Non-perturbative renormalization of the energy-momentum tensor in SU(3) Yang-Mills theory

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PLANT OF THE TALK

- Introduction
- A new perspective from relativistic thermodynamics
- The lattice setup and the numerical results
- Conclusions and perspectives

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Introduction

- Thermal quantum field theory: a bridge between particle and nuclear physics

**Equation of State of QCD**

- heavy ion collisions
- dynamics of the early universe
- properties of compact stars

- Quantum field theory described by an action $S$ in a space-time volume $V \times \beta$

$$Z(\beta, V) = \int \mathcal{D}\phi \ e^{-S(\phi)}$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \ \mathcal{O}(\phi) \ e^{-S(\phi)}$$

Temperature:
$$T = \frac{1}{\beta}$$

Free energy density:
$$f = -\frac{T}{V} \log Z(\beta, V)$$

Pressure:
$$p = T \frac{\partial \log Z(\beta, V)}{\partial V} = -f$$

Energy density:
$$e = \frac{T^2}{V} \frac{\partial \log Z(\beta, V)}{\partial T}$$

Entropy density:
$$s = \frac{e + p}{T} = \frac{\partial p}{\partial T}$$

$Z(\beta, V)$ contains all the information about the system but it cannot be computed directly

$$\log Z(\beta, V) \rightarrow \int_{x_0}^{x} dx' \ \frac{\partial \log Z}{\partial x'}$$

usually
$$x = \frac{2N}{g_0^2} \quad \longrightarrow \quad \frac{f}{T^4} - \frac{f_0}{T_0^4} = \beta^4 \int_{x_0}^{x} \frac{dx'}{x'} \left[ \langle s \rangle_0 - \langle s \rangle \right]$$

Engels, Fingberg, Karsch, Miller, and Weber
New problems for an old solution

theory of special relativity  \rightarrow  relativistic thermodynamics  \rightarrow  thermodynamics and statistical mechanics

\[ T = \gamma T_0 \quad s = \gamma s_0 \quad e = \gamma^2 (e_0 + p_0 \beta^2) \quad f = \gamma^2 f_0 \]

\[ V = \frac{V_0}{\gamma} \quad p = p_0 \quad w = e + p = \gamma^2 w_0 \quad \gamma(\beta) = \frac{1}{\sqrt{1 - \beta^2}} \]

energy-momentum tensor

\[ T_{\mu\nu} = \begin{pmatrix} e_0 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix} \]

\[ T_{ij} = \gamma^2 w_0 v_i v_j + p \delta_{ij} \]
\[ T_{00} = \gamma^2 w_0 - p = e \]
\[ T_{0k} = \gamma^2 w_0 v_k \]

Landau, Lifschitz, vol 6, “Fluid mechanics”

rewriting:

\[ s_0 = \frac{w_0}{T_0} = \frac{T_{0k}}{T_0 \gamma^2 v_k} \]
\[ \frac{s_0}{T_0^3} = \frac{T_{0k}}{T_0^4 \gamma^2 v_k} = \frac{\beta^4 (1 - \beta^2)^3}{v_k} T_{0k} \]
\[ \frac{1}{T_0} = \beta_0 = \beta \sqrt{1 - \beta^2} \]

\[ T_{0k} = \frac{v_k}{1 + v_k^2} (T_{00} + T_{kk}) \]

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New problems for an old solution

theory of special relativity \rightarrow \text{relativistic thermodynamics} \rightarrow \text{thermodynamics and statistical mechanics}

\[ T = \gamma T_0 \quad s = \gamma s_0 \quad e = \gamma^2 (e_0 + p_0 \bar{v}^2) \quad f = \gamma^2 f_0 \]

\[ V = \frac{V_0}{\gamma} \quad p = p_0 \quad w = e + p = \gamma^2 w_0 \quad \gamma(\bar{v}) = \frac{1}{\sqrt{1 - \bar{v}^2}} \]

energy-momentum tensor

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\[ T_{00} = \gamma^2 w_0 - p = e \]

\[ T_{0k} = \gamma^2 w_0 v_k \quad \text{Landau, Lifschiz, vol 6, “Fluid mechanics”} \]

rewriting:

\[ s_0 = \frac{w_0}{T_0} = \frac{T_{0k}}{T_0 \gamma^2 v_k} \]

\[ \frac{s_0}{T_0^3} = \frac{T_{0k}}{T_0^4 \gamma^2 v_k} = \frac{\beta^4 (1 - \bar{v}^2)^3}{v_k} T_{0k} \]

\[ \frac{1}{T_0} = \beta_0 = \beta \sqrt{1 - \bar{v}^2} \]

\[ T_{0k} = \frac{v_k}{1 + v_k^2} (T_{00} + T_{kk}) \]

the entropy density in the rest frame is related to the momentum \( T_{0k} \) and the time extent \( \beta \) in the moving frame

argument is general: applies to any system at equilibrium
This approach can be applied to a thermal quantum field theory in the euclidean path-integral formulation

$$\phi(\beta, \vec{x}) = \phi(0, \vec{x} - \beta \vec{\xi})$$

The energy-momentum tensor contains the currents associated to translation invariance

$$\mathcal{Z}(\beta, \vec{v}) = \text{Tr} \left( e^{-\beta (H - \vec{v} \cdot \vec{P})} \right)$$

$$\vec{v} \rightarrow i \vec{\xi}$$

Use the machinery of quantum fields to produce Ward identities generated by Lorentz invariance

$$\mathcal{Z}(\beta, \vec{\xi}) = \text{Tr} \left( e^{-\beta (H - i \vec{\xi} \cdot \vec{P})} \right)$$

$$f(\beta \sqrt{1 + \vec{\xi}^2}, 0) = -\frac{1}{\beta V} \log \mathcal{Z}(\beta, \vec{\xi})$$

$$\frac{s_0}{T_0^3} = -\frac{\beta^4 (1 + \vec{\xi}^2)^3}{\xi_k} \langle T_{0k} \rangle_\xi$$

$$s_0 = -\frac{1}{V \gamma^3 \xi_k} \frac{\partial}{\partial \xi_k} \log \mathcal{Z}(\beta, \vec{\xi})$$

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\[ f(\beta \sqrt{1 + \vec{\xi}^2}, 0) = -\frac{1}{\beta V} \log Z(\beta, \vec{\xi}) \]

\[ \frac{s_0}{T_0^3} = -\frac{\beta^4(1 + \vec{\xi}^2)^3}{\xi_k} \langle T_{0k} \rangle \xi Z \]

\[ s_0 = -\frac{1}{V \gamma^3 \xi_k} \left[ \frac{\partial}{\partial \xi_k} \log Z(\beta, \vec{\xi}) \right] \]

lattice regularization: breaking of translation invariance
Equation of State: entropy vs. temperature

- **Strategy 1:** two shifts $\xi$ and $\xi'$, measure the entropy density step-scaling function

\[
\Sigma(T, r) = \frac{T^3 s(T')}{T'^3 s(T)} = \frac{(1 + \xi'^2)^3 \xi_k}{(1 + \xi^2)^3 \xi'_k} \frac{\langle T_{0k} \rangle \xi'}{\langle T_{0k} \rangle \xi}
\]

\[
r = \frac{T'}{T} = \frac{\sqrt{1 + \xi'^2}}{\sqrt{1 + \xi^2}}
\]

the renormalization factor $Z$ drops.

- **Strategy 2:** only one shift and measure $\langle T_{0k} \rangle \xi$ and $Z$

\[
\frac{s_0}{T_0^3} = -\frac{\beta^4 (1 + \xi^2)^3}{\xi_k} \langle T_{0k} \rangle \xi Z
\]

need to know $Z$ for many values of $g_0^2$. 

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Strategy 1: entropy step-scaling function

we consider the shifts: $\xi=(1,1,1)$ and $\xi'=(1,0,0) \quad r = \frac{\sigma'}{\sigma} = \frac{\sqrt{1 + \xi'^2}}{\sqrt{1 + \xi^2}} = \sqrt{2}$

good compromise: small step (fine T dependence) and big step (broad range in T)

$$
\Sigma(T, \sqrt{2}) = \frac{1}{8} \frac{\langle T_{0k} \rangle_{\xi=(1,0,0)}}{\langle T_{0k} \rangle_{\xi=(1,1,1)}}
$$

$$
F_{\mu\nu}(x) = \frac{1}{8} [Q_{\mu\nu}(x) - Q_{\nu\mu}(x)]_{\text{traceless}}
$$

$$
T_{\mu\nu} = -\frac{2}{g_0^2} \left[ \text{Tr} (F_{\mu\alpha}(x)F_{\nu\alpha}(x)) - \frac{1}{4} \delta_{\mu\nu} \text{Tr} (F_{\alpha\beta}(x)F_{\alpha\beta}(x)) \right]
$$

$$
Q_{\mu\nu}(x) = \sum_{m=1}^{\infty} \chi \cdot \chi
$$

$v_0(T_0) = \frac{s(T_0)}{T_0^3}$

$v_{k+1}(T_{k+1}) = \Sigma(T_k, \sqrt{2}) \cdot v_k(T_k)$

$T_k = (\sqrt{2})^k T_c$
Strategy 2: direct measure of the entropy

We consider only one shift: $\xi = (1,0,0)$

$$\frac{s_0}{T_0^3} = -\frac{\beta^4(1 + \xi^2)^3}{\xi_k} \langle T_{0k}\rangle \xi Z$$

More appealing: no constraint on the temperature, any value is fine

- measurement of $\langle T_{0k}\rangle \xi$ : OK, no news.
- measurement of $Z$: need to know $Z(g_0^2)$ not only some values

conserved charge:

$$s_0 = -\frac{1}{V \gamma^3 \xi_k} \frac{\partial}{\partial \xi_k} \log Z(\beta, \xi)$$

$$Z = \frac{1}{V} \frac{1}{\langle T_{0k}\rangle \xi} \frac{\partial \log Z(\beta, \xi)}{\partial \xi_k} \Rightarrow Z = \frac{1}{2V} \frac{1}{\langle T_{0k}\rangle \xi} \log \frac{Z(\beta, \xi + \frac{a}{\beta} \hat{k})}{Z(\beta, \xi - \frac{a}{\beta} \hat{k})}$$

ratio of partition functions: very challenging to measure, hard for large volumes

$$\frac{Z_1}{Z_2} = \frac{Z_1}{Z_{1+\epsilon}} \frac{Z_{1+2\epsilon}}{Z_{1+3\epsilon}} \cdots \frac{Z_{2-2\epsilon}}{Z_{2-\epsilon}} \frac{Z_{2-\epsilon}}{Z_2}$$

$$S[U, r_i] = r_i S[U, \xi + \frac{a}{\beta} \hat{k}] + (1 - r_i) S[U, \xi - \frac{a}{\beta} \hat{k}] \Rightarrow \log \frac{Z(\beta, \xi + \frac{a}{\beta} \hat{k})}{Z(\beta, \xi - \frac{a}{\beta} \hat{k})} = \sum_{i=0}^{n-1} \log \frac{Z(\beta, r_i)}{Z(\beta, r_{i+1})}$$

$$r_i = \frac{i}{n} \quad i = 0, \ldots, n \quad \log \frac{Z(\beta, r_i)}{Z(\beta, r_{i+1})} = \langle \exp [S[U, r_{i+1}] - S[U, r_i]] \rangle_{r_i}$$
we have used the SU(3) Wilson action on a lattice with $L=16$

The method works well but:

- increasing the spatial volume is very challenging
- we need $Z(g^2_0)$ accurately for many values of $g^2_0$
Compute $Z$ in perturbation theory \( \Rightarrow \) function of $g_0^2$

\[
\frac{1}{2V} \log \frac{Z(\beta, \xi + \frac{g_0}{\beta} \hat{k})}{Z(\beta, \xi - \frac{g_0}{\beta} \hat{k})} = c_0 + c_1 g_0^2 + c_2 (g_0^2)^2 + \ldots
\]

\[
Z = \frac{c_0 + c_1 g_0^2 + c_2 (g_0^2)^2 + \ldots}{a_0 + a_1 g_0^2 + a_2 (g_0^2)^2 + \ldots} - \frac{c_0}{a_0} + 1
\]

\[
\langle T_{0k} \rangle_\xi = a_0 + a_1 g_0^2 + a_2 (g_0^2)^2 + \ldots
\]

Numerical Stochastic Perturbation Theory

\[
\phi(x) \rightarrow \phi(x, \tau)
\]

\[
\phi(x, \tau) = \sum_{n=0} \phi_n(x, \tau) g_0^n
\]

\[
\frac{\partial \phi(x, \tau)}{\partial \tau} = -\frac{\delta S[\phi]}{\delta \phi(x, \tau)} + \eta(x, \tau)
\]

\( \Rightarrow \) equal order in $g_0 \rightarrow \) cascade of eqs. for $\phi_n$

\[
\mathcal{O}[\phi(x)] \rightarrow \mathcal{O}[\phi(x, \tau)] = \sum_{n=0} \mathcal{O}_n[\phi(x, \tau)] g_0^n
\]

\( \Rightarrow \) \( \langle \mathcal{O} \rangle = \langle \mathcal{O}_0 \rangle + \langle \mathcal{O}_1 \rangle g_0 + \langle \mathcal{O}_2 \rangle g_0^2 + \ldots \)

Calculation of: $a_0, a_1, a_2, c_0, c_1, c_2$ at $\beta=3$ and $L=24$ and $48$


Compute $Z$ in perturbation theory $\Rightarrow$ function of $g_0^2$

$$\frac{1}{2V} \log \frac{Z(\beta, \xi + \frac{a}{\beta} \hat{k})}{Z(\beta, \xi - \frac{a}{\beta} \hat{k})} = c_0 + c_1 g_0^2 + c_2 (g_0^2)^2 + \ldots$$

$$Z = \frac{c_0 + c_1 g_0^2 + c_2 (g_0^2)^2 + \ldots}{a_0 + a_1 g_0^2 + a_2 (g_0^2)^2 + \ldots} - \frac{c_0}{a_0} + 1$$

$\langle T_{0k} \rangle_\xi = a_0 + a_1 g_0^2 + a_2 (g_0^2)^2 + \ldots$

**Numerical Stochastic Perturbation Theory**

$$\phi(x) \rightarrow \phi(x, \tau)$$

$$\phi(x, \tau) = \sum_{n=0} \phi_n(x, \tau) g_0^n$$

$$\frac{\partial \phi(x, \tau)}{\partial \tau} = - \frac{\delta S[\phi]}{\delta \phi(x, \tau)} + \eta(x, \tau)$$

equal order in $g_0$ $\Rightarrow$ cascade of eqs. for $\phi_n$

$$\mathcal{O}[\phi(x)] \rightarrow \mathcal{O}[\phi(x, \tau)] = \sum_{n=0} \mathcal{O}_n[\phi(x, \tau)] g_0^n$$

$$\langle \mathcal{O} \rangle = \langle \mathcal{O}_0 \rangle + \langle \mathcal{O}_1 \rangle g_0 + \langle \mathcal{O}_2 \rangle g_0^2 + \ldots$$

Calculation of: $a_0, a_1, a_2, c_0, c_1, c_2$ at $\beta=3$ and $L=24$ and 48

very strong finite size corrections $\Rightarrow$ relevant higher order corrections
alternative approach: non-perturbative measure of $Z + function of g_0^2$

$$\frac{1}{2V} \log \frac{Z(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{Z(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})} = c_0 + \frac{1}{2V} \int_0^{g_0^2} dg^2 \frac{\partial}{\partial g^2} \log \frac{Z(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{Z(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})} = c_0 + \int_0^{g_0^2} dg^2 \frac{18\beta}{(g^2)^2} \left( \langle \Box \rangle \vec{\xi} - \frac{a}{\beta} \hat{k} - \langle \Box \rangle \vec{\xi} + \frac{a}{\beta} \hat{k} \right)$$

Spatial size $L = 48$

$$Z = \frac{1}{2V} \frac{1}{\langle T_{0k} \rangle_\xi} \log \frac{Z(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{Z(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})}$$

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alternative approach: non-perturbative measure of $Z + \text{function of } g_0^2$

\[
\frac{1}{2V} \log \frac{Z(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{Z(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})} = c_0 + \frac{1}{2V} \int_0^{g_0^2} dg^2 \frac{\partial}{\partial g^2} \log \frac{Z(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{Z(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})} = c_0 + \int_0^{g_0^2} dg^2 \frac{18\beta}{(g^2)^2} \left( \langle \square \rangle \vec{\xi} - \frac{a}{\beta} \hat{k} - \langle \square \rangle \vec{\xi} + \frac{a}{\beta} \hat{k} \right)
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Spatial size $L=48$

\[
Z = \frac{1}{2V} \frac{1}{\langle T_{0k} \rangle_{\xi}} \log \frac{Z(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{Z(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})}
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alternative approach: non-perturbative measure of $Z$ + function of $g_0^2$

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\frac{1}{2V} \log \frac{Z(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{Z(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})} = c_0 + \frac{1}{2V} \int_0^{g_0^2} dg^2 \frac{\partial}{\partial g^2} \log \frac{Z(\beta, \vec{\xi} + \frac{a}{\beta} \hat{k})}{Z(\beta, \vec{\xi} - \frac{a}{\beta} \hat{k})} = c_0 + \int_0^{g_0^2} dg^2 \frac{18\beta}{(g^2)^2} \left( \langle \Box \rangle \xi_{\hat{k}} - \langle \Box \rangle \xi_{\hat{k}} \right)
\]

Spatial size $L=48$
non-perturbative measure of $Z$

Uncertainty less than 5 per-mille simulations in progress for $\beta = 4$ and 5

$g_0^2$
non-perturbative measure of $Z$

Uncertainty less than 5 per-mille simulations in progress for $\beta = 4$ and 5
Conclusions

- A new approach to thermal field theories based on relativistic thermodynamics is investigated: far more efficient than the state-of-the-art techniques
- The step-scaling function of the entropy density has been obtained with quite moderate computational effort \( \Rightarrow \) entropy density vs. T with 5 per-mille accuracy
- We have shown measurements of Z with an accuracy below 5 per-mille. This introduces also another method to compute the EoS.
- The approach used here is general and can be applied to any thermal field theory: in particular to QCD (extremely challenging with the standard techniques).

What next?

- Investigate the EoS in SU(3) YM with high accuracy using this new method
- Renormalization factor of the diagonal entries of the energy-momentum tensor:

\[
z = \frac{v_k}{1 + v_k^2} \frac{\langle T_{00} + T_{kk} \rangle}{\langle T_{0k} \rangle}
\]
- Thermodynamics of QCD