Non-perturbative renormalization of the energy-momentum tensor in SU(3) Yang-Mills theory

M I C H E L E P E P E

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IN COLLABORATION WITH

L E O N A R D O G I U S T I

University of M i l a n o – B i c o c c a

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PLAN OF THE TALK

- Introduction
- A new perspective from relativistic thermodynamics
- The lattice setup and the numerical results
- Conclusions and perspectives

Introduction

- Thermal quantum field theory: a bridge between particle and nuclear physics

Equation of State of QCD → heavy ion collisions
→ dynamics of the early universe
→ properties of compact stars

- Quantum field theory described by an action S in a space-time volume V×β

$$\mathcal{Z}(\beta, V) = \int \mathcal{D}\phi \, \mathrm{e}^{-S(\phi)} \qquad \langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \, \mathcal{O}(\phi) \, \mathrm{e}^{-S(\phi)}$$

temperature: $T = \dfrac{1}{\beta}$

free energy density: $f = -\dfrac{T}{V} \log \mathcal{Z}(\beta, V)$

pressure: $p = T \dfrac{\partial \log \mathcal{Z}(\beta, V)}{\partial V} = -f$

energy density: $e = \dfrac{T^2}{V} \dfrac{\partial \log \mathcal{Z}(\beta, V)}{\partial T}$

entropy density: $s = \dfrac{e + p}{T} = \dfrac{\partial p}{\partial T}$

$\mathcal{Z}(\beta, V)$ contains all the information about the system but it cannot be computed directly

Engels, Fingberg, Karsch, Miller, and Weber
Phys. Lett. B (1990)

$$\log \mathcal{Z}(\beta, V) \longrightarrow \int_{x_0}^{x} dx' \, \frac{\partial \log \mathcal{Z}}{\partial x'} \quad \text{usually} \quad x = \frac{2N}{g_0^2} \longrightarrow \frac{f}{T^4} - \frac{f_0}{T_0^4} = \beta^4 \int_{x_0}^{x} \frac{dx'}{x'} \left[\langle s \rangle_0 - \langle s \rangle \right]$$

New problems for an old solution

theory of special relativity → relativistic thermodynamics ← thermodynamics and statistical mechanics

$$T = \gamma T_0 \qquad s = \gamma s_0 \qquad e = \gamma^2(e_0 + p_0\vec{v}^2) \qquad f = \gamma^2 f_0$$

$$V = \frac{V_0}{\gamma} \qquad p = p_0 \qquad w = e + p = \gamma^2 w_0 \qquad \gamma(\vec{v}) = \frac{1}{\sqrt{1-\vec{v}^2}}$$

energy-momentum tensor

$$T_{\mu\nu} = \begin{pmatrix} e_0 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix} \Rightarrow \begin{aligned} T_{ij} &= \gamma^2 w_0 v_i v_j + p\delta_{ij} \\ T_{00} &= \gamma^2 w_0 - p = e \\ T_{0k} &= \gamma^2 w_0 v_k \end{aligned}$$

Landau, Lifschiz, vol 6, "Fluid mechanics"

rewriting:

$$s_0 = \frac{w_0}{T_0} = \frac{T_{0k}}{T_0\gamma^2 v_k} \Rightarrow \frac{s_0}{T_0^3} = \frac{T_{0k}}{T_0^4\gamma^2 v_k} = \frac{\beta^4(1-\vec{v}^2)^3}{v_k}T_{0k} \qquad \frac{1}{T_0} = \beta_0 = \beta\sqrt{1-\vec{v}^2}$$

$$T_{0k} = \frac{v_k}{1+v_k^2}(T_{00} + T_{kk})$$

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$$T_{0k} = \frac{v_k}{1+v_k^2}(T_{00} + T_{kk})$$

the entropy density in the rest frame is related to the momentum T_{0k} and the time extent β in the moving frame

argument is general: applies to any system at equilibrium

This approach can be applied to a thermal quantum field theory in the euclidean path-integral formulation

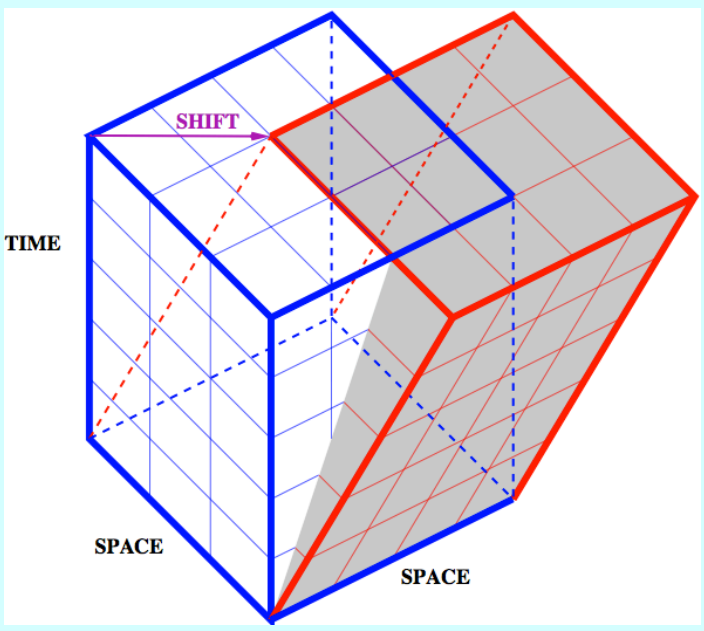
L. Giusti and H. Meyer,
PRL 2011, JHEP 2011 and 2013

$$\phi(\beta, \vec{x}) = \phi(0, \vec{x} - \beta\vec{\xi})$$

The energy-momentum tensor contains the currents associated to translation invariance

Use the machinery of quantum fields to produce Ward identities generated by Lorentz invariance

$$\mathcal{Z}(\beta, \vec{v}) = \mathrm{Tr}\left(\mathrm{e}^{-\beta(H - \vec{v}\cdot\vec{P})}\right) \xrightarrow[\vec{v} \to i\vec{\xi}]{} \mathcal{Z}(\beta, \vec{\xi}) = \mathrm{Tr}\left(\mathrm{e}^{-\beta(H - i\vec{\xi}\cdot\vec{P})}\right)$$



$$f(\beta\sqrt{1 + \vec{\xi}^2}, 0) = -\frac{1}{\beta V} \log \mathcal{Z}(\beta, \vec{\xi})$$

$$\frac{s_0}{T_0^3} = -\frac{\beta^4 (1 + \vec{\xi}^2)^3}{\xi_k} \langle T_{0k} \rangle_\xi$$

$$s_0 = -\frac{1}{V\gamma^3 \xi_k} \frac{\partial}{\partial \xi_k} \log \mathcal{Z}(\beta, \vec{\xi})$$

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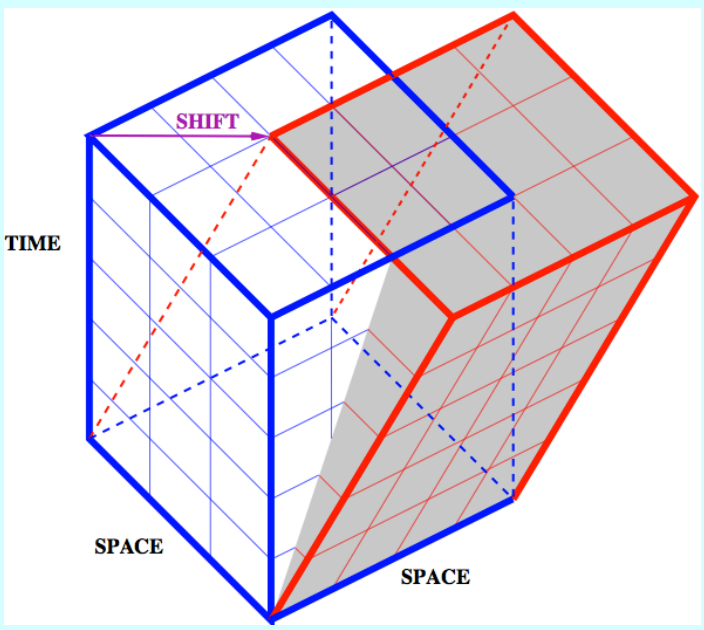


$$f(\beta\sqrt{1 + \vec{\xi}^2}, 0) = -\frac{1}{\beta V} \log \mathcal{Z}(\beta, \vec{\xi})$$

$$\frac{s_0}{T_0^3} = -\frac{\beta^4 (1 + \vec{\xi}^2)^3}{\xi_k} \langle T_{0k} \rangle_\xi Z$$

lattice regularization:
breaking of translation invariance

$$s_0 = -\frac{1}{V\gamma^3 \xi_k} \frac{\partial}{\partial \xi_k} \log \mathcal{Z}(\beta, \vec{\xi})$$

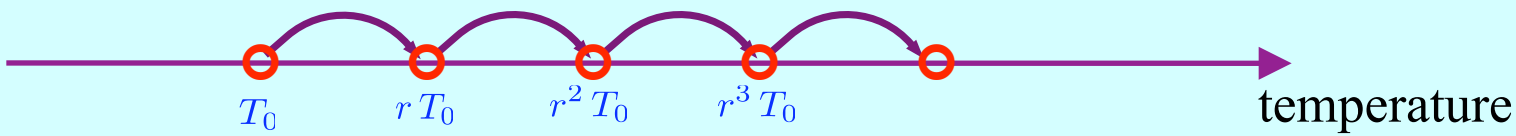
Equation of State: entropy vs. temperature

- Strategy 1: two shifts ξ and ξ', measure the entropy density step-scaling function

$$\Sigma(T,r) = \frac{T^3 s(T')}{T'^3 s(T)} = \frac{(1+\vec{\xi}'^2)^3 \xi_k}{(1+\vec{\xi}^2)^3 \xi'_k} \frac{\langle T_{0k}\rangle_{\xi'}}{\langle T_{0k}\rangle_{\xi}}$$

$$r = \frac{T'}{T} = \frac{\sqrt{1+\vec{\xi}^2}}{\sqrt{1+\vec{\xi}'^2}}$$

step in the temperature



the renormalization factor Z drops.

- Strategy 2: only one shift and measure $\langle T_{0k}\rangle_\xi$ and Z

$$\frac{s_0}{T_0^3} = -\frac{\beta^4(1+\vec{\xi}^2)^3}{\xi_k}\langle T_{0k}\rangle_\xi Z$$

need to know Z for many values of g_0^2.

Strategy 1: entropy step-scaling function

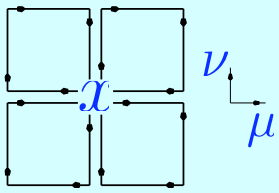
we consider the shifts: ξ=(1,1,1) and ξ'=(1,0,0) ➡ $r = \frac{T'}{T} = \frac{\sqrt{1+\vec{\xi}^2}}{\sqrt{1+\vec{\xi}'^2}} = \sqrt{2}$

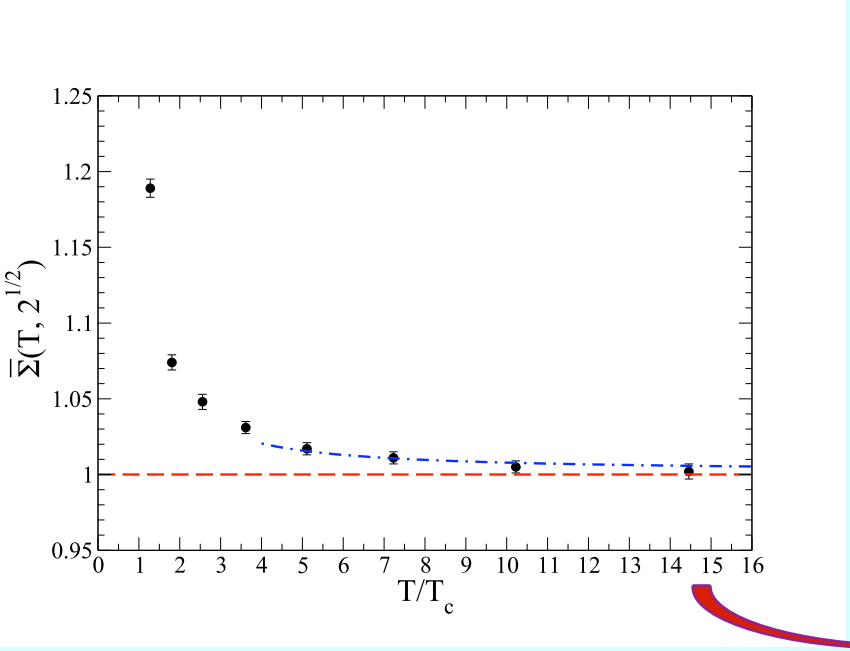
good compromise: small step (fine T dependence) and big step (broad range in T)

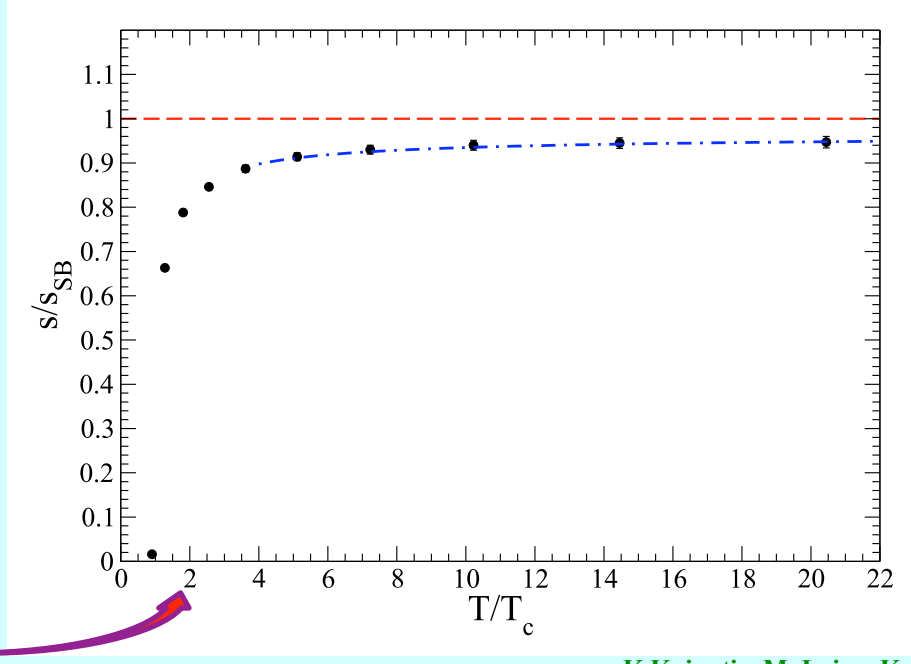
$$\Sigma(T,\sqrt{2}) = \frac{1}{8}\frac{\langle T_{0k}\rangle_{\xi=(1,0,0)}}{\langle T_{0k}\rangle_{\xi=(1,1,1)}}$$

$$T_{\mu\nu} = -\frac{2}{g_0^2}\left[\mathrm{Tr}\left(F_{\mu\alpha}(x)F_{\nu\alpha}(x)\right) - \frac{1}{4}\delta_{\mu\nu}\mathrm{Tr}\left(F_{\alpha\beta}(x)F_{\alpha\beta}(x)\right)\right]$$

$$F_{\mu\nu}(x) = \frac{1}{8}\left[Q_{\mu\nu}(x) - Q_{\nu\mu}(x)\right]_{traceless}$$

$Q_{\mu\nu}(x) = \sum$ 





$$v_0(T_0) = \frac{s(T_0)}{T_0^3} \qquad v_{k+1}(T_{k+1}) = \Sigma(T_k,\sqrt{2})\, v_k(T_k) \qquad T_k = (\sqrt{2})^k T_0$$

K.Kajantie, M. Laine, K. Rummukainen and Y. Schroder Phys. Rev. D (2003)

Strategy 2: direct measure of the entropy

We consider only one shift: ξ=(1,0,0)

$$\frac{s_0}{T_0^3} = -\frac{\beta^4(1+\vec{\xi}^2)^3}{\xi_k}\langle T_{0k}\rangle_\xi Z$$

More appealing: no constraint on the temperature, any value is fine

- measurement of $\langle T_{0k}\rangle_\xi$: OK, no news.
- measurement of Z : need to know $Z(g_0^2)$ not only some values

conserved charge: $$s_0 = -\frac{1}{V\gamma^3\xi_k}\frac{\partial}{\partial\xi_k}\log\mathcal{Z}(\beta,\vec{\xi})$$

$$Z = \frac{1}{V}\frac{1}{\langle T_{0k}\rangle_\xi}\frac{\partial\log\mathcal{Z}(\beta,\vec{\xi})}{\partial\xi_k} \quad \Rightarrow \quad Z = \frac{1}{2V}\frac{1}{\langle T_{0k}\rangle_\xi}\log\frac{\mathcal{Z}(\beta,\vec{\xi}+\frac{a}{\beta}\hat{k})}{\mathcal{Z}(\beta,\vec{\xi}-\frac{a}{\beta}\hat{k})}$$

D. Robaina and H. Meyer, PoS Lattice2013 (2013).

L. Giusti and M.P., arXiv:1403.0360

ratio of partition functions: very challenging to measure, hard for large volumes

$$\frac{Z_1}{Z_2} = \frac{Z_1}{Z_{1+\epsilon}}\frac{Z_{1+\epsilon}}{Z_{1+2\epsilon}}\frac{Z_{1+2\epsilon}}{Z_{1+3\epsilon}}\cdots\frac{Z_{2-2\epsilon}}{Z_{2-\epsilon}}\frac{Z_{2-\epsilon}}{Z_2}$$

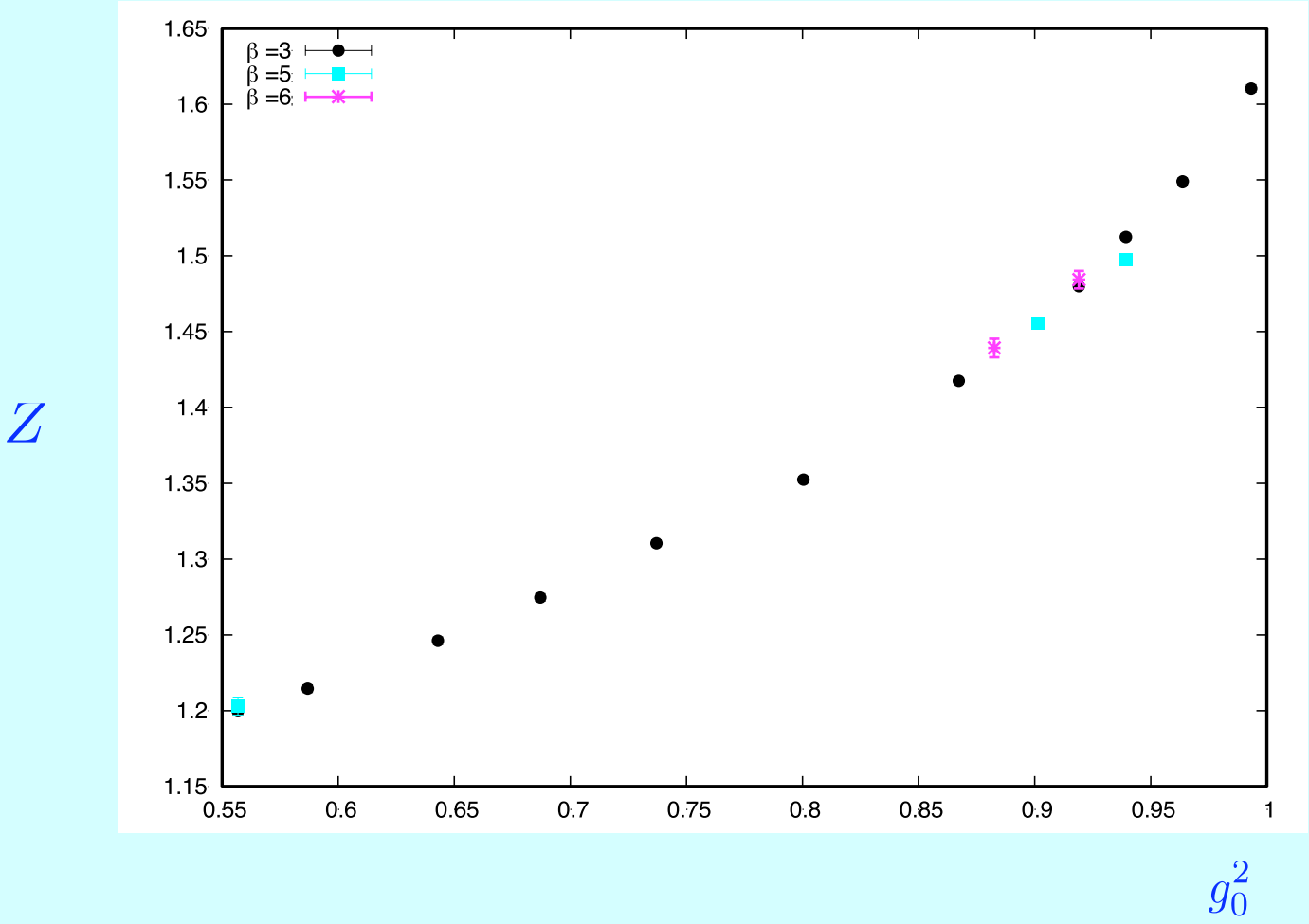
P. de Forcrand, M. D'Elia, M. P., Phys. Rev. Lett. 2001.

M. Della Morte, L. Giusti, Comp. Phys. Comm. 2009.

$$S[U,r_i] = r_i\, S[U,\vec{\xi}+\frac{a}{\beta}\hat{k}] + (1-r_i)\, S[U,\vec{\xi}-\frac{a}{\beta}\hat{k}] \quad \Rightarrow \quad \log\frac{\mathcal{Z}(\beta,\vec{\xi}+\frac{a}{\beta}\hat{k})}{\mathcal{Z}(\beta,\vec{\xi}-\frac{a}{\beta}\hat{k})} = \sum_{i=0}^{n-1}\log\frac{\mathcal{Z}(\beta,r_i)}{\mathcal{Z}(\beta,r_{i+1})}$$

$$r_i = \frac{i}{n} \qquad i=0,\ldots,n \qquad \log\frac{\mathcal{Z}(\beta,r_i)}{\mathcal{Z}(\beta,r_{i+1})} = \langle\, \exp\left[S[U,r_{i+1}] - S[U,r_i]\right]\,\rangle_{r_i}$$

we have used the SU(3) Wilson action on a lattice with L=16



The method works well but:

- increasing the spatial volume is very challenging
- we need $Z(g^2_0)$ accurately for many values of g^2_0

Compute Z in perturbation theory ➔ function of g_0^2

$$\frac{1}{2V}\log\frac{\mathcal{Z}(\beta,\vec{\xi}+\frac{a}{\beta}\hat{k})}{\mathcal{Z}(\beta,\vec{\xi}-\frac{a}{\beta}\hat{k})} = c_0 + c_1 g_0^2 + c_2 (g_0^2)^2 + \dots$$

$$Z = \frac{c_0 + c_1 g_0^2 + c_2 (g_0^2)^2 + \dots}{a_0 + a_1 g_0^2 + a_2 (g_0^2)^2 + \dots} - \frac{c_0}{a_0} + 1$$

$$\langle T_{0k}\rangle_\xi = a_0 + a_1 g_0^2 + a_2 (g_0^2)^2 + \dots$$

G. Parisi and Y. Wu, Sci. Sin. (1981)

F. DiRenzo, E. Onofri, G. Marchesini, P. Marenzoni, Nucl. Phys. B (1994)

Numerical Stochastic Perturbation Theory

$$\phi(x) \to \phi(x,\tau) \qquad \phi(x,\tau) = \sum_{n=0} \phi_n(x,\tau) g_0^n$$

$$\frac{\partial\phi(x,\tau)}{\partial\tau} = -\frac{\delta S[\phi]}{\delta\phi(x,\tau)} + \eta(x,\tau)$$

equal order in g_0 ➔ cascade of eqs. for ϕ_n

$$\mathcal{O}[\phi(x)] \to \mathcal{O}[\phi(x,\tau)] = \sum_{n=0} \mathcal{O}_n[\phi(x,\tau)] g_0^n \quad \Rightarrow \quad \langle\mathcal{O}\rangle = \langle\mathcal{O}_0\rangle + \langle\mathcal{O}_1\rangle g_0 + \langle\mathcal{O}_2\rangle g_0^2 + \dots$$

Calculation of: a_0, a_1, a_2, c_0, c_1, c_2 at β=3 and L=24 and 48

Compute Z in perturbation theory ➔ function of g_0^2

$$\frac{1}{2V}\log\frac{\mathcal{Z}(\beta,\vec{\xi}+\frac{a}{\beta}\hat{k})}{\mathcal{Z}(\beta,\vec{\xi}-\frac{a}{\beta}\hat{k})} = c_0 + c_1 g_0^2 + c_2 (g_0^2)^2 + \dots$$

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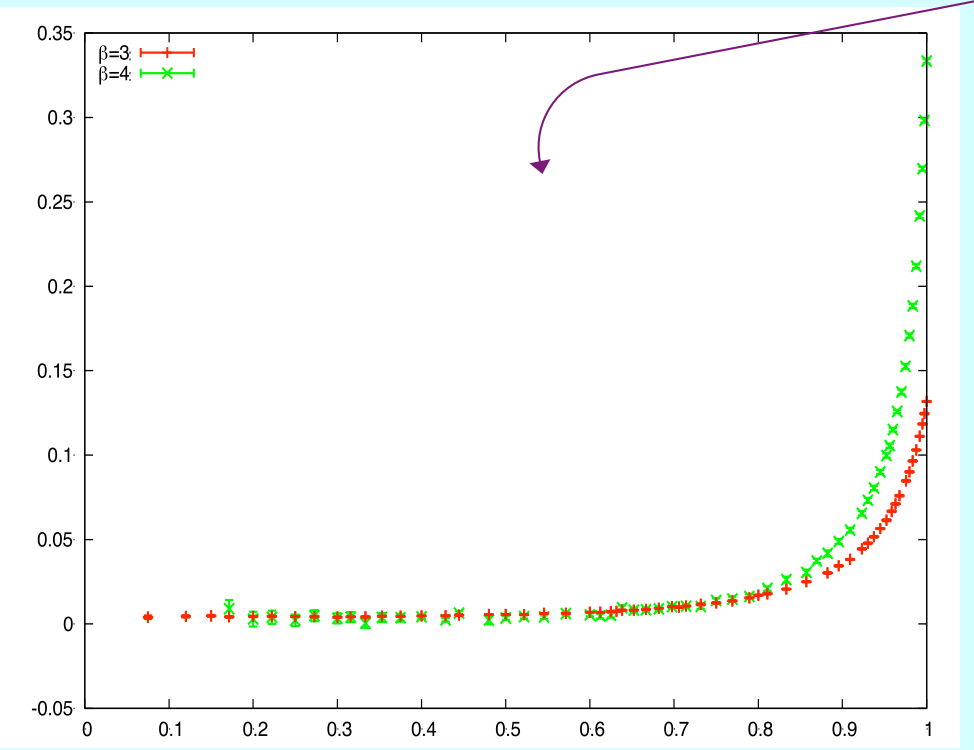
Calculation of: a_0, a_1, a_2, c_0, c_1, c_2 at β=3 and L=24 and 48

very strong finite size corrections

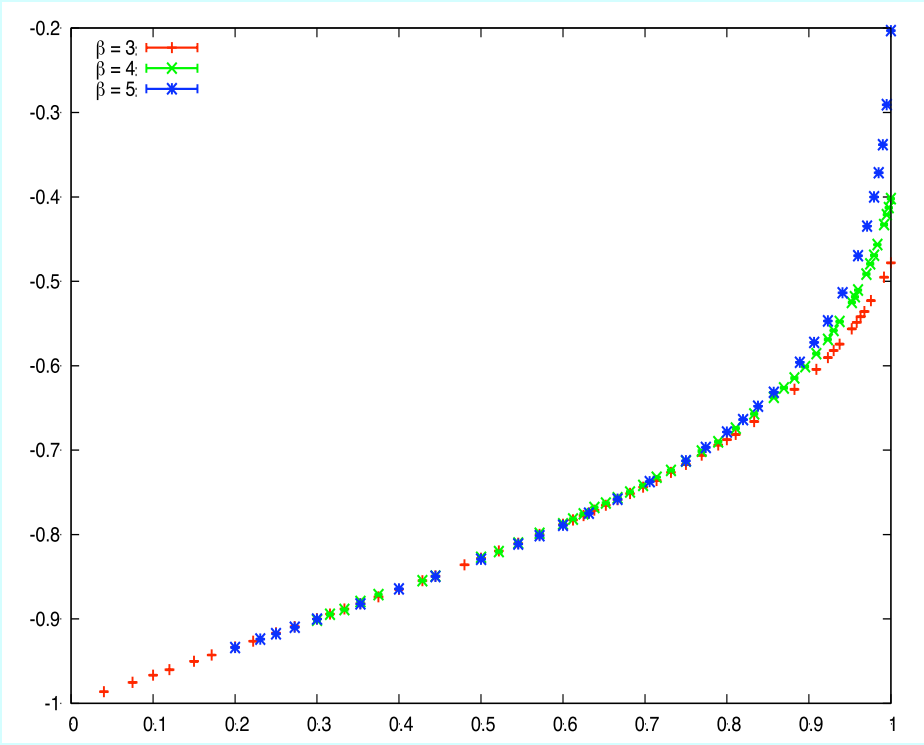
relevant higher order corrections

alternative approach: non-perturbative measure of Z + function of g_0^2

$$\frac{1}{2V}\log\frac{\mathcal{Z}(\beta,\vec{\xi}+\frac{a}{\beta}\hat{k})}{\mathcal{Z}(\beta,\vec{\xi}-\frac{a}{\beta}\hat{k})} = c_0 + \frac{1}{2V}\int_0^{g_0^2} dg^2 \frac{\partial}{\partial g^2}\log\frac{\mathcal{Z}(\beta,\vec{\xi}+\frac{a}{\beta}\hat{k})}{\mathcal{Z}(\beta,\vec{\xi}-\frac{a}{\beta}\hat{k})} = c_0 + \int_0^{g_0^2} dg^2 \frac{18\beta}{(g^2)^2}\left(\langle\Box\rangle_{\vec{\xi}-\frac{a}{\beta}\hat{k}} - \langle\Box\rangle_{\vec{\xi}+\frac{a}{\beta}\hat{k}}\right)$$



g_0^2



g_0^2

Spatial size L=48

$$Z = \frac{1}{2V}\frac{1}{\langle T_{0k}\rangle_\xi}\log\frac{\mathcal{Z}(\beta,\vec{\xi}+\frac{a}{\beta}\hat{k})}{\mathcal{Z}(\beta,\vec{\xi}-\frac{a}{\beta}\hat{k})}$$

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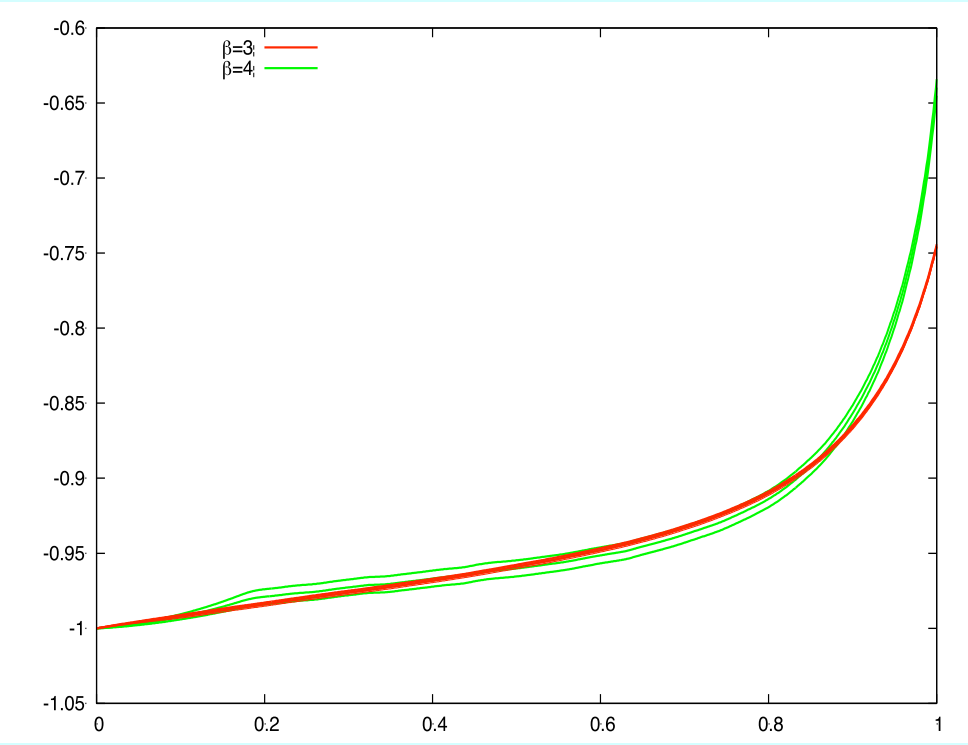

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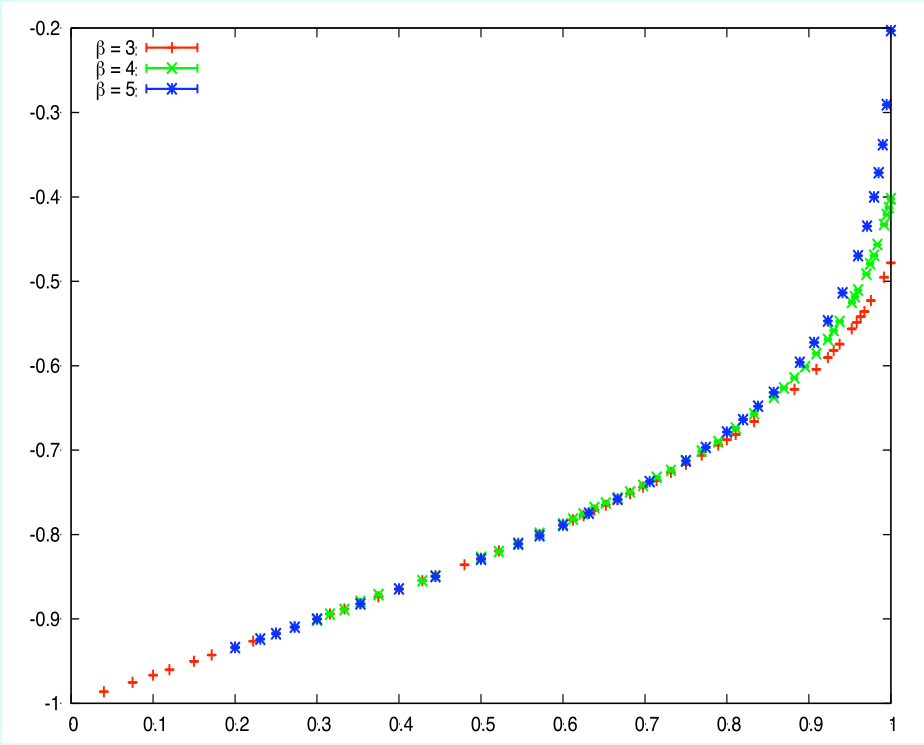

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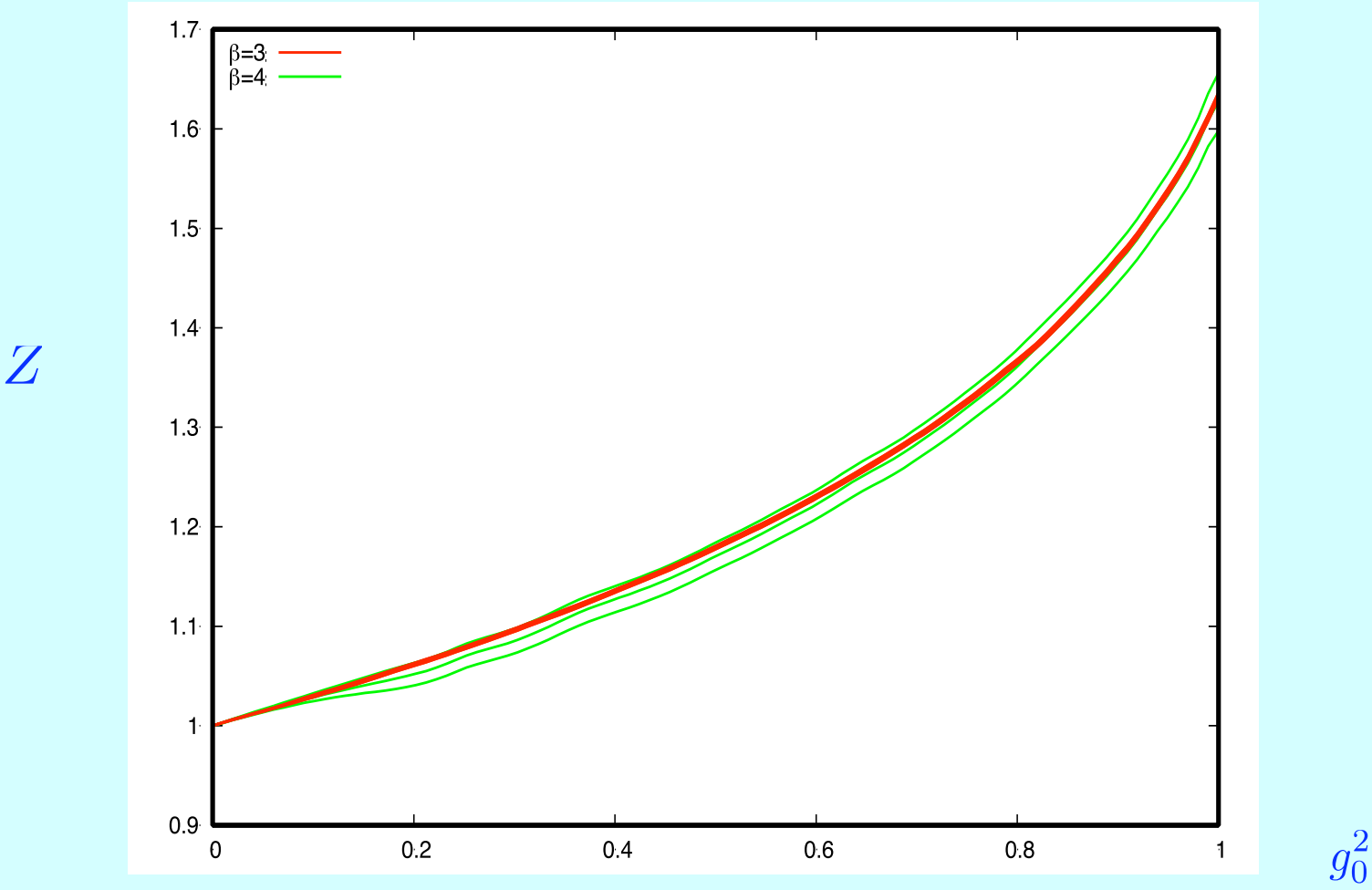
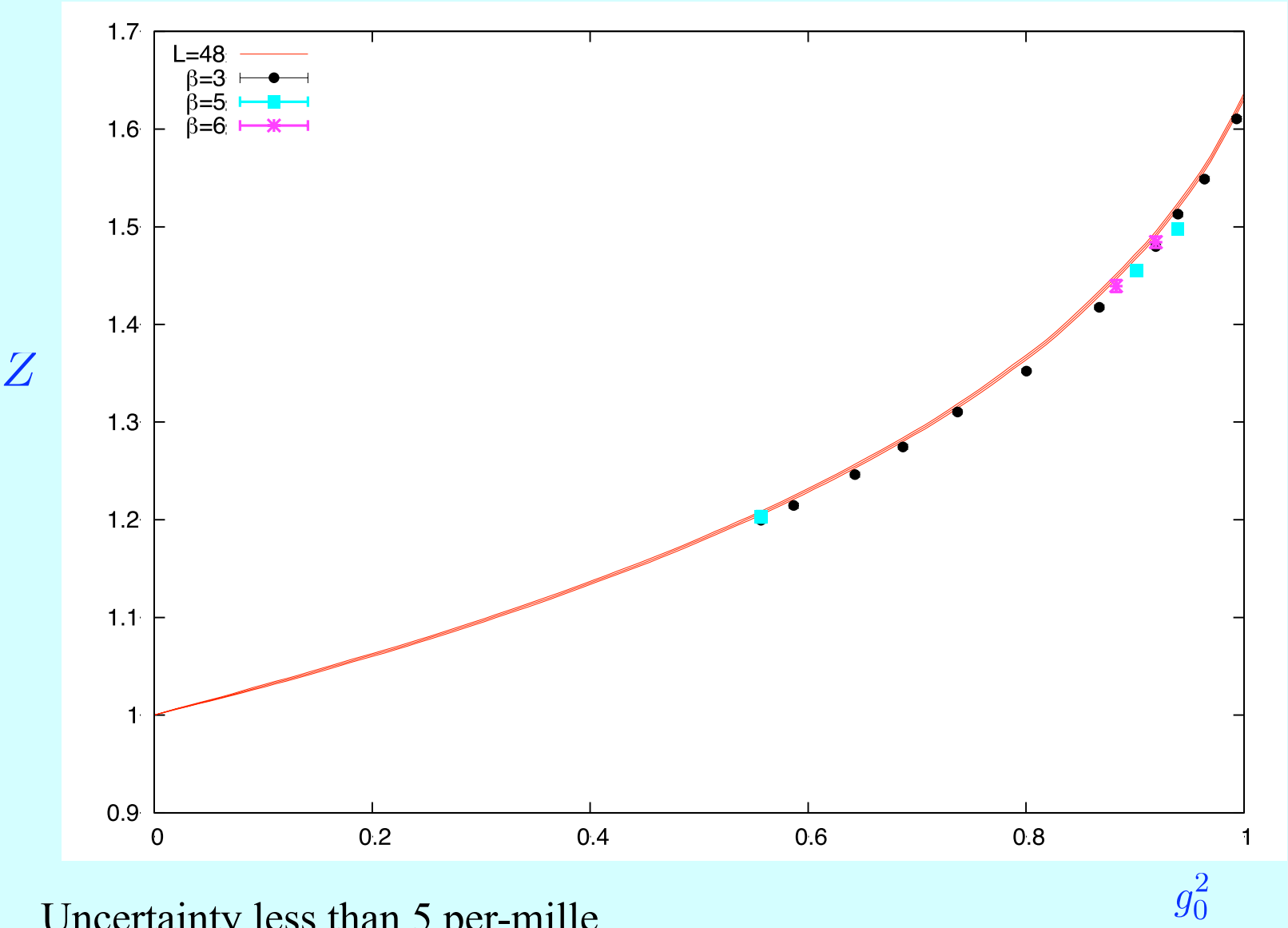


Spatial size L=48

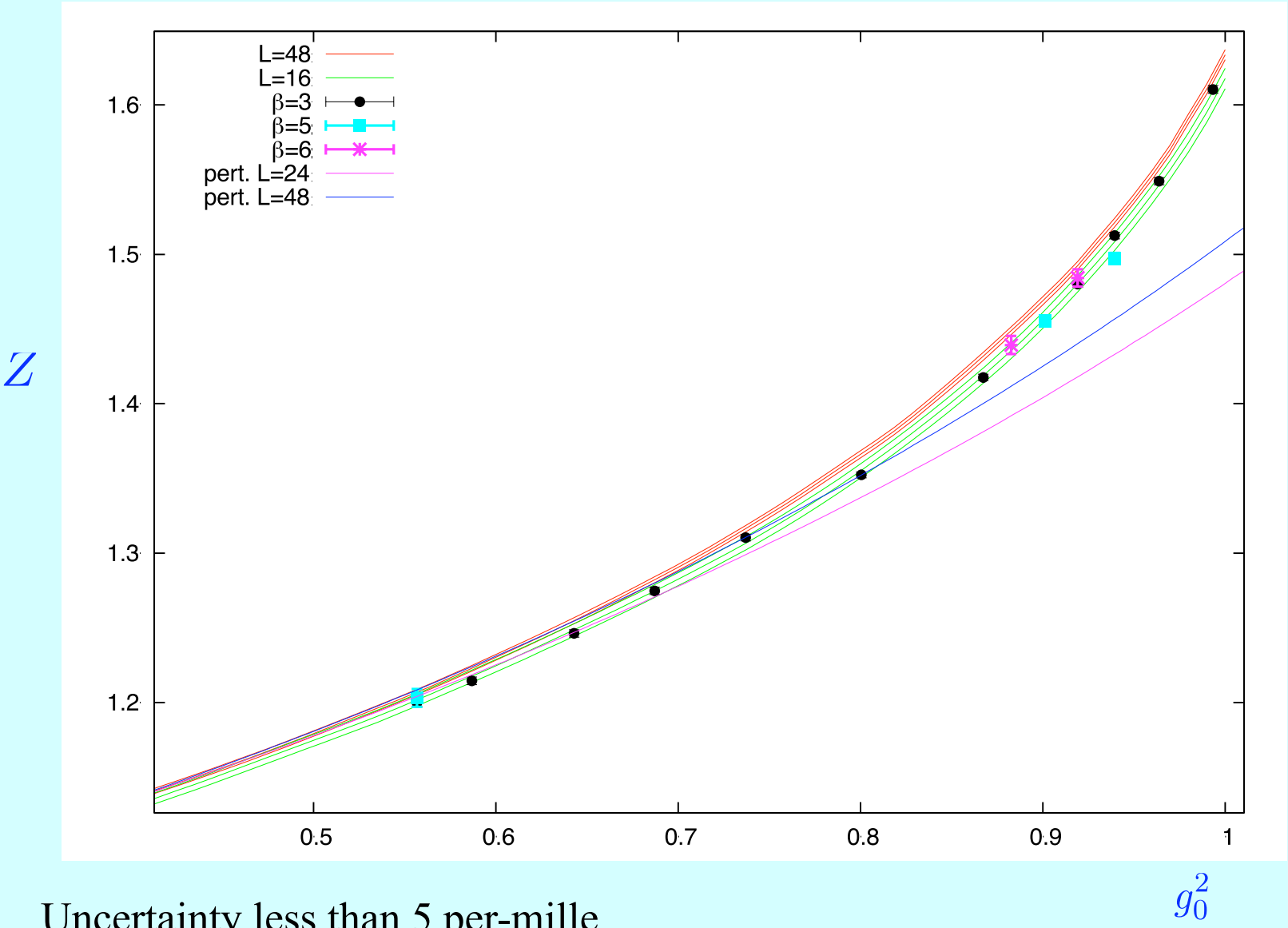
non-perturbative measure of Z



Uncertainty less than 5 per-mille

simulations in progress for β = 4 and 5

non-perturbative measure of Z



Uncertainty less than 5 per-mille

simulations in progress for β = 4 and 5

Conclusions

- A new approach to thermal field theories based on relativistic thermodynamics is investigated: far more efficient than the state-of-the-art techniques

- The step-scaling function of the entropy density has been obtained with quite moderate computational effort ➔ entropy density vs. T with 5 per-mille accuracy

- We have shown measurements of Z with an accuracy below 5 per-mille.
This introduces also another method to compute the EoS.

- The approach used here is general and can be applied to any thermal field theory: in particular to QCD (extremely challenging with the standard techniques).

What next?

- Investigate the EoS in SU(3) YM with high accuracy using this new method
- Renormalization factor of the diagonal entries of the energy-momentum tensor:

$$z = \frac{v_k}{1+v_k^2}\frac{\langle T_{00}+T_{kk}\rangle}{\langle T_{0k}\rangle}$$

- Thermodynamics of QCD