Chiral symmetry restoration at strong coupling?

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based on work with

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Outline

- Brief history of the QCD phase diagram as a function of N_f at $g = \infty$ and $T = 0$
- Calculating $\langle \bar{\psi}\psi \rangle$ diagrammatically
- Calculating group integrals using Young Projectors
- Results

Introduction: $\langle\bar{\psi}\psi\rangle$ at $g = \infty$

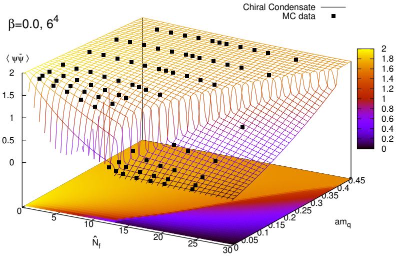
For $N_f = 0$

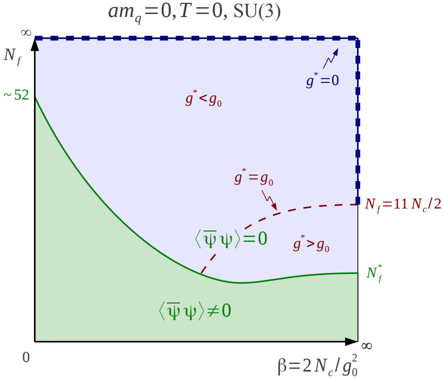
$$\langle\bar{\psi}\psi\rangle \neq 0\,.$$

What happens as N_f is increased?
Could the chiral symmetry be restored as we see from simulations at more moderate coupling strengths?

- Using a $1/d$ expansion to calculate $\langle\bar{\psi}\psi\rangle$ analytically [Kluberg-Stern, Morel, Petersson 1982] find that there is no phase transition to a phase in which $\langle\bar{\psi}\psi\rangle = 0$ for any N_f
- A mean field analysis based on [Damgaard, Hochberg, Kawamoto 1985] also suggests that the critical temperature $T_c \neq 0$ for all N_f
- Using Monte-Carlo simulations [de Forcrand, Kim, Unger 2013] find that a transition does occur, around $N_f \sim 13$ staggered fermion flavours
- Using a diagrammatic approach [Tomboulis 2013] also finds that a transition occurs, around $N_f \sim 10.7$ staggered flavours

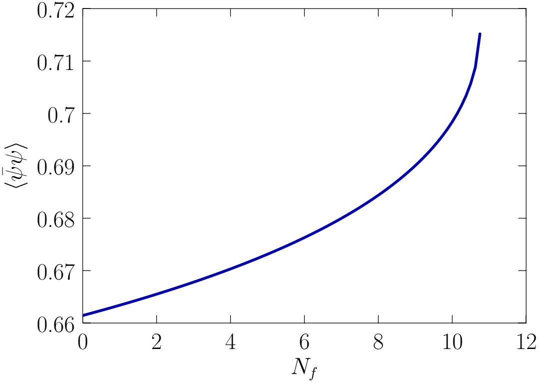
Simulation results [de Forcrand, Kim, Unger 2013]





$N_{fc} \sim 13$ staggered flavours.

Leading order strong coupling expansion [Tomboulis 2013]



$N_{fc} \sim 10.7$ staggered flavours for $N_c = 3$.

NOTE: This is a plot we created using the formulas below from [Tomboulis 2013].

$$\langle\bar{\psi}\psi\rangle = -\lim_{m\to 0} \mathrm{tr}\, G$$

$$\mathrm{tr}\, G = \left[m - \left(4d^2(d-1)\frac{N_f}{N_c}\left(\frac{g(m)}{2}\right)^9 - d\left(\frac{g(m)}{2}\right)\right)\right]^{-1}$$

$$g(m) = \left[m - \left(4d^2(d-1)\frac{N_f}{N_c}\left(\frac{g(m)}{2}\right)^9 - \left(\frac{2d-1}{2}\right)\left(\frac{g(m)}{2}\right)\right)\right]^{-1}$$

Calculating the chiral condensate

[Blairon, Brout, Englert and Greensite (1981); Martin and Siu (1983); Tomboulis (2013)]

The chiral condensate is obtained from

$$\langle\bar{\psi}(x)\psi(x)\rangle = -\mathrm{tr}\left[G(x,x)\right] = -\frac{1}{N_f}\partial_m \log Z\,.$$

Integrating out the fermion contribution results in

$$G(x,x) = \frac{\int \mathcal{D}U \det\left[1+K^{-1}M(U)\right]\left[\left[1+K^{-1}M(U)\right]^{-1}K^{-1}\right]_{xx}}{\int \mathcal{D}U \det\left[1+K^{-1}M(U)\right]}\,,$$

with

$$M_{xy} \equiv \frac{1}{2}\left[\gamma_\mu U_\mu(x)\delta_{y,x+\hat{\mu}} - \gamma_\mu U^\dagger_\mu(x-\hat{\mu})\delta_{y,x-\hat{\mu}}\right]\,,$$

$$K^{-1}_{xy} = m^{-1}\mathbb{I}_{N_f}\mathbb{I}_{N_c}\delta_{xy}\,.$$

The $K^{-1} \sim \frac{1}{m}$ suggests performing a hopping expansion.

Hopping expansion

Performing a hopping expansion on the fermion determinant leads to

$$\det\left[1+K^{-1}M\right] = \exp \mathrm{tr}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(K^{-1}M)^n\right] ,$$

which is a sum over closed loops.

Performing a hopping expansion on the contribution from the 2-pt correlator results in

$$\left[\left[1+K^{-1}M\right]^{-1}K^{-1}\right]_{xx} = \frac{1}{m}\left[\sum_{n=0}^{\infty}(-1)^n(K^{-1}M)^n\right]_{xx} .$$

which contains all loops that begin and end at site x.

Since $\mathrm{tr}\left[\text{odd \# of } \gamma_\mu\text{'s}\right]=0$, only contributions with n even contribute. For example, for $n=2$

$$\left[(K^{-1}M)^2\right]_{xx} = \frac{1}{(2m)^2}\sum_{\mu,\nu}\sum_{y}\left[\gamma_\mu\gamma_\nu\right]\left[U_\mu(x)\delta_{y,x+\hat{\mu}} - U_\mu^\dagger(x-\hat{\mu})\delta_{y,x-\hat{\mu}}\right]$$

$$\times\left[U_\nu(y)\delta_{x,y+\hat{\nu}} - U_\nu^\dagger(y-\hat{\nu})\delta_{x,y-\hat{\nu}}\right] .$$

Extending Martin and Siu

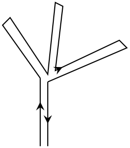
In general the chiral condensate takes the form

$$\frac{\mathrm{tr}[G(x,x)]}{N_s N_f d_R} = \frac{1}{m}\sum_{L=0}^{\infty}(-1)^L \frac{A(L)}{(2m)^{2L}}\,,$$

where $A(L)$ is the contribution of all diagrams with $2L$ links which start and end at $x = x_0$.

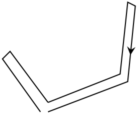
A general graph can be built out of irreducible graphs $I(l)$ of $2l$ links.

Irreducible



x_0

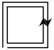
Reducible



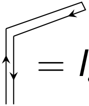
x_0

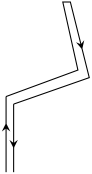
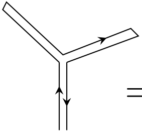
Irreducible graphs cannot be separated into smaller segments which start and end at x_0.

Irreducible diagrams

Irreducible graphs are built iteratively out of all possible combinations of smaller segments attached to a "base diagram" a) , or b) , or … .

$I(1) =$  $= I_a(1) = 2d$

$I(2) =$  $= I_a(2) = 2d\ \ [I_a(1)\ a_0']$

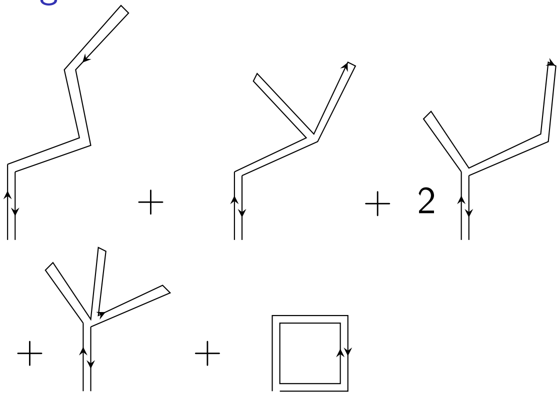
$I(3) =$  $+$  $= I_a(3)$

$$= 2d\left[I_a(2)a_0' + I_a(1)^2 a_0'^2\right]$$

with $a_0' = \frac{2d-1}{2d}$.

Irreducible diagrams

$I(4) =$



$$= I_a(4) + I_b(4)$$

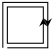
$$= 2d\left[I_a(3)a'_0 + 2I_a(1)I_a(2)a'^2_0 + I_a(1)^3 a'^3_0\right] - 4d(d-1)\frac{N_f}{N_c}$$

…

General diagrams

To obtain the contribution of all general diagrams $A(L)$ of a length $2L$, take all combinations of irreducible bits.

$$A(L) = \sum_{l=1}^{L} I(l) A(L-l)\,, \qquad L \geq 1\,; \qquad A(0) = 1\,,$$

where the irreducible graphs can begin with a) , or b) , or ... .

$$I(L) = 2dF_0(L-1) - 4d(d-1)\frac{N_f}{N_c}F_1(L-4)^7 + \ldots\,.$$

with $I(0) = 0$. $F_n(L)$ represents all possible graphs of length $2L$ which start and end on a site on a base diagram of area n.

$$F_n(L) = \sum_{\substack{l_i=1,2,\ldots,\\ k_j=4,8,\ldots,\\ \sum l_i + k_j = L-1}} I_a(l_1) I_a(l_2) \ldots I_a(l_p) I_b(k_1) I_b(k_2) \ldots I_b(k_q) a_n'^p b_n'^q\,,$$

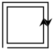
with $F_n(0) = 1$. $x_n' \equiv \frac{x_n}{d_x}$.

For example: $$a_0' = \frac{2d-1}{2d}\,, \qquad b_0' = \frac{4(d-1)^2}{4d(d-1)}\,.$$

Generating all irreducible graphs

The generating function for irreducible graphs, which gives the total contribution of all irreducible graphs including the mass dependence, is

$$W_I = \sum_{l=0}^{\infty} \left(-\frac{1}{4m^2}\right)^l I(l) = W_a + W_b + \ldots,$$

where W_a is all irreducible graphs starting with . W_b is all irreducible graphs starting with , etc. These take the form

$$W_a = 2dx \sum_{n=0}^{\infty} [a'_0 W_a + b'_0 W_b + \ldots]^n = \frac{2dx}{1 - a'_0 W_a - b'_0 W_b - \ldots},$$

$$W_b = -4d(d-1)\frac{N_f}{N_c}x^4 \left[\sum_{n=0}^{\infty} [a'_1 W_a + b'_1 W_b + \ldots]^n\right]^7 = \frac{-4d(d-1)\frac{N_f}{N_c}x^4}{(1 - a'_1 W_a - b'_1 W_b - \ldots)^7},$$

$$\ldots .$$

with $x \equiv -\frac{1}{4m^2}$. The chiral condensate is then obtained from

$$\frac{\mathrm{tr}[G(x,x)]}{N_s N_f N_c} = \lim_{m \to 0} \frac{1}{m}\left(\frac{1}{1 - W_I}\right).$$

Chiral limit $m \to 0$

To work directly in the massless limit it is convenient to introduce the variables $g_x \equiv -\frac{2mW_x}{d_x}$,

$$g \equiv d_a g_a + d_b g_b + \dots .$$

Taking $m \to 0$, the system of equations

$$g_a = \frac{1}{a_0 g_a + b_0 g_b + \dots},$$

$$g_b = \frac{\frac{N_f}{N_c}}{(a_1 g_a + b_1 g_b + \dots)^7},$$

$$g_c = \frac{\frac{N_f}{N_c}}{(a_2 g_a + b_2 g_b + \dots)^{11}},$$

$$\cdots,$$

can be solved numerically. The chiral condensate is then obtained from

$$\frac{\mathrm{tr}[G(x,x)]}{N_s N_f N_c} = \frac{2}{g}.$$

Calculating fundamental diagrams

To obtain the total contribution of a diagram, one must include the following

- A factor $\frac{1}{i!}(-N_f N_s)^i$, for a number of overlapping closed internal loops i,
- A mass factor $\left(-\frac{1}{4m^2}\right)^n$, for n pairs of links,
- $(-1)^k$ for k permutations of γ matrices,
- [...], containing the result obtained by performing the group integrations,
- {...}, containing the dimensionality of the graph.

Group integrals [Creutz, Cvitanovic]

Group integrals for overlapping links of the form , 
are nonzero $\forall\, N_c \equiv N$.

$$\int_{SU(N)} \mathrm{d}U \; U_a{}^d U_c^{\dagger\, b} = \frac{1}{N}\delta_c^d \delta_a^b \,,$$

$$\begin{aligned}\int_{SU(N)} \mathrm{d}U \; U_h^{\dagger\, a} U_g^{\dagger\, b} U_c{}^f U_d{}^e = \; & \frac{1}{2N(N+1)} \left(\delta_d^a \delta_c^b + \delta_c^a \delta_d^b\right) \left(\delta_h^e \delta_g^f + \delta_g^e \delta_h^f\right) \\ & + \frac{1}{2N(N-1)} \left(\delta_d^a \delta_c^b - \delta_c^a \delta_d^b\right) \left(\delta_h^e \delta_g^f - \delta_g^e \delta_h^f\right) \,.\end{aligned}$$

The group integral of  is nonzero for $SU(3)$

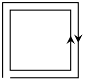
$$\int_{SU(3)} \mathrm{d}U \; U_i{}^j U_k{}^l U_m{}^n = \frac{1}{6}\epsilon_{ikm}\epsilon^{jln} \,.$$

Fundamental diagrams $L = 2, 4, 6$

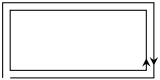
$L = 2$

 $= -\frac{1}{4m^2}\{2d\}$

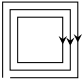
$L = 4$

 $= \left(-\frac{1}{4m^2}\right)^4 (-1)^2(-N_f)\left[\frac{1}{N_c}\right]\{4d(d-1)\}$

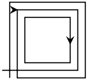
$L = 6$

 $= \left(-\frac{1}{4m^2}\right)^6 (-N_f)\left[\frac{1}{N_c}\right]\{12d(d-1)(2d-3)\}$

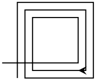
Fundamental diagrams $L = 6,\ N_c = 3$



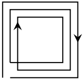
$$= \frac{1}{2!}\left(-\frac{1}{4m^2}\right)^6 (-1)^3(-N_f)^2 \left[\frac{1}{3}\right] \{4d(d-1)\}$$



$$= \left(-\frac{1}{4m^2}\right)^6 (-1)^3(-N_f) \left[-\frac{1}{3}\right] \{4d(d-1)\}$$



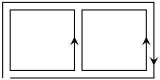
$$= \left(-\frac{1}{4m^2}\right)^6 (-1)^3 \left[\frac{1}{3}\right] \{4d(d-1)\}$$



$$= \left(-\frac{1}{4m^2}\right)^6 (-1)^3(-N_f) \left[-\frac{1}{3}\right] \{4d(d-1)\}$$

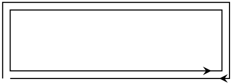
Fundamental diagrams $L = 7, 8$

$L = 7$

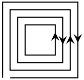


$$= \frac{1}{2!}\left(-\frac{1}{4m^2}\right)^7 (-1)^2(-N_f)^2 \left[\frac{1}{N_c^2}\right] \{12d(d-1)(2d-3)\}$$

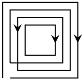
$L = 8$



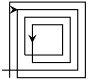
$$= \left(-\frac{1}{4m^2}\right)^8 (-1)^2(-N_f) \left[\frac{1}{N_c}\right] \{36d(d-1)(2d-3)^2\}$$



$$= \frac{3}{3!}\left(-\frac{1}{4m^2}\right)^8 (-1)^4(-N_f)^3 \left[\frac{2}{N_c}\right] \{4d(d-1)\}$$



$$= \frac{2}{2!}\left(-\frac{1}{4m^2}\right)^8 (-1)^4(-N_f)^2 [0] \{4d(d-1)\}$$



$$= \left(-\frac{1}{4m^2}\right)^8 (-1)^4(-N_f) \left[\frac{2}{N_c}\right] \{4d(d-1)\}$$

Fundamental diagrams $L > 9$

To obtain diagrams for $L > 9$ we need additional group integrals.

For example, to get to $L = 16$ for the fundamental and $N_c = 3$ we would need

$$\int_{SU(3)} \mathrm{d}U \; U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^{\dagger j} \,,$$

$$\int_{SU(N)} \mathrm{d}U \; U_a{}^b U_c{}^d U_e{}^f U_g^{\dagger h} U_i^{\dagger j} U_k^{\dagger l} \,,$$

$$\int_{SU(3)} \mathrm{d}U \; U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^j U_k{}^l \,,$$

$$\int_{SU(3)} \mathrm{d}U \; U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^j U_k^{\dagger l} U_m^{\dagger}{}^n \,,$$

$$\int_{SU(N)} \mathrm{d}U \; U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^{\dagger j} U_k^{\dagger l} U_m^{\dagger}{}^n U_o^{\dagger p} \,.$$

Note that each of the three $SU(3)$ integrals can be transformed into one of the $SU(N)$ integrals and Levi-Cevita tensors.

Fundamental diagrams $L > 9$

$$\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^{\dagger j} = \frac{1}{2}\epsilon_{gmn}\epsilon^{hkl} \boxed{\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_k^{\dagger m} U_l^{\dagger n} U_i^{\dagger j}}\,,$$

$$\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^j U_k{}^l$$

$$= \frac{1}{4}\epsilon_{im_1n_1}\epsilon^{ja_1b_1}\epsilon_{km_2n_2}\epsilon^{la_2b_2} \boxed{\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_{a1}^{\dagger m_1} U_{b1}^{\dagger n_1} U_{a2}^{\dagger m_2} U_{b2}^{\dagger n_2}}\,,$$

$$\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_i^j U_k^{\dagger l} U_m^{\dagger\ n}$$

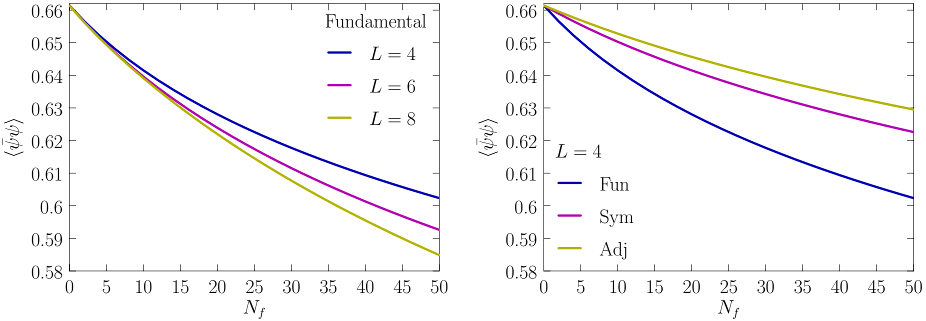
$$= \frac{1}{2}\epsilon_{iab}\epsilon^{jcd} \boxed{\int_{SU(3)} \mathrm{d}U\ U_a{}^b U_c{}^d U_e{}^f U_g{}^h U_c^{\dagger a} U_d^{\dagger b} U_k^{\dagger l} U_m^{\dagger\ n}}\,.$$

using [for $SU(3)$]

$$U_i^j = \frac{1}{2}\epsilon_{imn}\epsilon^{jkl} U_k^{\dagger m} U_l^{\dagger n}\,,$$

$$U_i^{\dagger j} = \frac{1}{2}\epsilon_{imn}\epsilon^{jkl} U_k{}^m U_l{}^n\,.$$

PRELIMINARY Results



Chiral condensate (normalised by $\frac{1}{d_R}$) for $SU(3)$ including ONLY area $n = 0$ and $n = 1$ diagrams

Conclusions and outlook

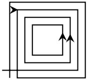
- We calculated the chiral condensate at $g = \infty$ for QCD with N_f flavours using a truncated diagrammatic expansion and find that $\langle\bar{\psi}\psi\rangle \neq 0$ at all N_f, though it approaches zero as $N_f \to \infty$.
- The expansion appears to converge for area $n = 0$ and $n = 1$ diagrams
- We calculated group integrals including up to 4 U's and 4 $U^\dagger$'s using the technique of Young projectors, which can be used to calculate diagrams up to $L = 8$ in the fundamental and $L = 4$ in the adjoint, symmetric, and antisymmetric.
- Area $n > 1$ diagrams have been calculated up to $L = 8$ but still need to be included in the calculation of the chiral condensate.

Backup slides

Issue: “diagram overlap problem”

More often than not, overlapping diagrams with nonzero area $(n > 0)$ are miscounted.

$L = 8$

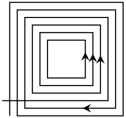
 $= \boxed{\frac{1}{2!}} \left(-\frac{1}{4m^2}\right)^{16} (-1)^4 (-N_f)^2 \boxed{[0]}$,

however, it gets counted as

$$\left(-\frac{1}{4m^2}\right)^{16} (-1)^4 (-N_f)^2 \left[\frac{1}{N_c^2}\right] .$$

Issue: "diagram overlap problem"

$L = 12$



$$= \left(-\frac{1}{4m^2}\right)^{24} (-1)^6 (-N_f)^3 \boxed{[0]} \,,$$

for $N_c \geq 3$. For $N_c = 2$ the result is $\left(-\frac{1}{4m^2}\right)^{24} (-1)^6 (-N_f)^3 \left[-\frac{1}{2}\right]$.

In either case it gets counted as

$$\left(-\frac{1}{4m^2}\right)^{24} (-1)^6 (-N_f)^3 \left[\frac{1}{N_c^3}\right] \,.$$

One can account for mis-counting at each order in L in which it appears (starting at $L = 8$).

Group integration with Young Projectors

All integrals we need can be converted to the form

$$\int_{SU(N)} \mathrm{d}U \; U_{\alpha_1}{}^{\beta_1} \ldots U_{\alpha_n}{}^{\beta_n} (U^\dagger)_{\gamma_1}{}^{\delta_1} \ldots (U^\dagger)_{\gamma_n}{}^{\delta_n}$$

Calculating the direct product of n U's ($U^\dagger$'s) leads to a direct sum of representations R (S).

The integral can be obtained from the Young Projectors $\mathbb{P}$ of these representations using

$$\int_{SU(N)} \mathrm{d}U \; R_a{}^b (S^\dagger)_c{}^d = \frac{1}{d_R} (\mathbb{P}^R)_a{}^d (\mathbb{P}^S)_c{}^b \; \delta_{RS} \, .$$

Young projectors $\mathbb{P}$

Consider for example the integral

$$I_2 \equiv \int_{\mathrm{SU}(N)} dU \, U_{\alpha_1}{}^{\beta_1} U_{\alpha_2}{}^{\beta_2} (U^\dagger)_{\gamma_1}{}^{\delta_1} (U^\dagger)_{\gamma_2}{}^{\delta_2} \, .$$

The direct product $\mathbf{N} \otimes \mathbf{N}$ is

$$\boxed{\alpha_1} \otimes \boxed{\alpha_2} = \begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \alpha_1 \\ \hline \alpha_2 \\ \hline \end{array} .$$

The Young projectors are thus formed by symmetrising, and antisymmetrising in α_1 and α_2,

$$\mathbb{P}^{S}_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \frac{1}{2}\left(\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} + \delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_2}\right) , \qquad \mathbb{P}^{AS}_{\alpha_1\alpha_2}{}^{\beta_1\beta_2} = \frac{1}{2}\left(\delta^{\beta_1}_{\alpha_1}\delta^{\beta_2}_{\alpha_2} - \delta^{\beta_2}_{\alpha_1}\delta^{\beta_1}_{\alpha_2}\right) .$$

The resulting integral is

$$I_2 = \frac{2}{N(N+1)} \mathbb{P}^{S}_{\alpha_1\alpha_2}{}^{\delta_1\delta_2} \mathbb{P}^{S}_{\gamma_1\gamma_2}{}^{\beta_1\beta_2} + \frac{2}{N(N-1)} \mathbb{P}^{AS}_{\alpha_1\alpha_2}{}^{\delta_1\delta_2} \mathbb{P}^{AS}_{\gamma_1\gamma_2}{}^{\beta_1\beta_2} \, .$$

Additional group integrals

$$I_3 \equiv \int_{\mathrm{SU}(N)} dU \, U_{\alpha_1}{}^{\beta_1} U_{\alpha_2}{}^{\beta_2} U_{\alpha_3}{}^{\beta_3} (U^\dagger)_{\gamma_1}{}^{\delta_1} (U^\dagger)_{\gamma_2}{}^{\delta_2} (U^\dagger)_{\gamma_3}{}^{\delta_3} \, .$$

with group decomposition

$$\begin{array}{|c|}\hline \alpha_1 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \alpha_2 \\ \hline \end{array} \otimes \begin{array}{|c|}\hline \alpha_3 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \end{array} \, (S) \oplus \begin{array}{|c|c|}\hline \alpha_1 & \alpha_2 \\ \hline \alpha_3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \, (M) \oplus \begin{array}{|c|c|}\hline \alpha_1 & \alpha_3 \\ \hline \alpha_2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \, (\tilde{M}) \oplus \begin{array}{|c|}\hline \alpha_1 \\ \hline \alpha_2 \\ \hline \alpha_3 \\ \hline \end{array} \, (AS) \, ,$$

results in

$$\begin{aligned} I_3 = {} & \\ & \frac{6}{N(N+1)(N+2)} \mathbb{P}^S{}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_2\delta_3} \mathbb{P}^S{}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_2\beta_3} + \frac{3}{N(N^2-1)} \mathbb{P}^M{}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_2\delta_3} \mathbb{P}^M{}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_2\beta_3} \\ & + \frac{3}{N(N^2-1)} \mathbb{P}^{\tilde{M}}{}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_2\delta_3} \mathbb{P}^{\tilde{M}}{}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_2\beta_3} + \frac{3}{N(N^2-1)} \mathbb{P}^M{}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_3\delta_2} \mathbb{P}^{\tilde{M}}{}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_3\beta_2} \\ & + \frac{3}{N(N^2-1)} \mathbb{P}^{\tilde{M}}{}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_3\delta_2} \mathbb{P}^M{}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_3\beta_2} + \frac{6}{N(N-1)(N-2)} \mathbb{P}^{AS}{}_{\alpha_1\alpha_2\alpha_3}{}^{\delta_1\delta_2\delta_3} \mathbb{P}^{AS}{}_{\gamma_1\gamma_2\gamma_3}{}^{\beta_1\beta_2\beta_3} \, . \end{aligned}$$

Higher dimensional representations

Higher dimensional representations can be written in terms of the fundamental and anti-fundamental. For example,

Symmetric

$$
\begin{aligned}
(U^S)_a{}^b = (U^S)_{(\alpha_1\alpha_2)}{}^{(\beta_1\beta_2)} &= (\mathbb{P}^S)_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2} U_{\gamma_1}{}^{\delta_1} U_{\gamma_2}{}^{\delta_2} (\mathbb{P}^S)_{\delta_1\delta_2}{}^{\beta_1\beta_2} \\
&= \frac{1}{2}\left(U_{\alpha_1}{}^{\beta_1} U_{\alpha_2}{}^{\beta_2} + U_{\alpha_1}{}^{\beta_2} U_{\alpha_2}{}^{\beta_1} \right)
\end{aligned}
$$

$a, b = 1, ..., d_S$.

Antisymmetric

$$
\begin{aligned}
(U^{AS})_m{}^n = (U^{AS})_{[\alpha_1\alpha_2]}{}^{[\beta_1\beta_2]} &= (\mathbb{P}^{AS})_{\alpha_1\alpha_2}{}^{\gamma_1\gamma_2} U_{\gamma_1}{}^{\delta_1} U_{\gamma_2}{}^{\delta_2} (\mathbb{P}^{AS})_{\delta_1\delta_2}{}^{\beta_1\beta_2} \\
&= \frac{1}{2}\left(U_{\alpha_1}{}^{\beta_1} U_{\alpha_2}{}^{\beta_2} - U_{\alpha_1}{}^{\beta_2} U_{\alpha_2}{}^{\beta_1} \right)
\end{aligned}
$$

$m, n = 1, ..., d_{AS}$.

Higher dimensional representations

Adjoint

$$(U^A)_a{}^b = 2\,\mathrm{Tr}\left(U t_a U^\dagger t^b\right) \,,$$

where the t_a are fundamental generators of $SU(N)$ satisfying

$$\mathrm{Tr}\left(t_a t_b\right) = \frac{1}{2}\delta_{ab}\,.$$

At leading order it is sufficient to use

$$\int_{SU(N)} \mathrm{d}U\; (U^R)_a{}^b (U^{R\dagger})_c{}^d = \frac{1}{d_R}\delta_a{}^d \delta_c{}^b\,.$$

$$\int_{SU(N)} \mathrm{d}U\; (U^A)_a{}^b (U^A)_c{}^d = \frac{1}{d_A}\delta_{ac}\delta^{bd}\,.$$

At the next order in the adjoint it is necessary to consider 3-link integrals.

3-link adjoint integrals

We are interested in integrals of the form

$$I_n^A \equiv \int dU\, U_{a_1}{}^{b_1} \cdots U_{a_n}{}^{b_n}$$

$$= 2^n\, (t_{a_1})_{\beta_1}{}^{\gamma_1} (t^{b_1})_{\delta_1}{}^{\alpha_1} \cdots (t_{a_n})_{\beta_n}{}^{\gamma_n} (t_{b_n})_{\delta_n}{}^{\alpha_n} \int dU\, U_{\alpha_1}{}^{\beta_1} \cdots U_{\alpha_n}{}^{\beta_n} U^{\dagger}_{\gamma_1}{}^{\delta_1} \cdots U^{\dagger}_{\gamma_n}{}^{\delta_n}$$

For example, for $n = 3$, plugging in the result for the fundamental integral and simplifying using the identity

$$t_a t_b = \frac{1}{2N} \delta_{ab} \mathbf{1}_N + \frac{1}{2} d_{abc} t_c + \frac{i}{2} f_{abc} t_c \, ,$$

results in

$$I_3^A = \frac{N}{(N^2-1)(N^2-4)} d_{a_1 a_2 a_3} d^{b_1 b_2 b_3} + \frac{1}{N(N^2-1)} f_{a_1 a_2 a_3} f^{b_1 b_2 b_3} \, .$$

where

$$i f_{abc} = 2\, \mathrm{Tr}\left([t_a, t_b] t_c\right) \, ,$$

$$d_{abc} = 2\, \mathrm{Tr}\left(\{t_a, t_b\} t_c\right) \, .$$

Bars and Green integrals [Bars and Green 1979]

Bars and Green calculate integrals of the form

$$F_n \equiv \int_{SU(N)} \mathrm{d}U \, [\mathrm{tr}(AU)]^n [\mathrm{tr}(A^\dagger U^\dagger)]^n$$

$$= \sum_{\substack{i_1,\ldots,i_n, \\ j_1,\ldots,j_n, \\ k_1,\ldots,k_n, \\ l_1,\ldots,l_n}} A_{i_1}{}^{j_1} \ldots A_{i_n}{}^{j_n} (A^\dagger)_{k_1}{}^{l_1} \ldots (A^\dagger)_{k_n}{}^{l_n} \int_{\mathrm{SU}(N)} \mathrm{d}U \, U_{j_1}{}^{i_1} \ldots U_{j_n}{}^{i_n} (U^\dagger)_{l_1}{}^{k_1} \ldots (U^\dagger)_{l_n}{}^{k_n}$$

This integral is a generating function for the types of integrals we are interested in.

One can obtain our integrals by separating out the $A_{i_1}{}^{j_1} \cdots A_{i_n}{}^{j_n} (A^\dagger)_{k_1}{}^{l_1} \cdots (A^\dagger)_{k_n}{}^{l_n}$ from each term in the results of [Bars and Green 1979], followed by symmetrising all of the i,j pairs, and k,l pairs.

The benefit of the Young projector technique is that the coefficients of each term are easier to determine. We have checked our results against Bars and Green up to $n = 4$.