Determination of $c_{SW}$ in $N_f = 3 + 1$ Lattice QCD with massive Wilson fermions

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Outline

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>[Lüscher et al. (1997)]</th>
<th>[Jansen et al. (1998)]</th>
<th>[Bulava et al. (2013)]</th>
<th>[Tekin et al. (2009)]</th>
</tr>
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<tr>
<td>0</td>
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<tr>
<td>2</td>
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<td>3</td>
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<td>Symanzik</td>
<td>Wilson</td>
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1. Symanzik Improvement of the Wilson Fermion action ($N_f = 4$, non-degenerate)
   - $O(a)$-Improvement terms
   - Massive scheme

2. Determination of $c_{sw}$ ($N_f = 3 + 1$)
   - Line of constant physics
   - Improvement condition

3. First results
Symanzik Improvement

- Wilson Fermion action

\[ S_F = a^4 \sum_{\Lambda} (\mathcal{L}^0_F + \mathcal{L}^M_F) \]

\[ \mathcal{L}^0_F = \bar{\Psi} D_W \Psi \]

\[ \mathcal{L}^M_F = \bar{\Psi} M \Psi \]

with

\[ M = \text{diag}(m_{q,u}, m_{q,d}, m_{q,s}, m_{q,c}) \]

- Add higher dimension terms that respect symmetries of QCD

  e.g. \( \bar{\Psi} \frac{i}{4} \sigma_{\mu\nu} F_{\mu\nu} \Psi \)

  e.g. \( \text{Tr}[M] \bar{\Psi} M \Psi \)

  \( \rightarrow \text{SU}(4)_f \) symmetry broken for non-degenerate masses

- on-shell improvement:

  use equations of motion to reduce number of terms
Symanzik Improvement

**Improved Wilson Fermion (& Gauge) action**

\[ S = a^4 \sum_\Lambda (\mathcal{L}_{F,I}^0 + \mathcal{L}_{F,I}^M + \mathcal{L}_G^0 + \mathcal{L}_G^M) \]

\[ \mathcal{L}_{F,I}^0 = \bar{\Psi} D_W \Psi + a \left( c_{sw} (g_0^2) \bar{\Psi} i \frac{\sigma_{\mu\nu} F_{\mu\nu}}{4} \Psi \right) \]

\[ \mathcal{L}_{F,I}^M = \bar{\Psi} M \Psi + a \left( \sigma_1 (g_0^2) \bar{\Psi} M^2 \Psi + \sigma_2 (g_0^2) \text{Tr}[M] \bar{\Psi} M \Psi + \sigma_3 (g_0^2) \text{Tr}[M^2] \bar{\Psi} \Psi + \sigma_4 (g_0^2) (\text{Tr}[M])^2 \bar{\Psi} \Psi \right) \]

\[ \mathcal{L}_G^M = a \left( \sigma_5 (g_0^2) \text{Tr}[M] \text{Tr}[F_{\mu\nu} F_{\mu\nu}] \right) \]

[Bhattacharya et al. (2006)]
Massless scheme

- \( O = \) dimensionless observable in light sector (no \( \Psi_c, \bar{\Psi}_c \))
- Dependence on mass of a light quark (\( l = u, d, s \))

\[
O(m_{q,l}) - O(0) \sim \frac{m_{q,l}}{\Lambda} + \mathcal{O}(a^2 m_{q,l}^2)
\]
\[
\approx \frac{m_{q,l}}{\Lambda}
\]

- mass effects can be computed and analyzed/extrapolated using ChPT
- small mass-dependent cutoff effects

- Dependence on mass of a charm quark

\[
O(m_{q,c}) - O(0) \sim ? + \mathcal{O}(a^2 m_{q,c}^2)
\]

- mass effects can’t be studied / are not of interest
- large mass-dependent cutoff effects (Problem 1)
Massless scheme

- Example for renormalization

\[
m_{R,i} = \lim_{a \to 0} Z_m(g_0^2, a\mu) \left[ m_{q,i} + (r_m(g_0^2) - 1) \right] \left( \text{Tr}[M] \right) \approx m_{q,c}
\]

- small error in \( r_m \) ⇒ huge error in \( m_{R,i} \)  

- Same problem exists for \( O(a) \) improvement:

\[
\tilde{m}_{q,i} = m_{q,i} + \left[ \sigma_3(g_0^2) + \sigma_4(g_0^2) \right] am_{q,c}^2 + \ldots
\]

- small error in \( \sigma_i \) ⇒ large error in \( \tilde{m}_{q,i} \)  
- non-perturbative determination of many improvement coefficients necessary

⇒ Massless scheme is impractical & unstable
⇒ Solution: massive (charm) scheme
Massive scheme

- **Improved Wilson Fermion (& Gauge) action**

\[ \mathcal{L}_I = \mathcal{L}_G + \bar{\Psi} D_W \Psi + \bar{\Psi} M \Psi \]

\[ + a \left( c_{sw}(g_0^2) \bar{\Psi} \frac{i}{4} \sigma_{\mu\nu} F_{\mu\nu} \Psi \right) \]

\[ + a \left( \sigma_1(g_0^2) \bar{\Psi} M^2 \Psi + \right. \]

\[ \left. \sigma_2(g_0^2) \text{Tr}[M] \bar{\Psi} M \Psi + \right. \]

\[ \left. \sigma_3(g_0^2) \text{Tr}[M^2] \bar{\Psi} \Psi + \right. \]

\[ \left. \sigma_4(g_0^2) (\text{Tr}[M])^2 \bar{\Psi} \Psi + \right. \]

\[ \left. \sigma_5(g_0^2) \text{Tr}[M] \text{Tr}[F_{\mu\nu} F_{\mu\nu}] \right) \]

- **\( \sigma_5 \)-term \( \sim S_G \)**

\[ \tilde{g}_0^2 = g_0^2 (1 + am_{q,c} b_g / N_f) \]

- **\( \sigma_{1-4} \)-terms**

Determine all renormalization and improvement constants at finite \( am_{q,c}^* \)

\[ Z^\text{light}_A (\tilde{g}_0^2, am_{q,c}^*) \]

\[ c_A (\tilde{g}_0^2, am_{q,c}^*) \]

... from light quark chiral symmetry
Massive scheme

- **Improved Wilson Fermion (\& Gauge) action**

\[
\mathcal{L}_I = \mathcal{L}_G + \bar{\Psi} D_W \Psi + \bar{\Psi} M \Psi \\
+ a \left( c_{sw}(g_0^2) \bar{\Psi} M^2 \Psi + \right) \\
+ a \left( \sigma_1(g_0^2) \bar{\Psi} M^2 \Psi + \right) \\
\sigma_2(g_0^2) \text{Tr}[M] \bar{\Psi} M \Psi + \\
\sigma_3(g_0^2) \text{Tr}[M^2] \bar{\Psi} \Psi + \\
\sigma_4(g_0^2) (\text{Tr}[M])^2 \bar{\Psi} \Psi + \\
\sigma_5(g_0^2) \frac{\text{Tr}[M] \text{Tr}[F_{\mu\nu} F_{\mu\nu}]}{2} \\
\right)
\]

- **\(\sigma_5\)-term \(\sim S_G\)**

\[
\tilde{g}_0^2 = g_0^2 \left( 1 + a m_{q,c} b_g/N_f \right)
\]

- **\(\sigma_{1-4}\)-terms**

Determine all renormalization and improvement constants at finite \(a m_{q,c}\)

\[
Z_A^{\text{light}} (\tilde{g}_0^2, a m_{q,c}) \\
c_A (\tilde{g}_0^2, a m_{q,c}) \\
\vdots
\]

from light quark chiral symmetry
Massive scheme

- Improved Wilson Fermion (& Gauge) action

\[ \mathcal{L}_I = \mathcal{L}_G + \bar{\Psi} D_W \Psi + \bar{\Psi} M \Psi \]
\[ + a \left( c_{sw} g_0^2, am_{q,c}^* \right) \bar{\Psi} i \frac{1}{4} \sigma_{\mu\nu} F_{\mu\nu} \Psi \]
\[ + a \left( \frac{\sigma_1 (g_0^2)}{2} \bar{\Psi} M^2 \Psi + \right) \]
\[ \frac{\sigma_2 (g_0^2)}{2} \bar{\Psi} \text{Tr}[M] \Psi + \]
\[ \frac{\sigma_3 (g_0^2)}{2} \bar{\Psi} \text{Tr}[M^2] \Psi + \]
\[ \frac{\sigma_4 (g_0^2)}{2} \text{Tr}[M]^2 \Psi \Psi + \]
\[ \frac{\sigma_5 (g_0^2)}{2} \bar{\Psi} \text{Tr}[M] \text{Tr}[F_{\mu\nu} F_{\mu\nu}] \]

- \( \sigma_5 \)-term \( \sim S_G \)

\[ \tilde{g}_0^2 = g_0^2 (1 + \frac{am_{q,c} b_g}{N_f}) \]

\( \sigma_{1-4} \)-terms

Determine all renormalization and improvement constants at finite \( am_{q,c}^* \)

\[ Z^\text{light}_A (\tilde{g}_0^2, am_{q,c}^*) \]
\[ c_A (\tilde{g}_0^2, am_{q,c}^*) \]

from light quark chiral symmetry
Massive scheme

- **Improved Wilson Fermion (& Gauge) action**

\[
\mathcal{L}_I = \mathcal{L}_G + \bar{\Psi} D_W \Psi + \bar{\Psi} M \Psi \\
+ a \left( c_{sw} g_0^2 , am^*_{q,c} \right) \bar{\Psi} \frac{i}{4} \sigma_{\mu\nu} F_{\mu\nu} \Psi \\
+ \mathcal{O}(am_{q,u}, am_{q,s}) \\
+ \mathcal{O}(a^2 (m_{q,c} - m^*_{q,c}))
\]

- **\(\sigma_5\)-term \(\sim S_G\)**

\[
\tilde{g}_0^2 = g_0^2 (1 + am_{q,c} b_g / N_f)
\]

- **\(\sigma_{1-4}\)-terms**

Determine all renormalization and improvement constants at finite \(am^*_{q,c}\)

\[
Z_A^{\text{light}} (\tilde{g}_0^2, am^*_{q,c}) \\
c_A (\tilde{g}_0^2, am^*_{q,c}) \\
\vdots
\]

from light quark chiral symmetry
Massive scheme

- **Improved Wilson Fermion (& Gauge) action**

\[
\mathcal{L}_I = \mathcal{L}_G + \bar{\Psi} D_W \Psi + \bar{\Psi} M \Psi \\
+ a \left( c_{\text{sw}}(g_0^2, a m_q^*, l, a m_q^*, c) \bar{\Psi} i \frac{\sigma_{\mu\nu} F_{\mu\nu}}{4} \Psi \right) \\
+ \mathcal{O}(a m_q, u, a m_q, s) \\
+ \mathcal{O}(a^2 (m_q, c - m_q^*, c)) \\
+ \mathcal{O}(a^2 (m_q, u - m_q^*, l)) \\
+ \mathcal{O}(a^2 (m_q, s - m_q^*, l))
\]

- **\(\sigma_5\)-term** \(\sim S_G\)

\[
\tilde{g}_0^2 = g_0^2 (1 + a m_q, c b_g / N_f)
\]

- **\(\sigma_{1-4}\)-terms**

Determine all renormalization and improvement constants at finite \(a m_q^*, c\)

\[
Z_A^{\text{light}}(\tilde{g}_0^2, a m_q^*, c) \\
c_A(\tilde{g}_0^2, a m_q^*, c) \\
: \\
\]

from light quark chiral symmetry

\[\rightarrow c_{\text{sw}}:\]

Determination at finite \(a m_q^*, l\) and \(a m_q^*, c\)

\((N_f = 3 + 1)\)
Determine $c_{sw}$ as function of 3 parameters $g_0^2, m_{q,l}^*, m_{q,c}^*$? × not feasible

↓

Fix light ($m_{q,l}^*$) and charm ($m_{q,c}^*$) mass to their approx. physical values

Determine $c_{sw}$ as function of 1 parameter $g_0^2$

↓

Line of constant physics (LCP)

1. $L \equiv$ fixed
2. $LM_l \equiv$ fixed
3. $LM_c \equiv$ fixed

→ fixes $g_0^2, \kappa_l, \kappa_c$ for given $L/a$
Line of constant physics

### LCP

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<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>$L$</td>
<td>$\overset{!}{=} \text{fixed}$</td>
</tr>
<tr>
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<td>$\overset{!}{=} \text{fixed}$</td>
</tr>
<tr>
<td>3</td>
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$\Leftrightarrow$

### LCP implicit

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<td>$\Phi_3 = L(\Gamma_{lc} - \Gamma_{\tilde{l}}/2)$</td>
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#### Gradient flow coupling

[Fritzsch, Ramos (2013)]

$$
\bar{g}_{GF}^2(L) = N^{-1} t^2 \left< E(t, x_0) \right|_{x_0=T/2} \left| t = c^2 L^2 / 8 \right. \text{ with } c = 0.3
$$

#### Effective meson masses

$$
\Gamma_{ij} = -\tilde{\partial}_0 \log \left( f_{A,I}^{ij}(x_0) \right) \bigg|_{x_0=T/2}
$$

with

$$
f_{A}^{ij}(x_0) \sim e^{-m_{PS}^{ij} \cdot x_0} (1 + \ldots)
$$
Line of constant physics

LCP

1. $L \overset{!}{=} \text{fixed}$
2. $LM_l \overset{!}{=} LM_s/3$
3. $LM_c \overset{!}{=} LM_c$

LCP implicit

1. $\Phi_1 = \bar{g}_{GF}^2 \overset{!}{=} \Phi_1^*$
2. $\Phi_2 = L\Gamma_{\bar{d}} \overset{!}{=} \Phi_2^*$
3. $\Phi_3 = L(\Gamma_{lc} - \Gamma_{\bar{d}}/2) \overset{!}{=} \Phi_3^*$

- Masses unknown in $N_f = 3 + 1$
- Resort to $N_f = 2$
- Difference is
  - $O(a)$ effect in $c_{sw}$
  - $O(a^2)$ effect in improved action
Line of constant physics

LCP

1. \( L \equiv \text{fixed} \)
2. \( L M_l \equiv L M_s / 3 \big|_{N_f=2} \)
3. \( L M_c \equiv L M_c \big|_{N_f=2} \)

LCP implicit

1. \( \Phi_1 = \bar{g}_{GF}^2 \equiv \Phi_1^* \)
2. \( \Phi_2 = L \Gamma_{\bar{u}} \equiv \Phi_2^* \)
3. \( \Phi_3 = L (\Gamma_{lc} - \Gamma_{\bar{u}} / 2) \equiv \Phi_3^* \)

- Masses unknown in \( N_f = 3 + 1 \)
- Resort to \( N_f = 2 \)
- Difference is
  - \( O(a) \) effect in \( c_{sw} \)
  - \( O(a^2) \) effect in improved action
Line of constant physics

LCP

1. $L \overset{!}{=} \text{fixed}$
2. $LM_l \overset{!}{=} \frac{LM_s}{3}|_{N_f=2}$
3. $LM_c \overset{!}{=} LM_c|_{N_f=2}$

LCP implicit

1. $\Phi_1 = \bar{g}^2_{GF} \overset{!}{=} \Phi^*_1$
2. $\Phi_2 = L\Gamma_l\bar{l} \overset{!}{=} \Phi^*_2$
3. $\Phi_3 = L(\Gamma_{lc} - \Gamma_{l\bar{l}}/2) \overset{!}{=} \Phi^*_3$

- Masses unknown in $N_f = 3 + 1$
- Resort to $N_f = 2$
- Difference is
  - $\mathcal{O}(a)$ effect in $c_{sw}$
  - $\mathcal{O}(a^2)$ effect in improved action
Line of constant physics

**LCP**

1. $L \overset{!}{\approx} 0.8 \text{ fm}$
2. $LM_l \overset{!}{=} LM_s/3|_{N_f=2}$
3. $LM_c \overset{!}{=} LM_c|_{N_f=2}$

**LCP implicit**

1. $\Phi_1 = \bar{g}_{GF}^2 \overset{!}{=} \Phi_1^* = 7.20$
2. $\Phi_2 = L\Gamma_{\tilde{l\tilde{l}}} \overset{!}{=} \Phi_2^*$
3. $\Phi_3 = L(\Gamma_{lc} - \Gamma_{\tilde{l\tilde{l}}}/2) \overset{!}{=} \Phi_3^*$

**Determination of $\Phi_1^*$**

- **Configurations** [Blossier et al. (2012)]
  - $N_f = 2, \kappa_{\text{sea}} = \kappa_{\text{crit}}$

**Continuum limit**

\[ \bar{g}_{GF}^2 \]

\[ (a/L)^2 \]

\[ \times 10^{-3} \]
Line of constant physics

**LCP**

1. \( L \approx 0.8 \text{ fm} \)
2. \( LM_l \mid_{N_f=2} = LM_s / 3 \)
3. \( LM_c \mid_{N_f=2} = LM_c \)

**LCP implicit**

1. \( \Phi_1 = \frac{g^2_{GF}}{\bar{m}} \quad \Rightarrow \quad \Phi_1^* = 7.20 \)
2. \( \Phi_2 = L \Gamma_{\bar{l}} \quad \Rightarrow \quad \Phi_2^* = 0.59 \)
3. \( \Phi_3 = L(\Gamma_{l_c} - \Gamma_{\bar{l}} / 2) \quad \Rightarrow \quad \Phi_3^* \)

**Determination of \( \Phi_2^* \)**

- **Configurations** [Blossier et al. (2012)]
  - \( N_f = 2, \kappa_{\text{sea}} = \kappa_{\text{crit}} \)
- **\( LM_s \)** [Fritzsch et al. (2012)]
  - \( \kappa_{\text{crit}}, Z, b_m, \ldots \rightarrow \kappa_l (L/a) \)
Line of constant physics

LCP

1. \( L \approx 0.8 \text{ fm} \)
2. \( LM_l \equiv LM_s/3|_{N_f=2} \)
3. \( LM_c \equiv LM_c|_{N_f=2} \)

LCP implicit

1. \( \Phi_1 = \frac{g^2}{G_F} \quad \equiv \Phi_1^* = 7.20 \)
2. \( \Phi_2 = L \Gamma_{\bar{u}\bar{u}} \quad \equiv \Phi_2^* = 0.59 \)
3. \( \Phi_3 = L(\Gamma_{lc} - \Gamma_{\bar{u}\bar{u}}/2) \quad \equiv \Phi_3^* = 5.96 \)

Determination of \( \Phi_3^* \)

- Configurations [Blossier et al. (2012)]
  \( N_f = 2, \kappa_{\text{sea}} = \kappa_{\text{crit}} \)
- \( LM_s \) [Fritzsch et al. (2012)]
  \( \kappa_{\text{crit}}, Z, b_m, \ldots \rightarrow \kappa_l(L/a) \)
- \( LM_c \) [Heitger et al. (2013)]
  \( \kappa_{\text{crit}}, Z, b_m, \ldots \rightarrow \kappa_c(L/a) \)

continuum limit

5.5 6.5 7
\( (a/L)^2 \) 0 2 4 6 8 10 x 10^{-3}
Improvement condition

PCAC relation

\[ \langle \partial_\mu A^{ij}_\mu (x) \, O \rangle = 2m^{ij}(x_0)\langle P^{ij}(x) \, O \rangle + \mathcal{O}(a^n) \]

- Use \( c_A \)-independent combination of \( m^{ij} \):

\[ m^{ij}(x_0), m^{ij}(y_0) \rightarrow M^{ij}(x_0, y_0) \]

- Improvement condition:

\[ \Delta M^{l\tilde{l}} = M^{l\tilde{l}}(3T/4, T/4) - M'^{l\tilde{l}}(3T/4, T/4) \stackrel{!}{=} 0 \]

- Use of light flavors \( l\tilde{l} \)

\[ \rightarrow \text{want to primarily improve light quark physics} \]
Strategy overview

\[ \frac{L}{a} \]

↓

LCP, \( \Phi_i \)

↓

\( g_0^2, \kappa_l, \kappa_c \)

↓

Impr. cond.

↓

\( c_{sw} \)

\[ \begin{array}{c|c}
T & \sim 0.8 \text{ fm} \\
L & \sim T/2 \\
\text{Bound. field} & \neq 0
\end{array} \]

The procedure may lead to something like:

(NO real data!!)

LCP determines maximal \( g_0^2 \)
(at smallest \( L/a \))

Felix Stollenwerk

Determination of \( c_{sw} \) in \( N_f = 3 + 1 \) LQCD
The procedure may lead to something like (NO real data!!)

LCP determines maximal $g_0^2$ (at smallest $L/a$)
\[ L/a = 8 \]

### Tune bare parameters to LCP

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<th>( g_0^2 )</th>
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LCP \( \Phi_i^* \) | 7.20 | 0.59 | 5.96

### Tuning criteria

\( \delta X = |X - X^*| \)

\[
\begin{align*}
\frac{\delta L}{L^*}, & \frac{\delta M_l}{M_l^*}, \frac{\delta M_c}{M_c^*} < 5\% \\
\uparrow \text{estimate} \\
\frac{\delta \Phi_i}{\Phi_i^*} < 5\%
\end{align*}
\]

keep cutoff effects under control \((M_c)\)

include statistical errors of \( \Phi_i \) conservatively
\( L/a = 8 \)

1. Tune bare parameters to LCP

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Tuning criteria \((\delta X = |X - X^*|)\)

\[
\frac{\delta L}{L^*}, \frac{\delta M_l}{M_l^*}, \frac{\delta M_c}{M_c^*} < 5\%
\]

\[\uparrow\] estimate

\[
\frac{\delta \Phi_i}{\Phi_i^*} + \Delta \left( \frac{\delta \Phi_i}{\Phi_i^*} \right) < 5\%
\]

- keep cutoff effects under control \((M_c!)\)
- include statistical errors of \( \Phi_i \) conservatively

\(i = 1, 2, 3\)
\( \frac{L}{a} = 8 \)

1. Tune bare parameters to LCP
2. Measure \( \Delta M^{l\bar{l}} \)

<table>
<thead>
<tr>
<th>( c_{SW} )</th>
<th>( g_0^2 )</th>
<th>( a\Delta M^{l\bar{l}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6</td>
<td>1.719</td>
<td>0.0135(10)</td>
</tr>
<tr>
<td>1.9</td>
<td>1.785</td>
<td>0.0060(10)</td>
</tr>
<tr>
<td>1.95</td>
<td>1.797</td>
<td>0.0037(14)</td>
</tr>
<tr>
<td>2.1</td>
<td>1.830</td>
<td>-0.0012(21)</td>
</tr>
<tr>
<td>2.2</td>
<td>1.852</td>
<td>-0.0045(20)</td>
</tr>
<tr>
<td>2.08(3)</td>
<td>1.825(7)</td>
<td>0</td>
</tr>
</tbody>
</table>

3. Interpolate in \( c_{SW} \)

4. Interpolate in \( g_0^2 \)
Help for tuning from perturbative evolution of lattice spacing $a$ with $g_0^2$:

$$
\frac{a(g_0^2)}{a((g_0')^2)} = e^{-[g_0^2 - (g_0')^2]/(2b_0)} [g_0^2/(g_0')^2]^{-b_1/(2b_0^2)} \left[ 1 + \mathcal{O}((g_0')^2) \right], \quad g_0 < g_0'
$$

Estimated range covered with $L/a = 8 - 24$:

$$
g_0^2 \in [1.472, 1.825]
$$
Summary

- Determination of $c_{sw}$
  - 1st step in $N_f = 3 + 1$ simulations with massive charm quark

- Mass-dependent renormalization scheme
  - avoid large number of improvement coefficients
  - avoid large mass-dependent corrections

- Line of constant physics
  - fix all relevant physical scales in finite volume
  - numerical values from $N_f = 2$

- Improvement condition in standard way (PCAC)

- First results for $L/a = 8$

Thank you!