An update on the status of NSPT computations

M. Brambilla       F. Di Renzo

Università degli Studi di Parma and INFN

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OUTLINE OF THE TALK

▶ Motivation

▶ Numerical Stochastic Perturbation Theory

▶ RI-MOM’ scheme

▶ Perturbative results

▶ Resummation of PT series

▶ Conclusions

Talk based on

(accepted on EPJC)
Non-perturbative computations has been the preferred choice for quite a long time, but:

▶ strictly speaking multiplicative renormalizability is proved only in Perturbation Theory; and
▶ fermion bilinears are either finite or only logarithmically divergent. Since there are no power divergences PT must work.

**Drawbacks of PT**

▶ perturbative series are badly convergent.
  ▶ go to high order
▶ diagrammatic Lattice PT is cumbersome;
  ▶ use an automated technique
A sketch of NSPT

- Let the system evolve according Langevin dynamic in a “fictitious” time $t$

$$\partial_t U(x, t) = \{-i\nabla S[U(x, t)] - i\eta(x, t)\} U(x, t)$$

where $\langle \eta(x, t) \rangle = 0 \quad \langle \eta(x, t)\eta(x', t') \rangle = 2\delta(x - x')\delta(t - t')$.

- By expanding the link in a power series one gets a system of equations to be truncated at a given order (Stochastic PT).

- The differential equations can be traded for integral ones (in this way one would get diagrams); in out approach the integration is performed numerically on a computer.

- Inverting the fermionic (Dirac) operator turns into inverting a series:

$$M[U(x, t)]^{-1} = M^{-1(0)} + \beta^{-1/2} M^{-1(1)} + \ldots$$

$$M^{-1(0)} = M^{0(-1)}, \quad M^{-1(n)} = -M^{0(-1)} \sum_{j=0}^{n-1} M^{(n-1)} M^{(j)-1}$$
**RI-MOM’ scheme**

Starting from Green functions (in Landau gauge)

\[ G_\Gamma(p) = \int dx \langle p | \bar{\psi}(x) \Gamma \psi(x) | p \rangle \]

vertex functions are obtained by amputation

\[ \Gamma_\Gamma(p) = S^{-1}(p) G_\Gamma(p) S^{-1}(p). \]

The quark field renormalization constant has to be computed from the condition

\[ Z_q(\mu, \alpha) = -i \frac{1}{12} \left. Tr(\not{p} S^{-1}(p)) \right|_{p^2 = \mu^2}. \]

After projecting on tree-level structure

\[ O_\Gamma(p) = Tr \left( \hat{P}_O \Gamma_\Gamma(p) \right), \]

one enforces renormalization conditions that read

\[ Z_{O_\Gamma}(\mu, \alpha) Z_q^{-1}(\mu, \alpha) O_\Gamma(p) |_{p^2 = \mu^2} = 1. \]
Zero quark mass and logarithmic divergencies

In order to have a mass-independent scheme, all this is defined at zero quark mass: this requires knowledge of the critical mass (known up to 2-loop, 3-loop as a byproduct).

Critical mass is computed from the propagator:

\[
\hat{S}(\hat{p}, \hat{m}_{cr}, \beta^{-1})^{-1} = \hat{i}\hat{p} + \hat{m}_W(\hat{p}) - \hat{\Sigma}(\hat{p}, \hat{m}_{cr}, \beta^{-1})
\]

\[
\hat{\Sigma}(0, \hat{m}_{cr}, \beta^{-1}) = \hat{m}_{cr}
\]

\[
\hat{m}^{(3),tls}_{cr} = -3.94(4) \quad \hat{m}^{(3),iwa}_{cr} = -0.78(2)
\]

Advantage of RI-MOM' scheme: logarithmic contributions to quark bilinears can be inferred from continuum computations \((l = \log(\mu a)^2)\)

\[
\gamma_{O_\Gamma} = \frac{1}{2} \frac{d}{dl} \log Z_{O_\Gamma} \quad \Rightarrow \quad Z_{O_\Gamma} = 1 + \alpha \left( c_1 - \gamma_{O_\Gamma}^{(1)} l \right) + \mathcal{O}(\alpha^2)
\]
**LATTICE ARTIFACTS**

A prototypal fitting form of ours reads:

$$\hat{O}_\Gamma(\hat{p}, pL, \nu) = c_1 + c_2 \sum_\sigma \hat{p}_\sigma^2 + c_3 \frac{\sum_\sigma \hat{p}_\sigma^4}{\sum_\rho \hat{p}_\rho^2} + c_4 \hat{p}_\nu^2 + \Delta \hat{O}_\Gamma(pL) + O(a^4)$$

- the $a \to 0$ limit can be obtained by means of the hypercubic expansion;
- by computing $\hat{O}_\Gamma(\hat{p}, pL, \nu)$ on different volumes we can account for finite size corrections;
- performing a combined fit we account for the limits $a \to 0$ and $L \to \infty$ simultaneously.
Results

- $n_f=2$ tree-level Symanzik [M. B., F. Di Renzo]

\[
\begin{array}{|c|c|c|c|}
\hline
 & \text{analytical one-loop} & \text{one-loop} & \text{two-loop} \\
\hline
Z_S & -0.6893 & -0.683(7) & -0.777(24) \\
Z_P & -1.1010 & -1.098(11) & -1.299(38) \\
Z_V & -0.8411 & -0.838(6) & -0.891(17) \\
Z_A & -0.6352 & -0.633(4) & -0.611(16) \\
\hline
\end{array}
\]

- $n_f=4$ Iwasaki [M. B., F. Di Renzo, M. Hasegawa]

\[
\begin{array}{|c|c|c|c|}
\hline
 & \text{analytical one-loop} & \text{one-loop} & \text{two-loop} \\
\hline
Z_S & -0.4488 & -0.442(6) & -0.170(11) \\
Z_P & -0.7433 & -0.739(7) & -0.202(13) \\
Z_V & -0.5623 & -0.561(7) & -0.067(12) \\
Z_A & -0.4150 & -0.419(6) & -0.033(12) \\
\hline
\end{array}
\]

(results are available also for $n_f=0$)
Summing the series

We can sum the series and compare with non perturbative results (Symanzik $\beta = 4.05$) [M. Constantinou et al. JHEP08(2010)068]

<table>
<thead>
<tr>
<th></th>
<th>$Z_V$</th>
<th>$Z_A$</th>
<th>$Z_S$</th>
<th>$Z_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NSPT</td>
<td>0.710(2)(28)</td>
<td>0.788(2)(18)</td>
<td>0.753(4)(30)</td>
<td>0.601(5)(48)</td>
</tr>
<tr>
<td>ETMC(M1)</td>
<td>0.659(4)</td>
<td>0.772(6)</td>
<td>0.645(6)</td>
<td>0.440(6)</td>
</tr>
<tr>
<td>ETMC(M2)</td>
<td>0.662(3)</td>
<td>0.758(4)</td>
<td>0.678(4)</td>
<td>0.480(4)</td>
</tr>
</tbody>
</table>

(Iwasaki $\beta = 2.10$) [arXiv:1403.4504 [hep-lat]]

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<th>$Z_A$</th>
<th>$Z_S$</th>
<th>$Z_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NSPT</td>
<td>0.677(9)(39)</td>
<td>0.769(9)(25)</td>
<td>0.712(14)(36)</td>
<td>0.538(15)(63)</td>
</tr>
<tr>
<td>ETMC(M1)</td>
<td>0.655(03)</td>
<td>0.762(04)</td>
<td>0.700(06)</td>
<td>0.516(02)</td>
</tr>
<tr>
<td>ETMC(M2)</td>
<td>0.657(02)</td>
<td>0.752(02)</td>
<td>0.749(03)</td>
<td>0.545(02)</td>
</tr>
</tbody>
</table>

- thee-loop contribution is relatively important: quite large truncation errors
- fair agreement between PT and non PT for Iwasaki action and finite Symanzik
- deviation between PT and non PT in Symanzik divergent
We can assess irrelevant effects by discarding the continuum limit and finite size contributions:

\[ \tilde{O}_\Gamma^{(i)}(\hat{p}, \nu) = c_2^{(i)} \sum_{\sigma} \hat{p}_\sigma^2 + c_3^{(i)} \sum_{\sigma} \frac{\hat{p}_\sigma^4}{\sum_\rho \hat{p}_\rho^2} + c_4^{(i)} \hat{p}_\nu^2 + \mathcal{O}(a^4) \]

The resummed quantity

\[ \sum_{i=1}^{3} \beta^{-i} \frac{1}{4} \sum_{\nu=1}^{4} \tilde{O}_\Gamma^{(i)}(\hat{p}, \nu) \]

can be regarded as the irrelevant contributions to \( Z_\Gamma \)

Finite size effects can be reconstructed to a fair accuracy provided one fits terms compliant to the lattice symmetries.
**Boosting the resummations**

Re-express the series as expansions in different couplings:

can we find better convergence proprieties?

<table>
<thead>
<tr>
<th></th>
<th>$x_0 = \frac{\beta^{-1}}{\sqrt{P}}$</th>
<th>$x_1 = -\frac{1}{P^{(0)}} \log(P)$</th>
<th>$x_2 = \frac{\beta^{-1}}{P}$</th>
<th>(M1)</th>
<th>(M2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_V$</td>
<td>0.686(21)</td>
<td>0.688(17)</td>
<td>0.661(55)</td>
<td>0.659(4)</td>
<td>0.662(3)</td>
</tr>
<tr>
<td>$Z_A$</td>
<td>0.773(12)</td>
<td>0.775(9)</td>
<td>0.763(26)</td>
<td>0.772(6)</td>
<td>0.758(4)</td>
</tr>
<tr>
<td>$Z_S$</td>
<td>0.727(29)</td>
<td>0.726(27)</td>
<td>0.705(49)</td>
<td>0.645(6)</td>
<td>0.678(4)</td>
</tr>
<tr>
<td>$Z_P$</td>
<td>0.558(45)</td>
<td>0.558(41)</td>
<td>0.526(73)</td>
<td>0.440(6)</td>
<td>0.480(4)</td>
</tr>
</tbody>
</table>

where $P$ is the $1 \times 1$ plaquette.
BPT apparently solves the problem of the discrepancies for $Z_V$ and $Z_A$;

discrepancies are still there for $Z_S$ and $Z_P$:
  - should even higher order terms be included?
  - could non-perturbative computations suffer from finite volume effects (any interplay between IR and UV effects)?

**SOME GENERAL REMARK**

- we put forward a method to assess finite size effects: there is in principle no reason why one should not attempt the same in the non-perturbative case;
- high-loop computations can provide a new handle to correct non-perturbative computations with respect to irrelevant contributions.
CONCLUSIONS

We computed 2 and 3-loop Renormalization Constants for quark bilinears in different regularizations.

- NSPT provides an approach independent w.r.t. non perturbative computations (different systematic effects);
- in principle there is no constraint on computing finite constants;
- in divergent constants we are limited to 3-loop order because of continuum computations;
- NSPT provides a new method to correct non-perturbative computations with respect to irrelevant contributions.

THANK YOU FOR YOUR ATTENTION
**TAMING THE LOGS**

Z’s expansion is in the form

$$Z(\mu, \alpha_0) = 1 + \sum_{n>0} \overbar{d}_n(l) \alpha_0^n \quad \overbar{d}_n(l) = \sum_{i=0}^{n} \overbar{d}_n^{(i)} l^i.$$

By differentiating w.r.t $\log(\mu a)^2$ one obtains the anomalous dimension

$$\gamma = \frac{1}{2} \frac{d}{dl} \log Z(\mu, \alpha) = \sum_{n>0} \gamma_n \alpha(\mu)^n$$

that depends only on the scheme.

**PROCEDURE**

- match the two expansion above (all log’s must cancel out);
- re-express the expansion in the bare coupling $\alpha_0$;
- subtract divergences from Z’s before performing fits.
Consider the case of quark field renormalization constant $Z_q$. Hypercubic symmetry fixes the (expected) form of self energy:

$$\frac{1}{4} \sum_{\mu} \gamma_\mu \text{Tr}_{\text{spin}}(\gamma_\mu \hat{\Sigma}) = i \sum_{\mu} \gamma_\mu \hat{p}_\mu \left( \hat{\Sigma}^{(0)}(\hat{p}) + \hat{p}^2 \hat{\Sigma}^{(1)}(\hat{p}) + \hat{p}^4 \hat{\Sigma}^{(2)}(\hat{p}) + \ldots \right)$$

$\hat{\Sigma}^{(i)}(\hat{p})$ can be expanded in hypercubic invariants

$$\hat{\Sigma}^{(i)}(\hat{p}) = c_1^{(i)} + c_2^{(i)} \sum_{\nu} \hat{p}^2_{\nu} + c_3^{(i)} \frac{\sum_{\nu} \hat{p}^4_{\nu}}{\sum_{\nu} \hat{p}^2_{\nu}} + \mathcal{O}(a^4).$$

The only term surviving the $a \to 0$ limit is $c_1^{(0)}$. 
Finite volume effects

If there were no finite size effects, point with the same \( p_\mu = \frac{2\pi}{L} n_\mu \) should join in a perfectly smooth way.

On a dimensional ground we expect a \( pL \) dependance. We can rewrite

\[
\hat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) = \hat{\Sigma}_\gamma(\hat{p}, \infty, \bar{\mu}) + \left( \hat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) - \hat{\Sigma}_\gamma(\hat{p}, \infty, \bar{\mu}) \right)
\]

\[
\equiv \hat{\Sigma}_\gamma(\hat{p}, \infty, \bar{\mu}) + \Delta \hat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu})
\]

to a first approximation we neglect corrections on top of corrections:

\[
\Delta \hat{\Sigma}_\gamma(\hat{p}, pL, \bar{\mu}) \sim \Delta \hat{\Sigma}_\gamma(pL).
\]

Since \( p_\mu L = \frac{2\pi n_\mu}{L} L = 2\pi n_\mu \): at fixed \( n \)-tuple different lattice sizes are affected by the \( pL \) effects