous spin

$$H_{xy} = \gamma^4 \Lambda \frac{1+s_x}{2} \delta_{xy} + i \vec{\gamma} \cdot \vec{\partial}_{xy} \quad s_x \in [-1, 1]$$

Ordered phase: “sea” of $s_x = 1$ with “islands” of $s_x \neq 1$ (magnetic field = 0+)

- $\frac{1+s_x}{2}$: 1 for aligned spins, 0 for anti-aligned $\approx$ effective spectral gap
- $\Lambda$: spin-fermion coupling $\approx$ Matsubara frequency (size of the gap)
- ordering of the spin configuration governed by $\beta_{\text{Ising}}$

Symmetry class: orthogonal ($L$ even) or symplectic ($L$ odd)

Test of viability of the sea/islands explanation
Low modes have small spectral density, which rapidly increases
Symmetry $\lambda \rightarrow -\lambda$ on average
Sharp gap in the spectrum
Participation Ratio

\[ PR = \left( \sum_x |\psi(x)|^4 \right)^{-1} / V \]

Modes change from localised to delocalised moving up in the spectrum
Spectral statistics changes from Poisson to RMT (orthogonal)

\[ I_\lambda = \int_0^{\bar{s}} ds \, p_\lambda(s) \quad s_i = \frac{\lambda_{i+1} - \lambda_i}{\langle \lambda_{i+1} - \lambda_i \rangle} \quad \bar{s} \simeq 0.5 \]

Critical \( \lambda_c \approx \) effective gap

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<tr>
<th>( \beta_{\text{Ising}} )</th>
<th>( \langle m \rangle )</th>
<th>( \langle \Lambda^{1+m/2} \rangle )</th>
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<tr>
<td>0.65</td>
<td>0.4854(2)</td>
<td>1.114(1)</td>
</tr>
<tr>
<td>0.70</td>
<td>0.5874(1)</td>
<td>1.1905(7)</td>
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Spatial fluctuations of the Polyakov loop provide a mechanism for localisation in QCD above $T_c$.

Preliminary results with the 3D Ising-Anderson model support the proposed mechanism.

Open issues:

- Larger volumes, check if there is a true phase transition.
- Tune the underlying spin model to study systems with larger correlation length.
References

- K. Slevin and T. Ohtsuki, *Phys. Rev. Lett.* **82** (1999) 382