Lattice Calculation of the Hadronic Light by Light Contributions to the Muon Anomalous Magnetic Moment

by Luchang Jin

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Columbia University

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1 Muon Anomalous Magnetic Moment

$$\boldsymbol{\mu}_\mu = -g_\mu \frac{e}{2m_\mu} \mathbf{s}_\mu$$

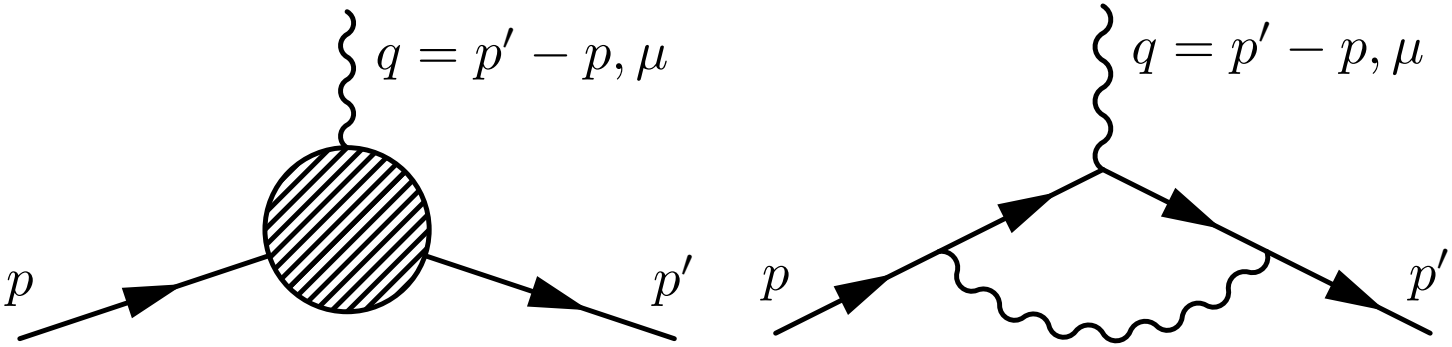

Figure 1. (L) Muon Vertex Function Diagram (R) Schwinger Term Diagram.

$$\bar{u}(p')\Gamma^\mu(p', p)u(p) = \bar{u}(p')\left[F_1(q^2)\gamma_\mu + i\frac{F_2(q^2)}{4m}[\gamma_\mu, \gamma_\nu]q_\nu\right]u(p)$$

$$F_2(0) = \frac{g_\mu - 2}{2} \equiv a_\mu$$

Schwinger's Term

Figure 2. The headstone of Julian Schwinger at Mt Auburn Cemetery in Cambridge, MA.

2 BNL E821 (0.54 ppm) and Standard Model Prediction

	Value $\pm$ Error	Reference
Experiment (0.54 ppm)	116592089 ± 63	E821, The $g-2$ Collab. 2006
Standard Model	116591828 ± 50	arXiv:1311.2198
Difference (Exp $-$ SM)	261 ± 78	
HVP LO	6949 ± 43	Hagiwara et al. 2011
Hadronic Light by Light	105 ± 26	Glasgow Consensus, 2007

Table 1. Standard model theory and experiment comparison [in units 10^{-11}]

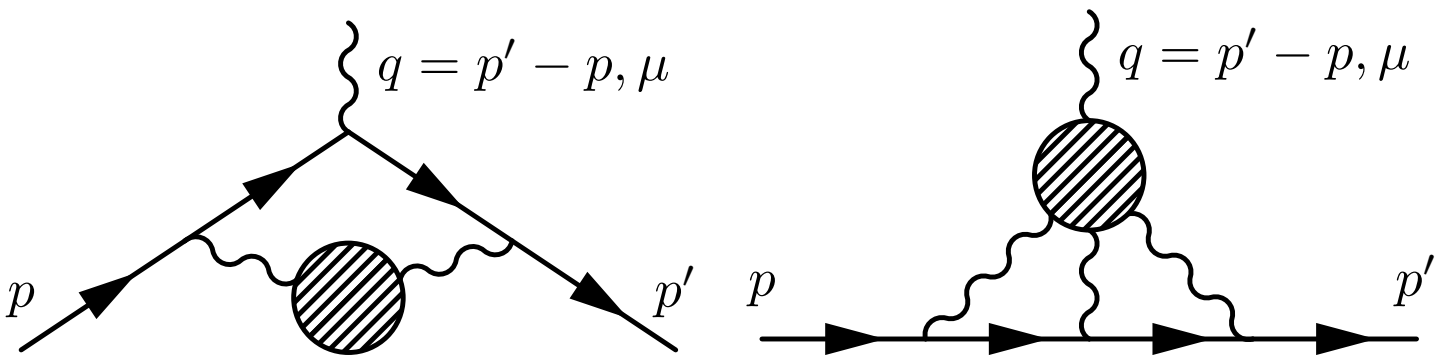

Figure 3. (L) Vaccum polarization diagram. (R) Light by light diagram.

There is 3.3σ deviation!

Future Fermilab E989 (0.14 ppm)

Figure 4. The 50-foot-wide Muon g-2 electromagnet being driven north on I-355 between Lemont and Downers Grove, Illinois, shortly after midnight on Thursday, July 25, 2013. *Credit: Fermilab.*

Almost 4 times more accurate then the previous experiment.

Connected Light by Light Diagram on Lattice

- In this talk, we focus on the calculation of connected light by light amplitude on lattice.
- This subject is started by T. Blum, M. Hayakawa, T. Izubuchi more than 5 years ago.

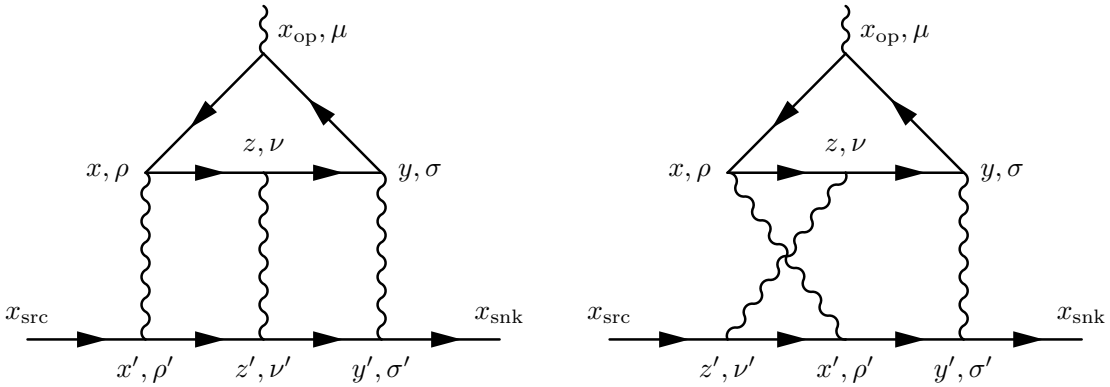

Figure 5. Light by Light diagrams. There are 4 other possible permutations.

$$
\begin{aligned}
\mathcal{M}_\mu^{\mathsf{LbL}} \;=\; & -(-ie)^6 \sum_{x,y,z} \mathrm{tr}(\gamma_\mu S_q(x_{\rm op};x)\gamma_\rho S_q(x;z)\gamma_\nu S_q(z;y)\gamma_\sigma S_q(y,x_{\rm op})) \\
& \cdot \sum_{x',y',z'} G_{\rho\rho'}(x;x')G_{\sigma\sigma'}(y;y')G_{\nu\nu'}(z;z') \\
& \cdot \Big[S(x_{\rm src};x')\gamma_{\rho'}S(x';z')\gamma_{\nu'}S(z';y')\gamma_{\sigma'}S(y';x_{\rm snk}) \\
& + S(x_{\rm src};z')\gamma_{\nu'}S(z';x')\gamma_{\rho'}S(x';y')\gamma_{\sigma'}S(y';x_{\rm snk}) \\
& + \text{other 4 permutations} \Big]
\end{aligned}
\tag{1}
$$

Outline

1. Muon Anmalous Magetic Moment
2. BNL E821 (0.54 ppm) and Standard Model Prediction
3. **Lattice QED with Schwinger Term as an Example**
   i. Stochastic Phton
   ii. Exact Photon
   iii. Finite Volume Effects
   iv. Discretization Errors
4. Light by Light Evaluation Stragtegy
   i. Computation Cost
   ii. Evaluation Formula
5. QED Light by Light Simulations
6. Lessons Learned
7. Future Plans

3 Lattice QED with Schwinger Term as an Example

We would like to do a standard Euclidean-space lattice calculation with a muon source and sink, well separated in Euclidean time.

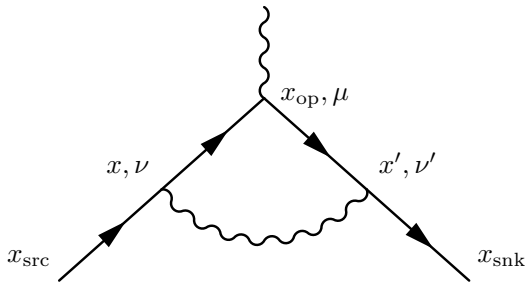

Figure 6. Schwinger term diagram.

$$\begin{aligned} \mathcal{M}_\mu^{\text{1-loop}} &= (-ie)^2 \sum_{x,x'} S(x_{\rm src}; x)\gamma_\nu S(x; x_{\rm op})\gamma_\mu S(x_{\rm op}; x')\gamma_{\nu'} S(x'; x_{\rm snk}) \\ &\cdot \; G_{\nu\nu'}(x; x') \end{aligned} \tag{2}$$

Naively, the sum would require $\mathcal{O}(\text{Volume}^2)$ computation, which is not affordable. We discussion two strategies:

- Calculate the sum stochasticly.
- Fast Fourior Transformation.

Both approaches make the problem $\mathcal{O}(\text{Volume})$.

3.1 Stochastic Photon

Evaluate the photon propagator with N stochastic sample.

$$G_{\mu\nu}(x;y) \approx \frac{1}{M}\sum_{m=1}^{M} A_\nu^m(x)A_{\nu'}^m(y) \tag{3}$$

$$A_\mu^m(x) = \frac{1}{\sqrt{V}}\sqrt{2}\mathrm{Re}\sum_k \frac{\epsilon_\mu^m(k)}{\sqrt{|k^2|}}e^{i\,k\cdot x} \qquad \frac{1}{M}\sum_{m=1}^{M} \epsilon_\mu^m(k)\epsilon_\nu^{m*}(k') \approx \delta_{\mu\nu}\delta_{kk'} \tag{4}$$

$$\begin{aligned}\mathcal{M}_\mu^{\text{1-loop}} &= (-ie)^2\frac{1}{M}\sum_{m=1}^{M} \\ &\cdot \left[\sum_x S(x_{\mathrm{src}};x)\gamma_\nu A_\nu^m(x)S(x;x_{\mathrm{op}})\right]\gamma_\mu\left[\sum_{x'} S(x_{\mathrm{op}};x)\gamma_{\nu'}A_{\nu'}^m(x')S(x';x_{\mathrm{snk}})\right]\end{aligned} \tag{5}$$

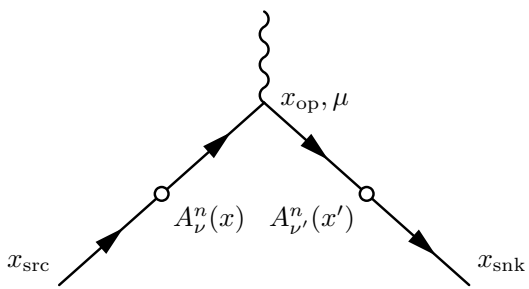

Figure 7. Schwinger term diagram calculated with stochastic photon.

3.2 Exact Photon

$$G_{\mu\nu}(x;y) \;=\; \frac{1}{V}\sum_k \frac{\delta_{\mu\nu}}{k^2} e^{i\,k\cdot(x-y)} \tag{6}$$

$$\begin{aligned}\mathcal{M}_\mu^{\text{1-loop}} \;&=\; (-i\,e)^2 \frac{1}{V}\sum_k \frac{\delta_{\nu\nu'}}{k^2} \\ &\cdot\; \left[\sum_x S(x_{\text{src}};x)\gamma_\nu e^{i\,k\cdot x} S(x;x_{\text{op}})\right]\gamma_\mu \left[\sum_{x'} S(x_{\text{op}};x')\gamma_{\nu'} e^{-i\,k\cdot x'} S(x';x_{\text{snk}})\right]\end{aligned} \tag{7}$$

Evaluate the express in brackets with Fast Fourier Transformation.

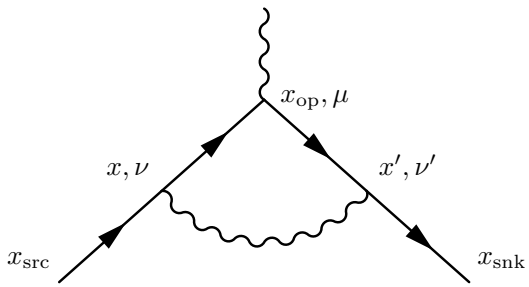

Figure 8. Schwinger term diagram calculated with exact photon.

3.3 Finite Volume Effects

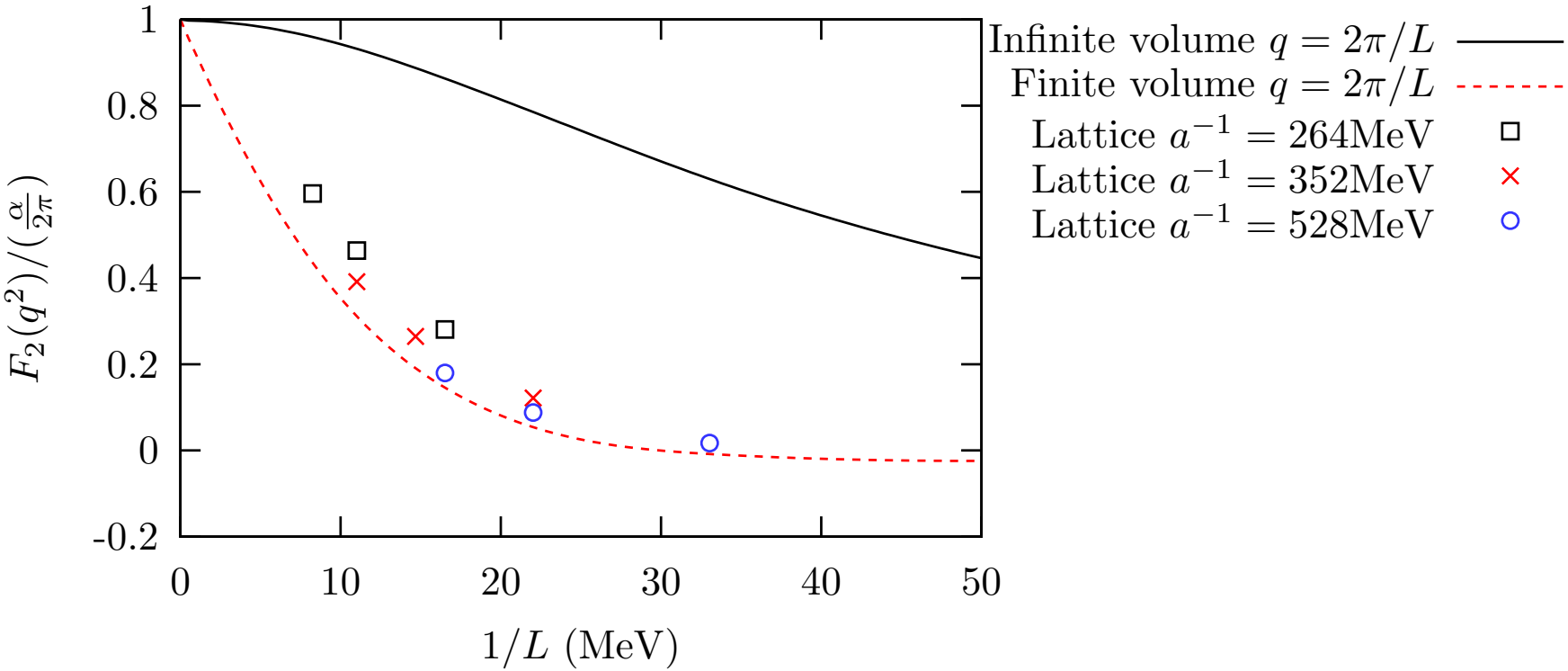

Figure 9. Finite volume effects on F_2. The data points are obtained using exact photon method.

- The solid line represent the analytic result in infinite volume and momentum transfer $q = 2\pi/L$. The dashed line represent the analytic result in L^3 volume and momentum transfer $q = 2\pi/L$.
- Lattice sizes are $32^3 \times 128$, $24^3 \times 96$, $16^3 \times 64$ with $L_s = 8$ and $t_{\rm snk} - t_{\rm op} = t_{\rm op} - t_{\rm src} = T/4$.
- Muon mass is $m_\mu = 105\text{MeV}$. a is the lattice spacing.

3.4 Discretization Errors

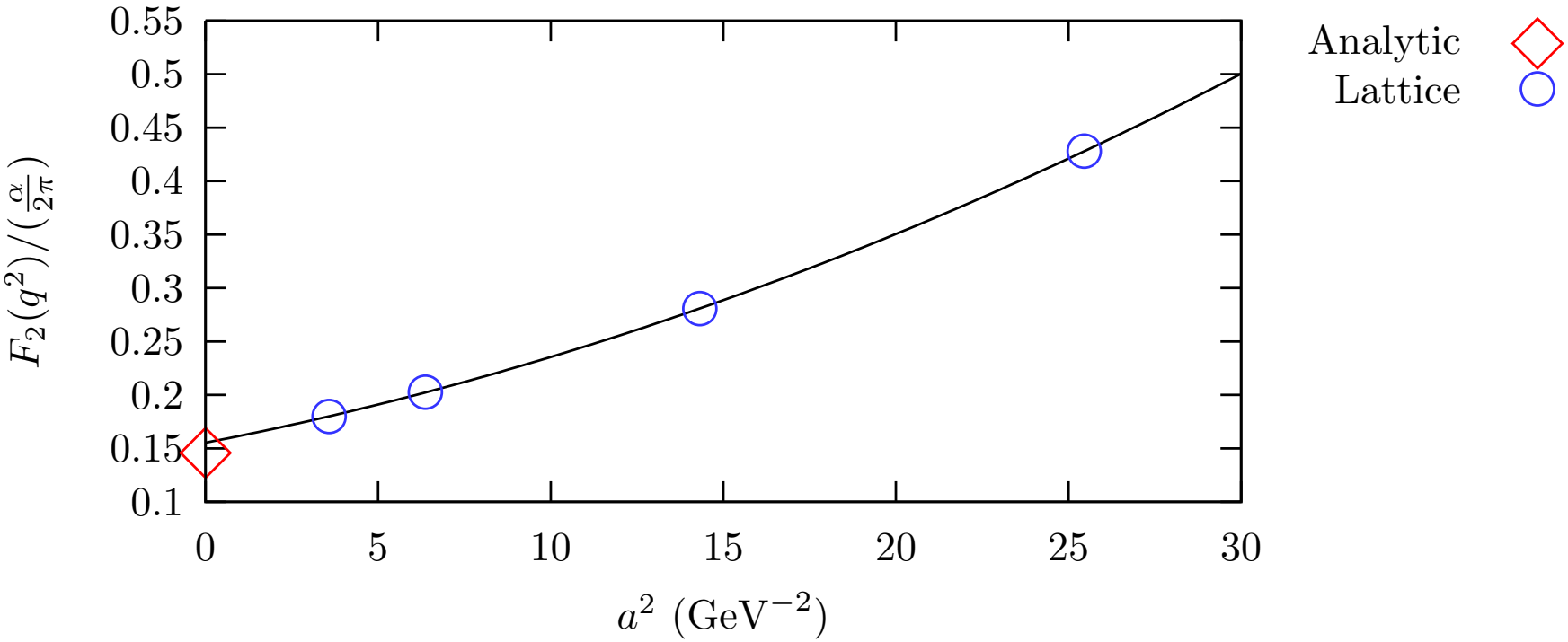

Figure 10. Discretization errors on F_2. The data points are obtained using exact photon method.

- $m_\mu L = 6.4$ and lattice sizes are $32^3 \times 128$, $24^3 \times 96$, $16^3 \times 64$, $12^3 \times 48$ with $L_s = 8$ and $t_{\rm snk} - t_{\rm op} = t_{\rm op} - t_{\rm src} = T/4$.
- $q = 2\pi/L$ is the momentum of the external photon.
- The line is 2nd order polynomial obtained by fitting the results from lattice calculations.
- Muon mass is $m_\mu = 105\text{MeV}$. a is the lattice spacing. An a^4 term is visible.

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4 Light by Light Evaluation Strategy

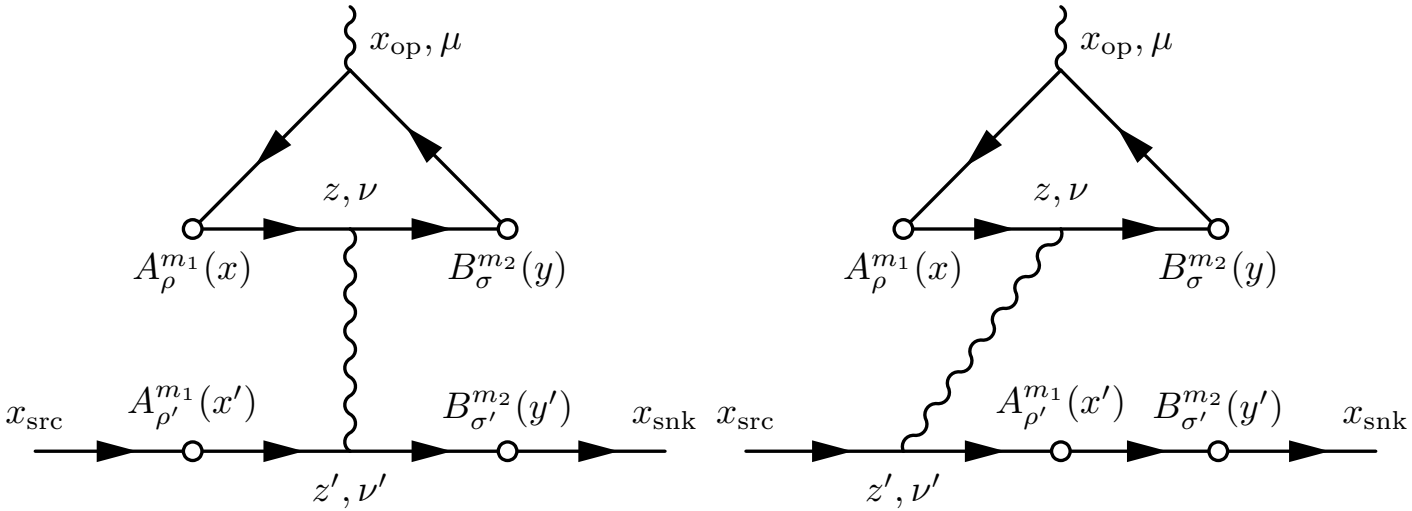


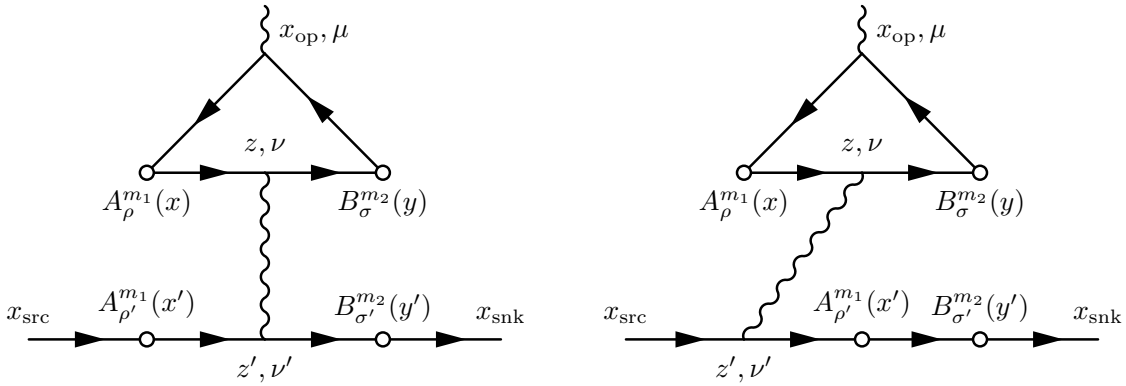
Figure 11. Light by Light diagrams calculated with one exact photon and two stochastic photon. There are 4 other possible permutations.

- $M = 12$ stochastic photon fields for both A and B.
- $S = 18$ random wall sources for the external local current.

4.1 Computation Cost

- $2 \times S \times M$ times inversion for the quark loop.
- $8 \times M^2$ times inversion for muon line.
- Statistics roughly proportion to $S \times M^2$
- Cost grows as $\mathcal{O}(\text{Volume})$ not $\mathcal{O}(\text{Volume}^2)$.

4.2 Evaluation Formula



$$
\begin{aligned}
\mathcal{M}_\mu^{\mathsf{LbL}} \;=\; & -(-ie)^6 \frac{1}{M^2} \sum_{m_1,m_2=1}^{M} \frac{1}{V} \sum_k \frac{\delta_{\nu\nu'}}{k^2} \\
& \cdot \sum_z \mathrm{tr}\left\{ \gamma_\mu \left[\sum_x S_q(x_{\rm op};x)\gamma_\rho A_\rho^{m_1}(x) S_q(x;z) \right] \gamma_\nu e^{i\,k\cdot z} \left[\sum_y S_q(z;y)\gamma_\sigma B_\sigma^{m_2}(y) S_q(y,x_{\rm op}) \right] \right\} \\
& \cdot \sum_{z'} \left\{ \left[\sum_{x'} S(x_{\rm src};x')\gamma_{\rho'} A_{\rho'}^{m_1}(x') S(x';z') \right] \gamma_{\nu'} e^{-i\,k\cdot z'} \left[\sum_{y'} S(z';y')\gamma_{\sigma'} B_{\sigma'}^{m_2}(y') S(y';x_{\rm snk}) \right] \right. \\
& + S(x_{\rm src};z')\gamma_{\nu'} e^{-i\,k\cdot z'} \left[\sum_{x'} S(z';x')\gamma_{\rho'} A_{\rho'}^{m_1}(x') \left(\sum_{y'} S(x';y')\gamma_{\sigma'} B_{\sigma'}^{m_2}(y') S(y';x_{\rm snk}) \right) \right] \\
& \left. + \text{other 4 permutations} \right\}
\end{aligned}
\tag{8}
$$

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5 QED Light by Light Simulations

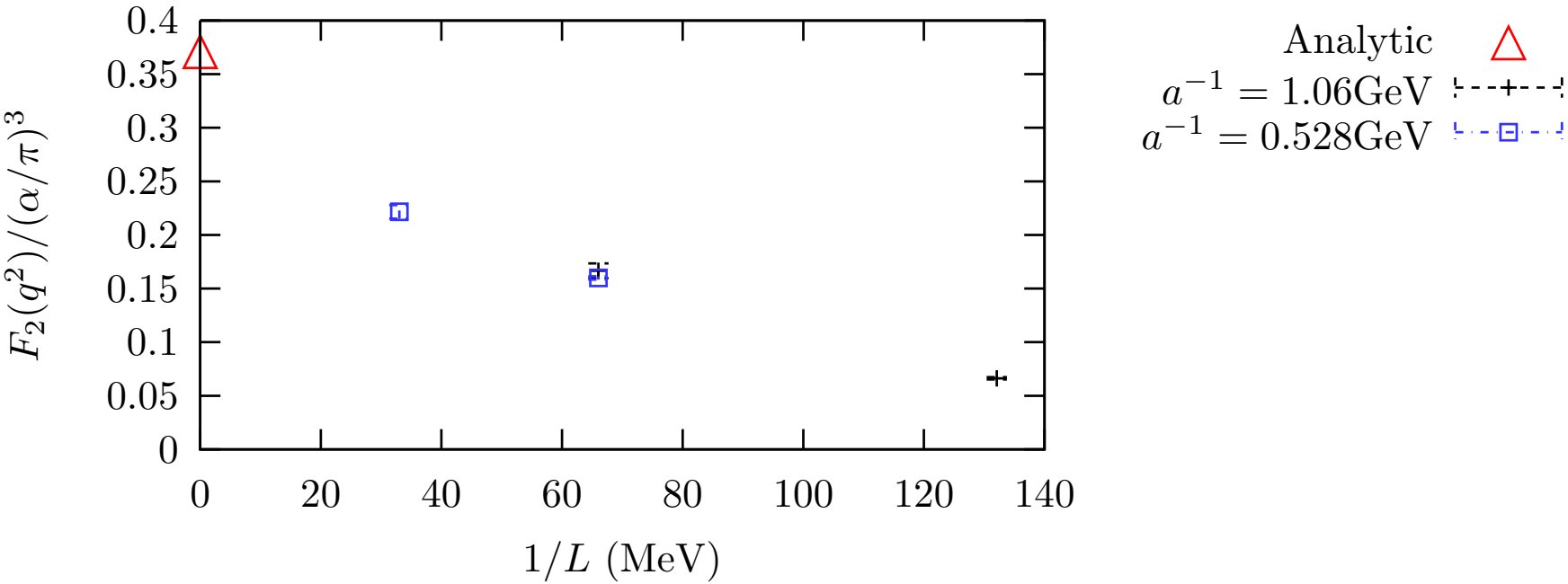

Figure 12. Finite volume effect on F_2.

- Replace quark loop by muon loop.
- Lattice sizes are $16^3 \times 64$, $8^3 \times 32$ with $L_s = 8$ and $t_{\text{snk}} - t_{\text{op}} = t_{\text{op}} - t_{\text{src}} = T/4$.
- The simulations were done in L^3 volume and momentum transfer $q = 2\pi/L$.
- Muon mass is $m_\mu = 105\text{MeV}$. a is the lattice spacing.

Detailed Simulation Data

Lattice Size	m_μ	$\frac{\text{Result} \pm \text{Err}}{(\alpha/\pi)^3}$	$N \times S \times M^2$ confs	$\frac{\text{Var}}{(\alpha/\pi)^3}$
$8^3 \times 64$	0.2	0.1604 ± 0.0025	$355 \times 36 \times 12^2$	3.4
$8^3 \times 32$	0.05	0.0194 ± 0.0004	$147 \times 36 \times 12^2$	0.34
$8^3 \times 32$	0.1	0.0663 ± 0.0011	$201 \times 36 \times 12^2$	1.07
$8^3 \times 32$	0.2	0.1599 ± 0.0016	$571 \times 36 \times 12^2$	2.7
$8^3 \times 32$	0.4	0.1762 ± 0.0038	$213 \times 36 \times 12^2$	4.0
$16^3 \times 64$	0.05	0.0663 ± 0.0013	$307 \times 18 \times 12^2$	1.62
$16^3 \times 64$	0.1	0.1666 ± 0.0069	$88 \times 18 \times 12^2$	3.3
$16^3 \times 64$	0.2	0.2216 ± 0.0063	$299 \times 18 \times 12^2$	5.6

Figure 13. M stands for the number of stochastic A, B fields, S stands for the number of random wall sources x_{op} that we use to calculate the external current. The calculation is repeated N times. $\text{Var} = \text{Err} \times \sqrt{N \times S \times M^2}$ stands for the projected variance according to the uncertainty of the result and the total number of confs.

- We have good control of the excited state effects.

6 Lessons Learned

- Average over different combinations of A, B electromagnetic field helps reducing the statistical errors. This trick contribute 10 times the statistics, limited only by memory of the machine.
- Random wall source at the location of the external current works very well. This trick contribute around 10 times the statistics for $16^3 \times 64$ lattice compare with point source, works better at larger lattice.
- Using symmetric kinematics significantly reduces the statistial error as both the initial and the final state are the lowest energy state possible. We use antiperiodic boundry condition in z direction and set the momenta of initial and final muon to be $\pm\pi/L$.

7 Future Plans

- Connected QCD Light by Light Diagram
- Disconnected QCD Light by Light Diagram