Background Field Method and NRQED matching

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JWL & B.C.Tiburzi, in progress

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Low-energy EM properties of hadrons
(magnetic moment, charge radius, electric & magnetic polarizabilities, ...)

Compton scattering

Lamb shift (2S(1/2)-2P(1/2))

Proton radius puzzle

muonic hydrogen spectrum

Proton polarizability effect is the main source of uncertainty.

e-p scattering & hydrogen spectrum

7σ

O(α^5 m^4)
Lattice QCD approach

Current insertion method

\[ F(q^2) = Z + \frac{1}{3!} q^2 \langle r^2 \rangle + \cdots \]

Background field method

(1) calculate lattice two-point correlation functions in the presence of classical external EM fields.
(2) compare with the EFT predictions.

e.g) Energy shift by a uniform electric field for neutron

- Zero E field
- Nonzero E field

NRQED matching for hadron

**NRQED action density**

$$\mathcal{L} = \phi^\dagger \left[ iD_0 + c_2 \frac{\vec{D}^2}{2M} + c_D \frac{\vec{\nabla} \cdot \vec{E}}{8M^2} + c_4 \frac{\vec{D}^4}{8M^3} + ic_M \frac{\{D^i, [\vec{\nabla} \times \vec{B}]^i\}}{8M^3} \right] \phi$$

Caswell & Lepage (1986)
Kinoshita & Nio (1996)
Manohar (1997)

**One- & two-photon matching**

$$c_D = \frac{4}{3} M^2 \langle r \rangle^2$$

$$16\pi M^3 \alpha_E = Z c_D - c_A_1 - \frac{1}{2} c_A_2$$

Hill & Paz (2011)
Lee & Tiburzi (2014)

**Two-point function**

$$-\delta E(\vec{E}) = - \left( c_{A_1} + \frac{1}{2} c_{A_2} \right) \frac{\vec{E}^2}{8M^3}$$

$$-\delta E(\vec{E}) = \frac{1}{2} \left[ 4\pi \alpha_E - \frac{Z}{3M} \langle r^2 \rangle \right] \vec{E}^2$$

charge radius
NRQED matching for hadron

NRQED action density

\[ \mathcal{L} = \phi^\dagger \left[ iD_0 + c_2 \frac{\vec{D}^2}{2M} + c_D \frac{\vec{\nabla} \cdot \vec{E}}{8M^2} + c_4 \frac{\vec{D}^4}{8M^3} + ic_M \{ D^i, [\vec{\nabla} \times \vec{B}]^i \} \right. \]

\[ + c_{A_1} \frac{\vec{B}^2 - \vec{E}^2}{8M^3} - c_{A_2} \frac{\vec{E}^2}{16M^3} \left. \right] \phi \]

One- & two-photon matching

\[ c_D = \frac{4}{3} M^2 \langle r \rangle^2 \]

\[ 16\pi M^3 \alpha_E = Z c_D - c_{A_1} - \frac{1}{2} c_{A_2} \]

Two-point function

\[ -\delta E(\vec{E}) = - \left( c_{A_1} + \frac{1}{2} c_{A_2} \right) \frac{\vec{E}^2}{8M^3} \]

\[ -\delta E(\vec{E}) = \frac{1}{2} \left[ 4\pi \alpha_E - \frac{Z}{3M} \langle r^2 \rangle \right] \vec{E}^2 \]

charge radius (virtual)

\[ \text{Inconsistent!} \]
EFT for Background field calculation

- Philosophy of EFT
  - Symmetry  
  - Power countings
  - No surface terms
  - Equations of motion

Minimize the number of parameters

Conventional effective theories were developed with an assumption of the on-shell condition, where operators related by equations of motion are redundant and eliminated.

With background external fields, the on-shell condition might be lost.
Equation-of-motion operators

- **Toy model: relativistic scalar QED**

\[ \mathcal{L} = D_\mu \Phi^\dagger D^\mu \Phi - M^2 \Phi^\dagger \Phi + \frac{C_1}{2M^4} \Phi^\dagger \Phi \partial^2 F^2 + \frac{C_2}{M^4} F^2 \left( D_\mu \Phi^\dagger D^\mu \Phi - M^2 \Phi^\dagger \Phi \right) \]

**On-shell:** Fields satisfy the classical equations of motion.

**Off-shell:** Unclear at a first glance. Loop calculations (complicate) are involved.
Equation-of-motion operators

- Toy model: relativistic scalar QED

\[
\mathcal{L} = D_\mu \Phi^\dagger D^\mu \Phi - M^2 \Phi^\dagger \Phi + \frac{C_1}{2M^4} \Phi^\dagger \Phi \partial^2 F^2 + \frac{C_2}{M^4} F^2 \left( D_\mu \Phi^\dagger D^\mu \Phi - M^2 \Phi^\dagger \Phi \right)
\]

Field redefinition

\[
\Phi = \left(1 - \frac{C_2}{2M^4} F^2 \right) \Phi'
\]

\[
C' = C_1 + C_2
\]

\[
\mathcal{L}' = D_\mu \Phi' D^\mu \Phi' - M^2 \Phi'^\dagger \Phi' + \frac{C'}{2M^4} \Phi'^\dagger \Phi' \partial^2 F^2
\]

\[
C'
\]
is chosen for the reduced action to reproduce S-matrix elements.
Equation-of-motion operators

**External field:** Explicit time-dependence may prevent for the two-point function to develop a single particle pole. We should appeal to the Green’s function to resolve the parameters $C_1$ and $C_2$.

\[
\mathcal{G}'(x, y) = \langle 0 | T\{\Phi'(x)\Phi'^+(y)\} | 0 \rangle
\]

\[
\mathcal{G}(x, y) = \langle 0 | T\{\Phi(x)\Phi^+(y)\} | 0 \rangle
\]

\[
= \left[ 1 - \frac{C_2}{2M^4} [F^2(x) + F^2(y)] \right] \mathcal{G}'(x, y)
\]

Field redefinition

\[
\Phi = \left( 1 - \frac{C_2}{2M^4} F^2 \right) \Phi'
\]

Time dependence of Green’s function could be altered if the external filed is time dependent.

*General theory ≠ Reduced theory*
Equation-of-motion operators

External field: Explicit time-dependence may prevent for the two-point function to develop a single particle pole. We should appeal to the Green’s function to resolve the parameters $C_1$ and $C_2$.

\[
G'(x, y) = \langle 0 | T \{ \Phi'(x) \Phi'^+(y) \} | 0 \rangle
\]

\[
G(x, y) = \langle 0 | T \{ \Phi(x) \Phi^+(y) \} | 0 \rangle = \left[ 1 - \frac{C^2}{M^4 F^2} \right] G'(x, y)
\]

Fortunately, the modifications are overall constant for uniform EM fields.

\[
\Phi = \left(1 - \frac{C^2}{2M^4 F^2}\right) \Phi'
\]

field redefinition

General theory = Reduced theory

For uniform EM fields, it turns out that the relativistic effective theory for hadron is free from the subtlety of EoM operators.

No surface term & Weak field limit
EoM operators in NRQED

NRQED action density including EoM operators

$$\mathcal{L} = \phi^\dagger \left[ iD_0 + c_2 \frac{\vec{D}^2}{2M} + c_D \frac{[\vec{\nabla} \cdot \vec{E}]}{8M^2} + c_4 \frac{\vec{D}^4}{8M^3} + ic_M \left\{ D^i, [\vec{\nabla} \times \vec{B}]^i \right\} \right.$$ $$+ c_{A_1} \frac{\vec{B}^2 - \vec{E}^2}{8M^3} - c_{A_2} \frac{\vec{E}^2}{16M^3} + c_{x_0} \frac{[iD_0, \vec{D} \cdot \vec{E} + \vec{E} \cdot \vec{D}]}{8M^3} \left\{ \right\} \phi$$

On-shell process

$$\phi^\dagger \frac{[iD_0, \vec{D} \cdot \vec{E} + \vec{E} \cdot \vec{D}]}{8M^3} \phi = \phi^\dagger \frac{\vec{D}^2, \vec{D} \cdot \vec{E} + \vec{E} \cdot \vec{D}}{16M^4} \phi$$

Uniform E field

$$\vec{A} = -\mathcal{E} \tau \hat{x}_3$$

$$[iD_0, \vec{D} \cdot \vec{E} + \vec{E} \cdot \vec{D}] = -2Z\vec{E}^2$$

$$[\vec{D}^2, \vec{D} \cdot \vec{E} + \vec{E} \cdot \vec{D}] = 0$$

The time dependence of Green’s function is altered by field redefinition.

$$\phi = \left( 1 - c_{x_0} \frac{\vec{D} \cdot \vec{E} + \vec{E} \cdot \vec{D}}{8M^3} \right) \phi'$$

$$G_{\mathcal{E}}(\tau, 0) = \left( 1 + c_{x_0} \frac{ZE^2 \tau}{4M^3} \right) G'_{\mathcal{E}}(\tau, 0)$$
NRQED matching with EOM ops

Determination of $c_{X_0}$

Matching relativistic and nonrelativistic Green’s function
Nonrelativistic expansion of relativistic QED - keeping EOM ops

$$c_{X_0} = -Z - \frac{1}{2} c_D$$

Lee & Tiburzi (2014)

One- & two-photon matching

$$c_D = \frac{4}{3} M^2 \langle r \rangle^2$$

$$16\pi M^3 \alpha_E = Z c_D - c_{A_1} - \frac{1}{2} c_{A_2}$$

Two-point function

$$-\delta E(\vec{E}) = - \left( c_{A_1} + \frac{1}{2} c_{A_2} + 2Z c_{X_0} \right) \frac{\vec{E}^2}{8M^3}$$

Correct!

Extension to a spin-half hadron is straightforward by including

$$c_{X_0} \frac{[iD_0, \vec{D} \cdot \vec{E} + \vec{E} \cdot \vec{D}]}{8M^3} + c'_{X_0} \frac{[iD_0, i\vec{\sigma} \cdot (\vec{D} \times \vec{E} + \vec{E} \times \vec{D})]}{8M^3}$$
Nonrelativistic correlation functions

- Correlation functions using NRQED
  Easy & Well developed power counting
  EoM ops. are necessary.
- However, no direct comparison with lattice correlators.

Scalar case
\[ \phi(x) = \mathcal{N}(x) \Phi(x) \]
\[ \mathcal{N}(\vec{x}, \tau) = [4(M^2 - \vec{D}^2)]^{1/4} e^{M\tau} \]
\[ G_{NR}(\tau) = \sum_{\vec{x}} \langle 0 | \mathcal{N}(\vec{x}, \tau) \Phi(\vec{x}, \tau) \Phi^{\dagger}(0, 0) | 0 \rangle \]

Normalization

Spinor case
\[ \psi(\vec{x}, \tau) = e^{iS(\vec{x}, \tau)} \Psi(\vec{x}, \tau) \]
\[ G_{NR}(\tau) = \sum_{\vec{x}} \langle 0 | \mathcal{T}_+(\vec{x}, \tau) \Psi(\vec{x}, \tau) \Psi^{\dagger}(0, 0) \mathcal{T}_-(0, 0) | 0 \rangle \]
\[ \mathcal{T}_+(\vec{x}, \tau) = \frac{1 + \gamma_4}{2} e^{M\tau} e^{iS(\vec{x}, \tau)} \]
\[ \mathcal{T}_-(0, 0) = \frac{1 + \gamma_4}{2} \]

Foldy-Wouthuysen transformation
\[ S_2 = \frac{Z + \kappa}{4M^2} \gamma^0 \tilde{\gamma} \cdot \vec{E} \]

\( \mathcal{N} \) and \( \mathcal{T}_+ \) include measured parameters
\[ \rightarrow \] systematic errors

mass, anomalous magnetic moment, ...
Relativistic correlation functions

- Correlation functions using relativistic QED

Direct comparisons with lattice correlators are possible.

**Uniform & Weak EM fields:** *Life becomes much simpler thanks to equations of motion.*

**Field redefinition**

**Uniform E**

\[ \mathcal{L} = \bar{\Psi} \left[ i\dot{\Phi} - M + \frac{\kappa}{4M} \sigma_{\mu\nu} F^{\mu\nu} + 2\pi \alpha_E E^2 \right] \Psi \]

*boost projection, integral representation*


**Uniform B**

\[ \mathcal{L} = \bar{\Psi} \left[ i\dot{\Phi} - M + \frac{\kappa}{4M} \sigma_{\mu\nu} F^{\mu\nu} + 2\pi \beta_M B^2 \right] \Psi \]

*spin projection, Landau level*

Lee & Tiburzi (2014)

\( \alpha_E \) - and \( \beta_M \) - terms can be added to M trivially.

Calculate the correlation functions by treating \( \kappa \)-term exactly.
Summary & Outlook

• In presence of external fields, EoM ops can modify the time-dependence of Green’s function.

  Lack of on-shell condition

• Background field method + NRQED matching consistent

  Retaining EoM ops

• Relativistic correlation functions are practically useful for lattice QCD calculations.

  Uniform & Weak E, B fields No EoM ops, Power counting

  EFT is reduced by field redefinitions.

• Extend to hadrons with higher spins.

  spin-1, spin-3/2, ...

• Extend to non-uniform E, B field? Spin-polarizabilities

  \[ \mathcal{L}_{\text{eff}}^{(3),\text{spin}} = \psi^\dagger(\vec{x}, t)2\pi i \left[ \gamma_{E_1} \vec{\sigma} \cdot \vec{E} \times \vec{E} - \gamma_{M_1} \vec{\bar{\sigma}} \cdot \vec{B} \times \vec{B} - \gamma_{E_2} \sigma^i E^{ij} B^j - \gamma_{M_2} \sigma^i B^{ij} E^j \right] \psi(\vec{x}, t) \]