Update on staggered Wilson fermions

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Introduction

Will discuss the current situation for staggered versions of Wilson fermions (“staggered Wilson fermions”) regarding

- Theoretical viability
- Computational efficiency vs usual Wilson fermions
- Usability

As illustration of usability, will discuss construction of pseudoscalar meson operators for 2-flavor staggered Wilson fermions.
What are staggered Wilson fermions?

**Usual Wilson fermion:**
Naïve fermion (16 species) + Wilson term
→ 1 physical species, 15 doublers

**Staggered Wilson fermion (the idea):**
Staggered fermion (4 species) + “staggered Wilson term”
→ 1 or 2 physical species, 3 or 2 doublers
Origin of staggered Wilson fermions

• 2-flavor staggered version of overlap fermions arose from spectral flow approach to the index of the staggered Dirac operator in [DA, PRL (2010), PLB (2011)]. Has 2-flavor staggered Wilson fermion as kernel.

• 1-flavor version proposed later by C. Hoelbling [PLB (2011)]

• The staggered Wilson terms in both cases are combinations of the “flavored mass terms” of Golterman & Smit [NPB (1984)] which lift the degeneracy of the 4 staggered fermion flavors.
Motivation: Staggered more efficient than Wilson

Consider the momentum space Brillouin zone in 2D

<table>
<thead>
<tr>
<th>Wilson</th>
<th>Staggered</th>
</tr>
</thead>
<tbody>
<tr>
<td>junk (0, π/a)</td>
<td>phys (0, π/a)</td>
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<tr>
<td>junk (π/a, π/a)</td>
<td>phys (π/a, π/a)</td>
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<td>phys (0,0)</td>
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</tbody>
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+ spinor index \( \alpha = 1, 2 \)

- 1 physical species
- 3 doublers (junk)
- 2 physical species
- no junk, no waste
- smaller Dirac matrix!
Staggered formalism is more efficient, but what to do about the extra species?

Possibilities:

• Live with the 4 species (‘tastes’) for each quark.
  -- The current approach for Asqtad and HISQ.

• Use staggered fermion as a description of 4 quarks (flavors)
  -- Problems arise when the mass degeneracy of the 4 species is lifted.

• Add a ‘Wilson term’ to reduce the number of physical species
  [DA, C. Hoelbling].
Staggered Wilson fermion formulation

Fermion action \( \bar{\chi}(D_{sw} + m)\chi \) where

\[
D_{sw} = D_{st} + \frac{1}{a} M \quad \text{staggered Wilson Dirac operator}
\]

\[
D_{st} = \eta_\mu \nabla_\mu \quad \text{usual staggered Dirac operator}
\]

\( M = \text{flavored mass terms [Golterman-Smit]}\)

In the spin-flavor interpretation of the free field momentum space rep have

\[
\hat{M}(q) = 1 \otimes M + O(a^2 q^2)
\]

Zero-eigenvector of \( M \rightarrow \text{massless physical fermion species} \)

Nonzero-eigenvector of \( M \rightarrow \text{heavy species, mass } \sim 1/a \text{ (doublers)} \)
Some details of the flavored mass terms

\[ M = M^{(0)} + M^{(2)} + M^{(4)} \]

\[ M^{(n)} = n\text{-link term} \]

In the free field momentum space spin-flavor interpretation have

\[ \hat{M}(q) = r_0 (1 \otimes 1) + ir_{\mu\nu} c_{\mu}(aq)c_v(aq)(1 \otimes \xi_\mu \xi_v) + r_5 c_5(aq)(1 \otimes \xi_5) \]

\[ c_{\mu}(aq) \equiv \cos(aq_\mu) \], \quad c_5(aq) \equiv \prod_{\mu=1,...,4} \cos(aq_\mu) \]

\[ \xi_\mu : \text{Dirac matrices on flavor space} \]

\[ \Rightarrow \hat{M}(q) = 1 \otimes (r_0 1 + ir_{\mu\nu} \xi_\mu \xi_v + r_5 \xi_5) + O(a^2 q^2) \]

Lifts the degeneracy of the 4 staggered fermion flavors.

Can get zero-eigenvector (\(\rightarrow\) massless fermion) by tuning \(r_0\).
2-flavor version of staggered Wilson fermion

\[ M = M^{(0)} + M^{(4)}, \quad M^{(2)} \text{ is absent} \]

\[ \hat{M}(q) = 1 \otimes (r_0 1 + r_5 \xi_5) + O(a^2 q^2) \]

→ Set \( r_0 = -r_5 = r \), then

\[ \hat{M}(q) = r 1 \otimes (1 - \xi_5) + O(a^2 q^2) \]

Zero-eigenspace of this is the 2-dim flavor subspace on which \( \xi_5 = 1 \)
Chirality aspect of staggered Wilson fermions

Recall: staggered fermion has an exact \textit{flavored} chiral symmetry

\[ \varepsilon D_{st} = -D_{st} \varepsilon \]

\[ \varepsilon \chi(x) = (-1)^{x_1 + \cdots + x_4} \chi(x) \quad \varepsilon \cong \gamma_5 \otimes \xi_5 \]

Crucial fact:

\varepsilon acts as \textit{unflavored} \( \gamma_5 \otimes 1 \) on the physical species of the staggered Wilson fermion.

This is because \( \xi_5 = 1 \) on the zero-eigenspace of \( \mathbf{M} \).
Chirality aspect (continued)

→ Use $\epsilon$ as the unflavored $\gamma_5$ for staggered Wilson fermions.

Note:

$$D_{sw} = \epsilon D_{sw} \epsilon$$  \hspace{1cm} (\epsilon\text{-hermiticity})

$$\epsilon^2 = 1$$

These don’t hold for the usual unflavored $\gamma_5$ in the staggered formalism!

Get staggered versions of domain wall and overlap fermions by
$$D_w \rightarrow D_{sw}, \ \gamma_5 \rightarrow \epsilon \ [DA, PLB (2011)]$$
Theoretical viability of staggered Wilson fermions?

- $M^{(2)}$ and $M^{(4)}$ break the staggered `shift’ symmetries
- $M^{(2)}$ also breaks the flavored lattice rotation symmetry of staggered fermions.

→ Questions:

- Is the staggered Wilson spin-flavor structure preserved at the quantum level?
- New counter-terms appear? Fine-tunings required?
Results on theoretical viability  [DA, to appear]

• There are new fermionic counterterms, but their only effect on the physical species is a wavefunction renormalization. → No fine-tunings needed for these.

• For the 1-flavor versions of staggered fermions, which break lattice rotation symmetry, a new gluonic counter-term arises from the fermion loop contributions to the gluonic 2-, 3- and 4-point functions. → Needs to be included in the bare action and fine-tuned.

Conclusion: 1-flavor versions of staggered Wilson fermions are problematic, but 2-flavor version is fine.

Will restrict considerations to 2-flavor staggered Wilson fermions henceforth.
Computational efficiency

• Previous study on quenched $16^3 \times 32$ lattice with $\beta = 6$ found a speed-up factor of 4-6 for staggered Wilson compared to usual Wilson for inverting the Dirac operator on a source at fixed pion masses. [DA, D. Nogradi & C. Zielinski, Lattice 2013 proc.]

• New study on $20^3 \times 40$ lattice with $\beta = 6$ and $\beta = 6.14$ tests the dependence of the speed-up factor on physical volume and lattice spacing.

Results:

• Increased speed-up for smaller lattice spacing at fixed volume
• Speed-up mostly unchanged (slight increase) for larger volume at fixed lattice spacing.

Details in C. Zielinski’s talk.
Usability of 2-flavor staggered Wilson fermions

- Need meson & baryon operators for the 2 flavors
- Situation is complicated by the fact that only a subset of the flavor symmetries are unbroken.
- However, meson & baryon operators for the 4 degenerate flavors described by usual staggered fermion are already known. [Golterman & Smit, Sharpe, ...]
- Can adapt these to the staggered Wilson case – will illustrate for pseudoscalar mesons.
Pseudoscalar mesons with staggered Wilson

• For usual staggered fermions (4 flavors) have 16 pseudoscalar mesons

• Their operators $\bar{\chi}(\gamma_5 \otimes \xi_F)\chi$ form irreps of the flavored lattice rotation symmetry group as follows:

$$\xi_F = 1 \text{ (scalar)} \quad \xi_F = \xi_\mu \text{ (vector)} \quad \xi_F = \xi_\mu \xi_\nu \text{ (tensor)}$$

$$\xi_F = \xi_\mu \xi_5 \text{ (pseudovector)} \quad \xi_F = \xi_5 \text{ (pseudoscalar)}$$

Use same operators for staggered Wilson case. They continue to form irreps of the flavored rotation symmetry group, but their physical meaning changes...
Pseudoscalar mesons with staggered Wilson (cont’d)

\[
\xi_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \xi F = \begin{pmatrix} \text{light - light} & \text{light - heavy} \\ \text{heavy - light} & \text{heavy - heavy} \end{pmatrix}
\]

The light-light part of \( \xi_F \) determines the physical meaning of \( \bar{\chi}(\gamma_5 \otimes \xi_F)\chi \) for staggered Wilson fermions.

\( \rightarrow \) The tensor irrep \( \xi_F = \xi_\mu \xi_\nu \) gives the 3 pions. They are degenerate since these pion operators belong to the same irrep.

E.g.

\[
\xi_1 \xi_2 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}
\]

\( \rightarrow \) \( -i\bar{\chi}(\gamma_5 \otimes \xi_1 \xi_2)\chi \) becomes operator for \( \pi_0 \).
Conclusions

The situation for 2-flavor staggered Wilson fermions:

- Theoretically viable
- Computationally efficient
- Usable

Potentially advantageous for

- QCD thermodynamics
- High precision computation of the eta’ mass.

C. Zielinski’s talk:
Details for the computational efficiency, including spectrum of the staggered Wilson Dirac operator for a range of lattices.