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Phase Diagram of Wilson and Twisted Mass Fermions at finite isospin chemical potential

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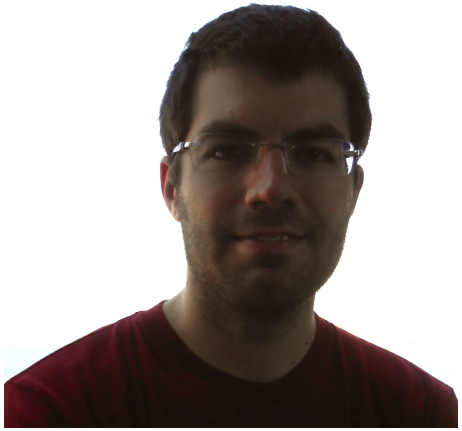
My Collaborators



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The setting: Wilson Dirac Operator

$$D_{\mathrm{W}} = \gamma^{\alpha} D_{\alpha} + a D_{\alpha} D^{\alpha} + m \cos\omega + \imath m \sin\omega \gamma_5 \tau_3 + \mu \gamma_0 \tau_3$$

- Dirac matrices: γ_α
- two flavours $\rightarrow$ Pauli matrices: τ_i
- covariant derivative: D_α
- degenerate quark mass: m
- isospin chemical potential: μ
- lattice spacing: a
- twisting angle: $\omega \in [0, \pi/2]$ (Frezzotti, Rossi (2000))

$N_{\mathrm{f}} = 2$, $T = \mu_{\mathrm{B}} = 0$, and $\mu_{\mathrm{I}}, a \neq 0$

Effective Theory: p-Regime

- lattice volume (space-time volume) $V = L^4 \to \infty$
- lattice spacing $a^4 V$, quark masses $m^2 V$, momenta $p^4 V$ fixed

! slightly different counting scheme in contrast to Aoki et al. (2003/11)

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Effective action:

$$S = \int_V d^4x \mathcal{L}(U(x)) = V \mathcal{L}_{\text{micro}}(U_0) + \int_V d^4x [\mathcal{L}_1(U_0, \Pi(x)) + \mathcal{L}_2(U_0, \Pi(x))]$$

- microscopic Lagrangian: $\mathcal{L}_{\text{micro}}(U_0) \sim O(1)$
- Lagrangian linear in the pions Π: $\mathcal{L}_1(U_0, \Pi(x)) \sim O(1/\sqrt{V})$
  Should vanish at the saddlepoint U_0!
- Lagrangian quadratic in the pions Π: $\mathcal{L}_2(U_0, \Pi(x)) \sim O(1/V)$
  Yields the pion masses!

Chiral Lagrangian (two flavours)

$$V\mathcal{L}_{\text{micro}} = \frac{\Sigma m V}{2}\left[\cos\omega\,\mathrm{tr}\,(U+U^{-1}) + \imath\sin\omega\,\mathrm{tr}\,\tau_3(U-U^{-1})\right]$$
$$-F^2\mu^2 V\mathrm{tr}\,U\tau_3U^{-1}\tau_3 - \overbrace{(W_6+W_8/2)}^{C_2/16}a^2 V\mathrm{tr}^2(U+U^{-1})$$

with $U \in \mathrm{SU}(2)$

- part of continuum QCD: $\widehat{m} = \Sigma m V$
- part of isospin chemical potential: $\widehat{\mu}^2 = F^2\mu^2 V$
- part of discretization: $\widehat{a}^2 = (W_6 + W_8/2)a^2 V = C_2 a^2/16\, V$
- low energy constants (to be fixed): $\Sigma,\ F,\ W_{6/8}$

- $\mu = \omega = 0$: Sharpe, Singleton (1998); Bär, Rupak, Shoresh (2004); Sharpe (2006); Bär, Necco, Schaefer (2009);
- $\mu = 0$: Sharpe, Wu (2004) (Note a slightly different counting scheme)

Thermodynamic limit

$$V\mathcal{L}_{\text{micro}} = \frac{\widehat{m}}{2}\left[\cos\omega\,\mathrm{tr}\,(U+U^{-1}) + \imath\sin\omega\,\mathrm{tr}\,\tau_3(U-U^{-1})\right] - \widehat{\mu}^2\mathrm{tr}\,U\tau_3U^{-1}\tau_3 - \widehat{a}^2V\mathrm{tr}^2(U+U^{-1})$$

with

$$U = \cos\varphi\,\mathbb{1}_2 + \imath\sin\varphi\begin{bmatrix}\cos\vartheta_1 & e^{\imath\vartheta_2}\sin\vartheta_1\\ e^{-\imath\vartheta_2}\sin\vartheta_1 & -\cos\vartheta_1\end{bmatrix}$$

$\varphi, \vartheta_1 \in [0,\pi]$ and $\vartheta_2 \in [0, 2\pi]$

$\widehat{m}, \widehat{\mu}^2, \widehat{a}^2 \to \infty$ (all parameters of the same order)

Good quantities:

- chiral condensate: $\Sigma \propto \langle\bar{\psi}\psi\rangle \propto \cos\varphi$
- π_0 condensate: $\langle\bar{\psi}\gamma_5\tau_3\psi\rangle \propto \sin\varphi\cos\vartheta_1$
- isospin charge density: $\partial\mathcal{L}/\partial\widehat{\mu} \propto \langle\bar{\psi}\gamma_0\tau_3\psi\rangle \propto \sin^2\varphi\cos^2\vartheta_1$
- we also considered $\pi^\pm$ condensate: $\langle\bar{\psi}\gamma_5\tau_{1/2}\psi\rangle \propto \sin\varphi\sin\vartheta_1(e^{\imath\vartheta_2} \pm e^{-\imath\vartheta_2})$

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$$\cos\omega - \beta_3 \sin\omega) + 16\widehat{a}^2(\beta_3^2 - \alpha^2),$$

$$\cos\omega - \beta_3 \sin\omega) - 16\widehat{a}^2\alpha^2 \pm 2\widehat{\mu},$$

$$U^{(III)}|_{\omega=0} = \frac{\widehat{m}}{16\widehat{a}^2}\mathbb{1}_2 - \imath \operatorname{sign}\widehat{m}\sqrt{1-\left(\frac{\widehat{m}}{16\widehat{a}^2}\right)^2}\,\tau_3$$

$$c_1 = \frac{1}{2}(\operatorname{tr} H_0 + 16|\underline{u}|^2\widehat{\mu}^2)$$

$$= \widehat{\mu}^2\left(4 + \frac{6\widehat{m}^2\sin^2\omega}{(4\widehat{\mu}^2)^2} + \frac{6\widehat{m}^2\cos^2\omega}{(16\widehat{a}^2+4\widehat{\mu}^2)^2}\right) + 8\widehat{a}^2\left(1 - \frac{\widehat{m}^2\cos^2\omega}{(16\widehat{a}^2+4\widehat{\mu}^2)^2}\right) \geq \sqrt{c_2} \geq 0$$

$$\frac{\widehat{m}^2}{(16\widehat{a}^2+4\widehat{\mu}^2)^2}\begin{bmatrix} 0 & e^{\imath\vartheta_2} \\ e^{-\imath\vartheta_2} & 0 \end{bmatrix}_2 = \frac{1}{4}(\operatorname{tr} H_0 + 16|\underline{u}|^2\widehat{\mu}^2)^2 - \operatorname{tr} C(H_0) - 16\langle \underline{u}, H_0\underline{u}\rangle\widehat{\mu}^2$$

$$= \left[6\widehat{\mu}^2\left(\frac{\widehat{m}^2\cos^2\omega}{(16\widehat{a}^2+4\widehat{\mu}^2)^2} + \frac{\widehat{m}^2\sin^2\omega}{(4\widehat{\mu}^2)^2}\right) - 8\widehat{a}^2\left(1 - \frac{\widehat{m}^2\cos^2\omega}{(16\widehat{a}^2+4\widehat{\mu}^2)^2}\right)\right]^2$$

$$\int_0^1 \Theta\left(|\widehat{m}|(\sqrt{1-y^2}\cos\omega - y\sin\omega) - 16\widehat{a}^2 y\sqrt{1-y^2}\right) dy\,\mathbb{1}_2 \qquad \text{(A.16)} \quad \frac{\widehat{m}^2\sin^2\omega}{(4\widehat{\mu}^2)^2}\Bigg) \geq 0.$$

$$\operatorname{sgn}\widehat{m}\sqrt{1-\left[\int_0^1 \Theta\left(|\widehat{m}|(\sqrt{1-y^2}\cos\omega - y\sin\omega) - 16\widehat{a}^2 y\sqrt{1-y^2}\right) dy\right]^2}\,\tau_3.$$

I, $\omega \in [0, \pi/2]$	$4\widehat{\mu}^2 \geq \lvert\widehat{m}\rvert \sin\omega \geq 0$ (if $\omega = 0$ then $\widehat{\mu}^2 > 0$) and $16\widehat{a}^2 + 4\widehat{\mu}^2 \geq \frac{4\widehat{\mu}^2\lvert\widehat{m}\rvert\cos\omega}{\sqrt{(4\widehat{\mu}^2)^2 - \widehat{m}^2\sin^2\omega}} \geq 0$
$II_\pm$, $\omega \in]0, \pi/2[$	$4\widehat{\mu}^2 \geq \lvert\widehat{m}\rvert\sin\omega \geq 0$ and $\frac{4\lvert\widehat{m}\rvert\lvert\widehat{\mu}\rvert^2\cos\omega}{\sqrt{(4\widehat{\mu}^2)^2 - \widehat{m}^2\sin^2\omega}} > 16\widehat{a}$ or $\lvert\widehat{m}\rvert\sin\omega > 4\widehat{\mu}^2$
$III_\pm^{\omega=0}$	$\widehat{\mu}^2 < 0$ and $16\widehat{a}^2 > \lvert\widehat{m}\rvert > 0$
$IV_\pm^{\omega=0}$	$\widehat{\mu}^2 > 0$ and $\lvert\widehat{m}\rvert > 16\widehat{a}^2 + 4\widehat{\mu}^2$; or $0 > \widehat{\mu}^2$ and $\lvert\widehat{m}\rvert > 16\widehat{a}^2$
Aoki phase	$\widehat{\mu}^2 = 0$ and $16\widehat{a}^2 > \lvert\widehat{m}\rvert > 0$
$III_\pm^{\omega=\pi/2}$	$\widehat{\mu}^2 \leq 0$ and $-16\widehat{a}^2 > \lvert\widehat{m}\rvert > 0$ or $-16\widehat{a}^2 > \lvert\widehat{m}\rvert > 0$ and $-4\widehat{a}^2 > \widehat{\mu}^2 > 0$
$IV_\pm^{\omega=\pi/2}$	$\widehat{\mu}^2 \leq 0$ and $\lvert\widehat{m}\rvert > -16\widehat{a}^2$; or $\lvert\widehat{m}\rvert > 4\widehat{\mu}^2 > 0$ and $\lvert\widehat{m}\rvert > -16\widehat{a}^2$

$$U^{(I)} = \frac{\widehat{m}\cos\omega}{16\widehat{a}^2 + 4\widehat{\mu}^2}\mathbb{1}_2 + \imath\begin{bmatrix} -\frac{\widehat{m}\sin\omega}{4\widehat{\mu}^2} & e^{\imath\vartheta_2}\sqrt{1 - \frac{\widehat{m}^2\cos^2\omega}{(16\widehat{a}^2+4\widehat{\mu}^2)^2} - \frac{\widehat{m}^2\sin^2\omega}{(4\widehat{\mu}^2)^2}} \\ e^{-\imath\vartheta_2}\sqrt{1 - \frac{\widehat{m}^2\cos^2\omega}{(16\widehat{a}^2+4\widehat{\mu}^2)^2} - \frac{\widehat{m}^2\sin^2\omega}{(4\widehat{\mu}^2)^2}} & \frac{\widehat{m}\sin\omega}{4\widehat{\mu}^2} \end{bmatrix}$$

$$U^{(\text{Aoki})} = \frac{\widehat{m}}{16\widehat{a}^2}\mathbb{1}_2 + \imath\sqrt{1-\left(\frac{\widehat{m}}{16\widehat{a}^2}\right)^2}\begin{bmatrix} \cos\vartheta_1 & e^{\imath\vartheta} \\ e^{-\imath\vartheta_2}\sin\vartheta_1 & -\cos\vartheta_1 \end{bmatrix}$$

$$U^{(III)}|_{\omega=\pi/2} = \operatorname{sign}\widehat{m}\sqrt{1-\left(\frac{\widehat{m}}{16\widehat{a}^2}\right)^2}\,\mathbb{1}_2 + \imath\frac{\widehat{m}}{16\widehat{a}^2}\tau_3.$$

phase	$\frac{\Sigma(\widehat{m})}{\Sigma} = \frac{\langle\bar{\psi}\psi\rangle}{\Sigma}$	$\frac{C_{\pi^0}}{\Sigma} = \imath\frac{\langle\bar{\psi}\gamma_5\tau_3\psi\rangle}{\Sigma}$	$\frac{\lvert\Delta C_{\pi^0}\rvert}{\Sigma^2} = -\frac{\langle(\bar{\psi}\gamma_5\tau_3\psi)^2}{\Sigma^2}$
(B.17) I, $\in \left[0, \frac{\pi}{2}\right]$	$\frac{\widehat{m}\cos\omega}{8\widehat{a}^2 + 2\widehat{\mu}^2}$	$\frac{\widehat{m}\sin\omega}{2\widehat{\mu}^2}$	0
$II_\pm$, $\in \left]0, \frac{\pi}{2}\right[$	$2\operatorname{sign}\widehat{m}\,F\left(\frac{\lvert\widehat{m}\rvert}{16\widehat{a}^2}\right)$	$2\operatorname{sign}\widehat{m}\sqrt{1 - F^2\left(\frac{\lvert\widehat{m}\rvert}{16\widehat{a}^2}\right)}$	0
(B.18) $III_\pm^{\omega=0}$	$\frac{\widehat{m}}{8\widehat{a}^2}$	$2\operatorname{sign}\widehat{m}\sqrt{1-\left(\frac{\widehat{m}}{16\widehat{a}^2}\right)^2}$	0
$IV_\pm^{\omega=0}$	$2\operatorname{sign}\widehat{m}$	0	0
Aoki phase $\widehat{\mu} = \omega = 0$	$\frac{\widehat{m}}{8\widehat{a}^2}$	0	$2 - \frac{\widehat{m}^2}{128\widehat{a}^4}$

(A.7) $\frac{\widehat{m}}{\widehat{a}^2}$ 0

$\operatorname{sgn}\widehat{m}$ 0

$$\widehat{E}_{\pi,2}$$

$$U^{(IV)}|_{\omega=\pi/2} = \operatorname{sign}\widehat{m}\,\mathbb{1}_2$$

$$\widehat{m}_{\pi^0} = \sqrt{|\widehat{m}|(|\alpha|\cos\omega + \sqrt{1-\alpha^2}\sin\omega) + 16\widehat{a}^2(1-2\alpha^2)},$$

$$\widehat{m}_{\pi^\pm} = \sqrt{|\widehat{m}|(|\alpha|\cos\omega + \sqrt{1-\alpha^2}\sin\omega) - 16\widehat{a}^2\alpha^2} \pm 2\widehat{\mu}.$$

$$\omega=\pi/2 = \imath\begin{bmatrix} \\ e^{-\imath\vartheta_2}\sqrt{} \end{bmatrix}$$

$$\Pi_{\pi,1} \propto \frac{\widehat{m}\sin\omega}{4\widehat{\mu}^2}\tau_1 + \frac{\widehat{m}\cos\omega}{16\widehat{a}^2 + 4\widehat{\mu}^2}\tau_2 + \sqrt{1 - \frac{\widehat{m}^2\cos^2\omega}{(16\widehat{a}^2+4\widehat{\mu}^2)^2} - \widehat{m}}$$

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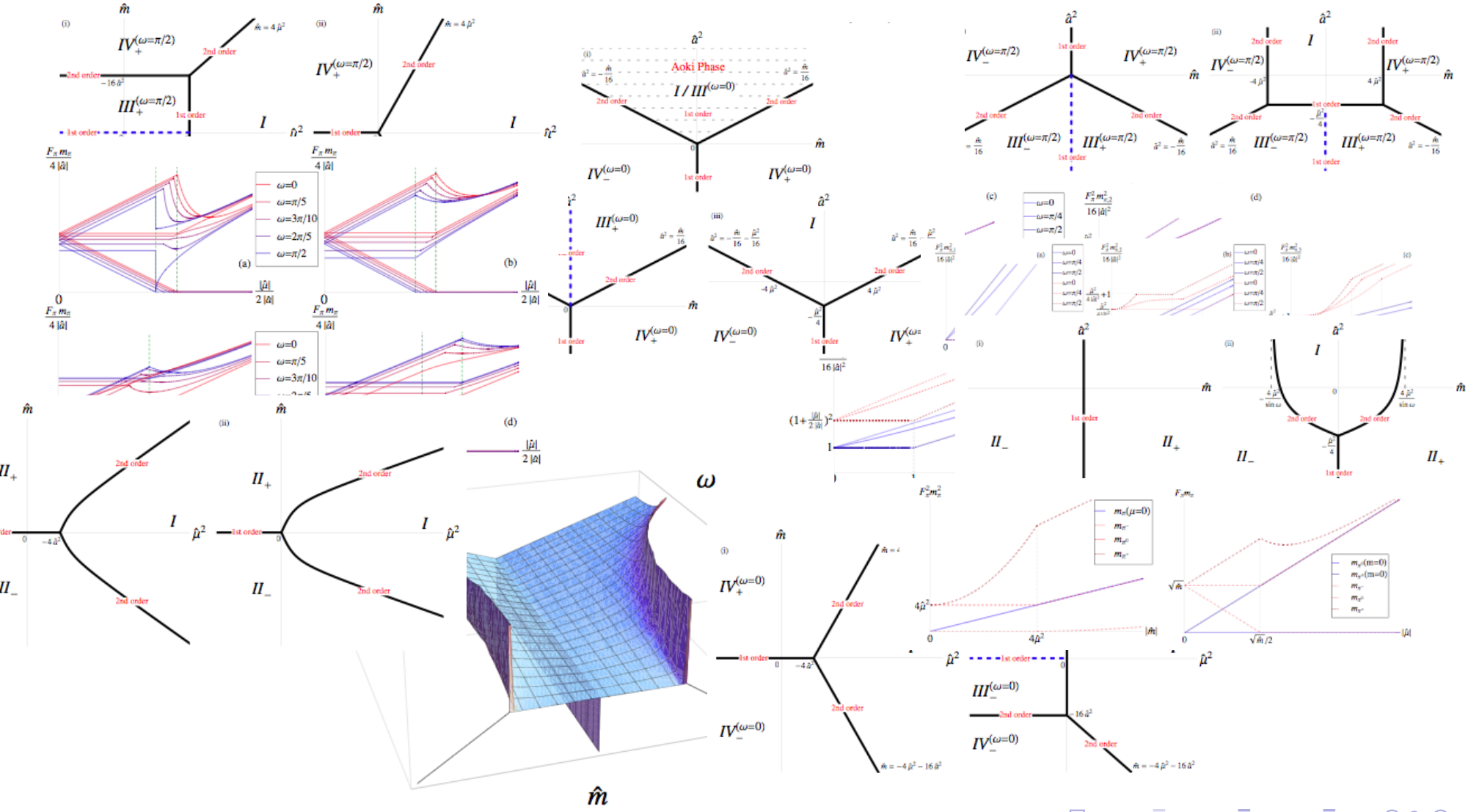
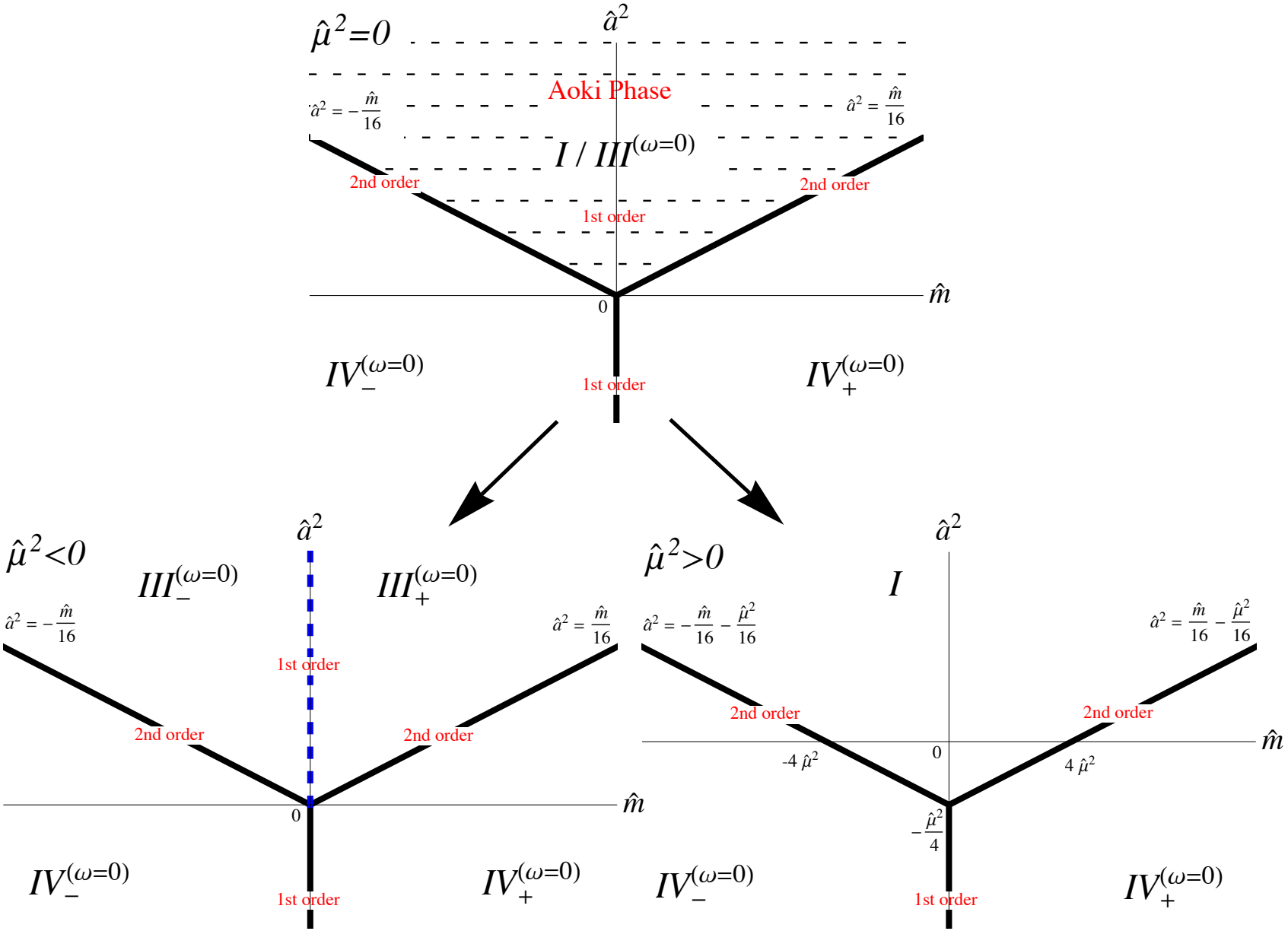
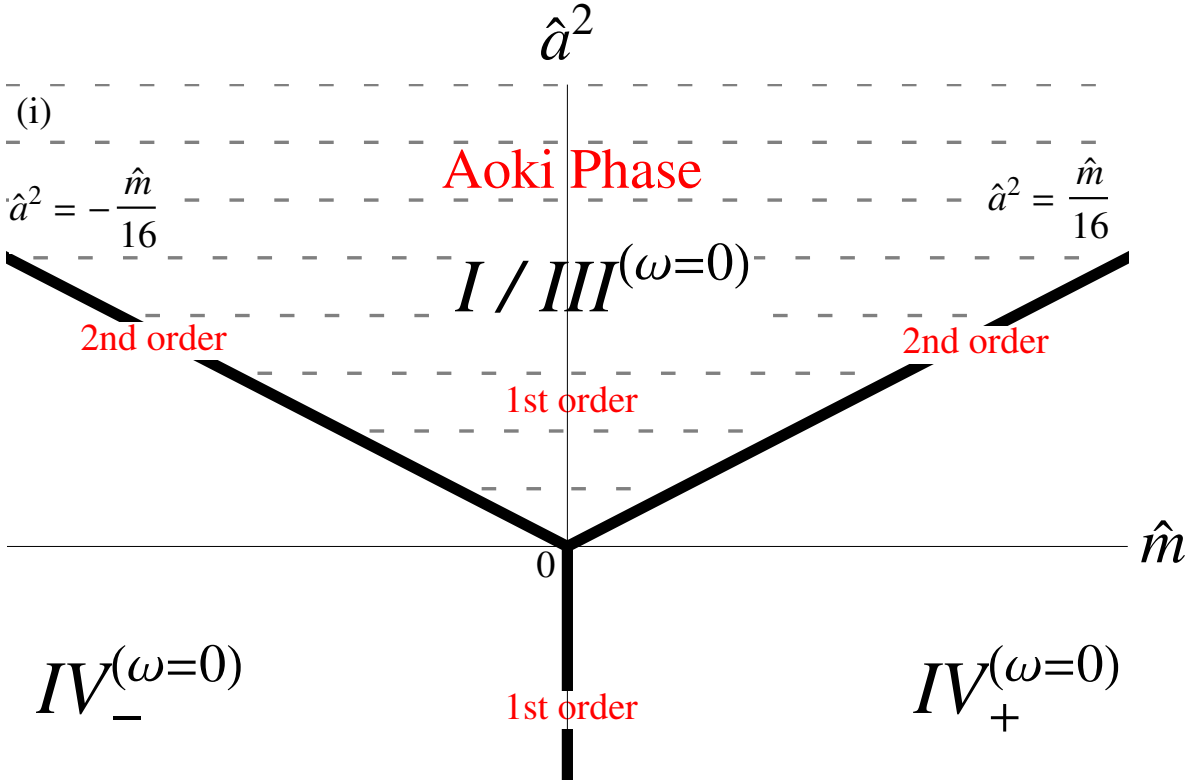


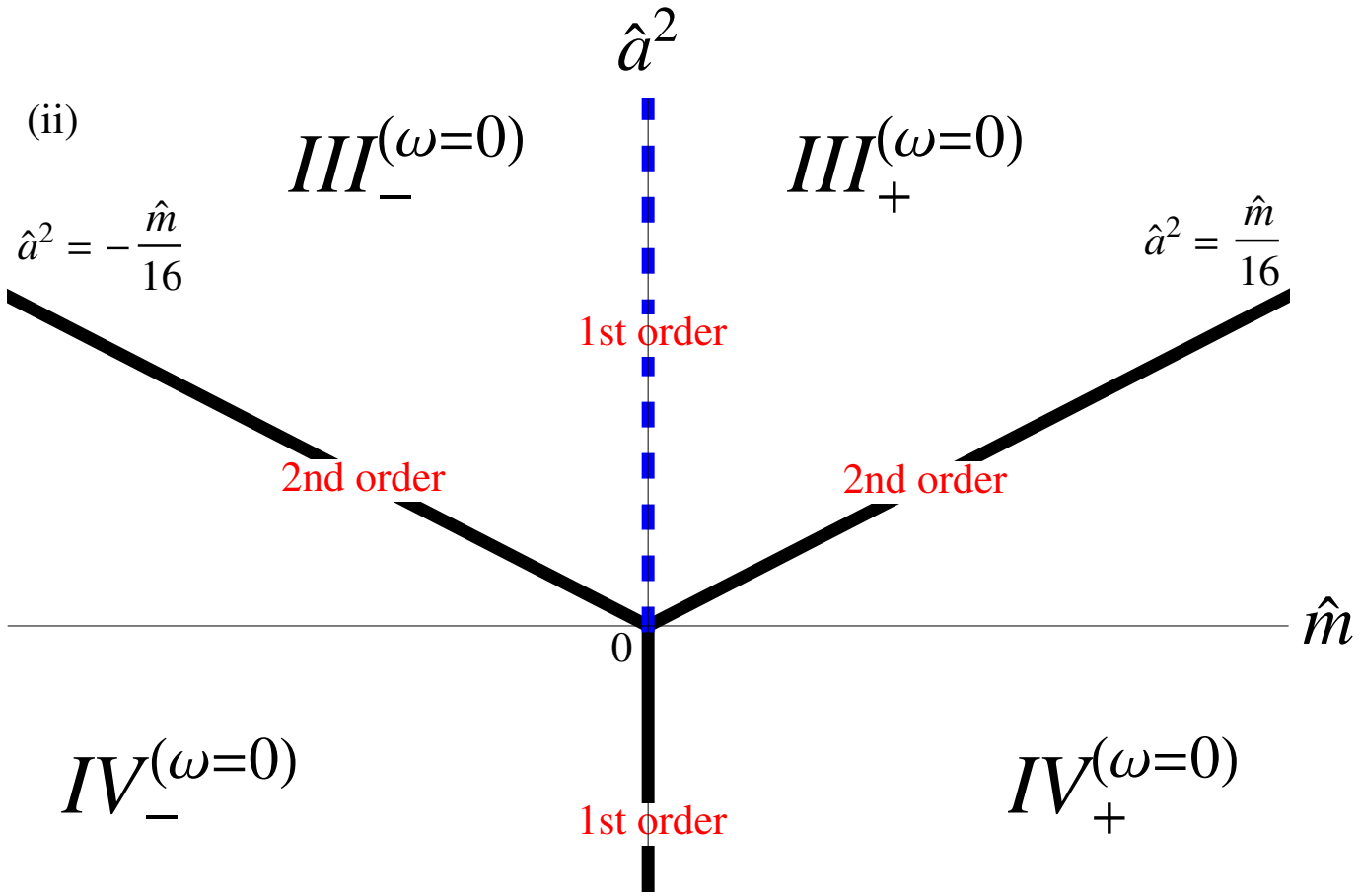
Read This: Phase Diagrams at $\omega = 0$



$\mu^2 = 0$:



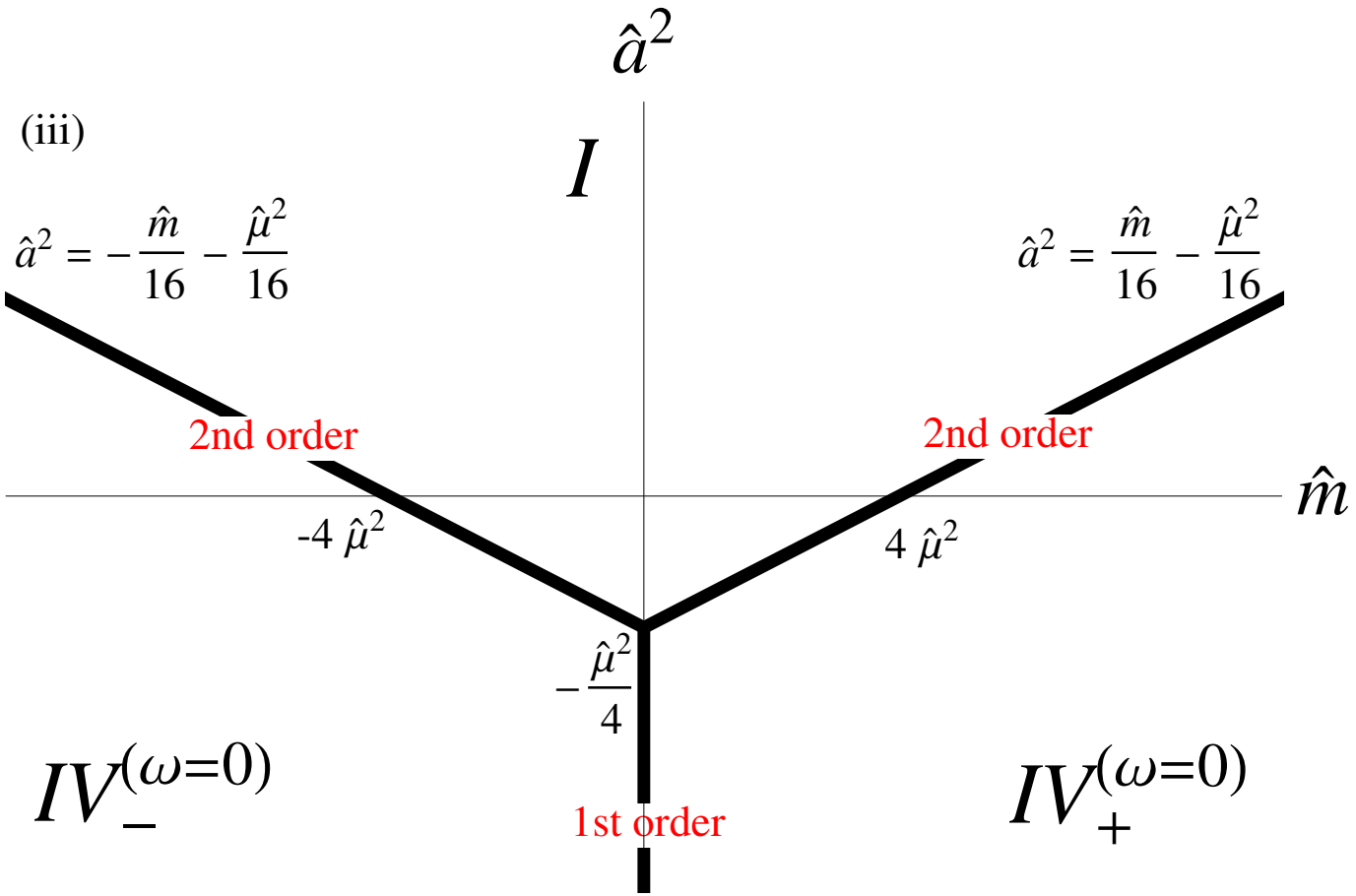
- I/III (Aoki phase): $\langle\bar{\psi}\psi\rangle \propto \widehat{m}$
  Spontaneous breaking of parity & flavour symmetry
  $\Rightarrow \langle\bar{\psi}\gamma_5\tau_3\psi\rangle = 0$ while $\lim_{\mu^2\to 0}\langle\bar{\psi}\gamma_5\tau_3\psi\rangle, \lim_{\omega\to 0}\langle\bar{\psi}\gamma_5\tau_3\psi\rangle \neq 0$
- $IV_\pm^{(\omega=0)}$: $\langle\bar{\psi}\psi\rangle \propto \mathrm{sign}\widehat{m}$; $\langle\bar{\psi}\gamma_5\tau_3\psi\rangle = 0$



$\mu^2 < 0$:

- $III_\pm^{(\omega=0)}$: $\langle\bar{\psi}\psi\rangle \propto \widehat{m}$; $\langle\bar{\psi}\gamma_5\tau_3\psi\rangle \propto \mathrm{sign}\widehat{m}$
- $IV_\pm^{(\omega=0)}$: $\langle\bar{\psi}\psi\rangle \propto \mathrm{sign}\widehat{m}$; $\langle\bar{\psi}\gamma_5\tau_3\psi\rangle = 0$

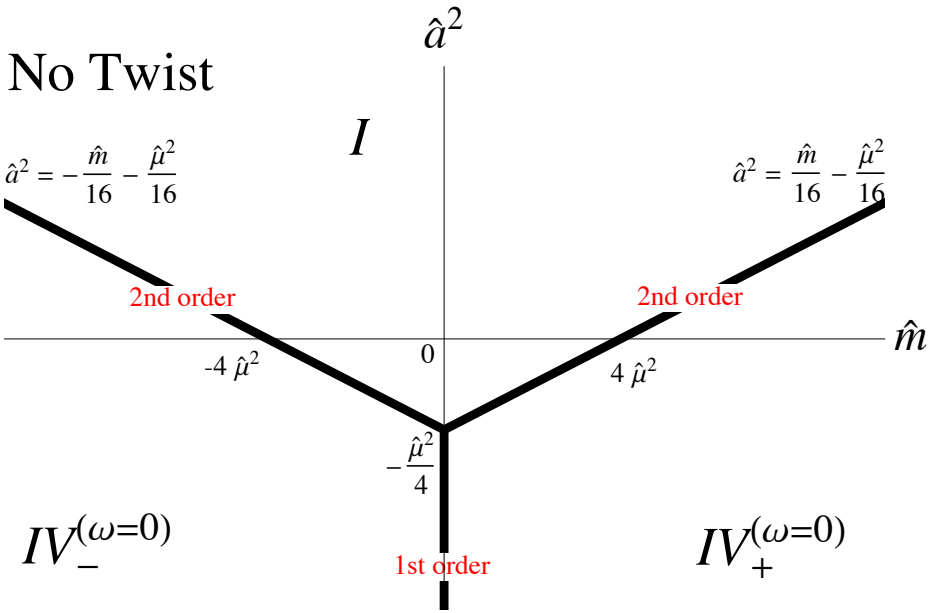
$\mu^2 > 0$:

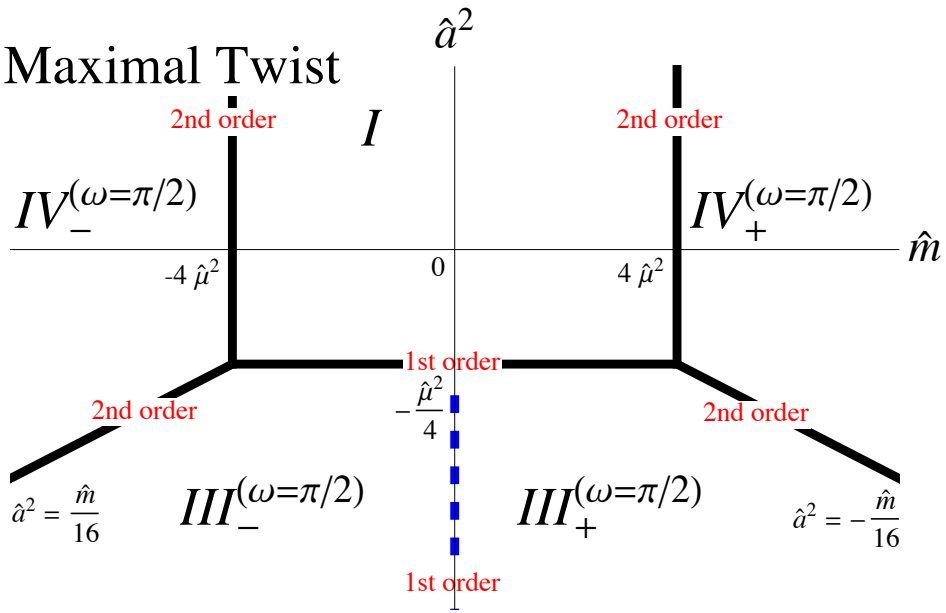


- I: $\langle\bar{\psi}\psi\rangle \propto \widehat{m}$; $\langle\bar{\psi}\gamma_5\tau_3\psi\rangle \propto \widehat{m}$
- $IV_\pm^{(\omega=0)}$: $\langle\bar{\psi}\psi\rangle \propto \mathrm{sign}\widehat{m}$; $\langle\bar{\psi}\gamma_5\tau_3\psi\rangle = 0$

Summary

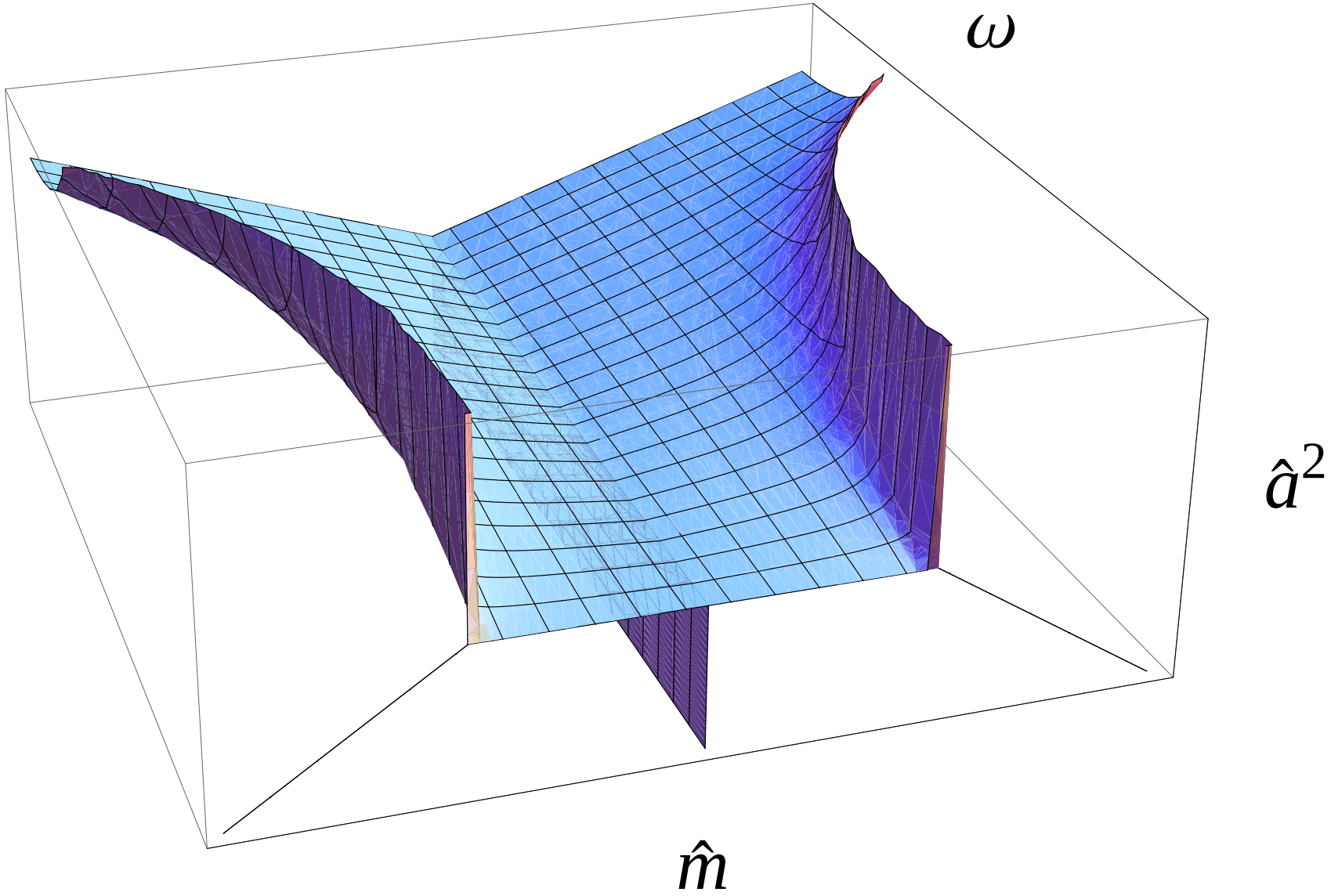
- Aoki phase is a phase boundary (1st order phase transition)
- rich phase structure (5 phases at no twist as well as at maximal twist)
- drastic change of phase diagram under twist (also in the physical situation $\widehat{\mu}^2 > 0$)





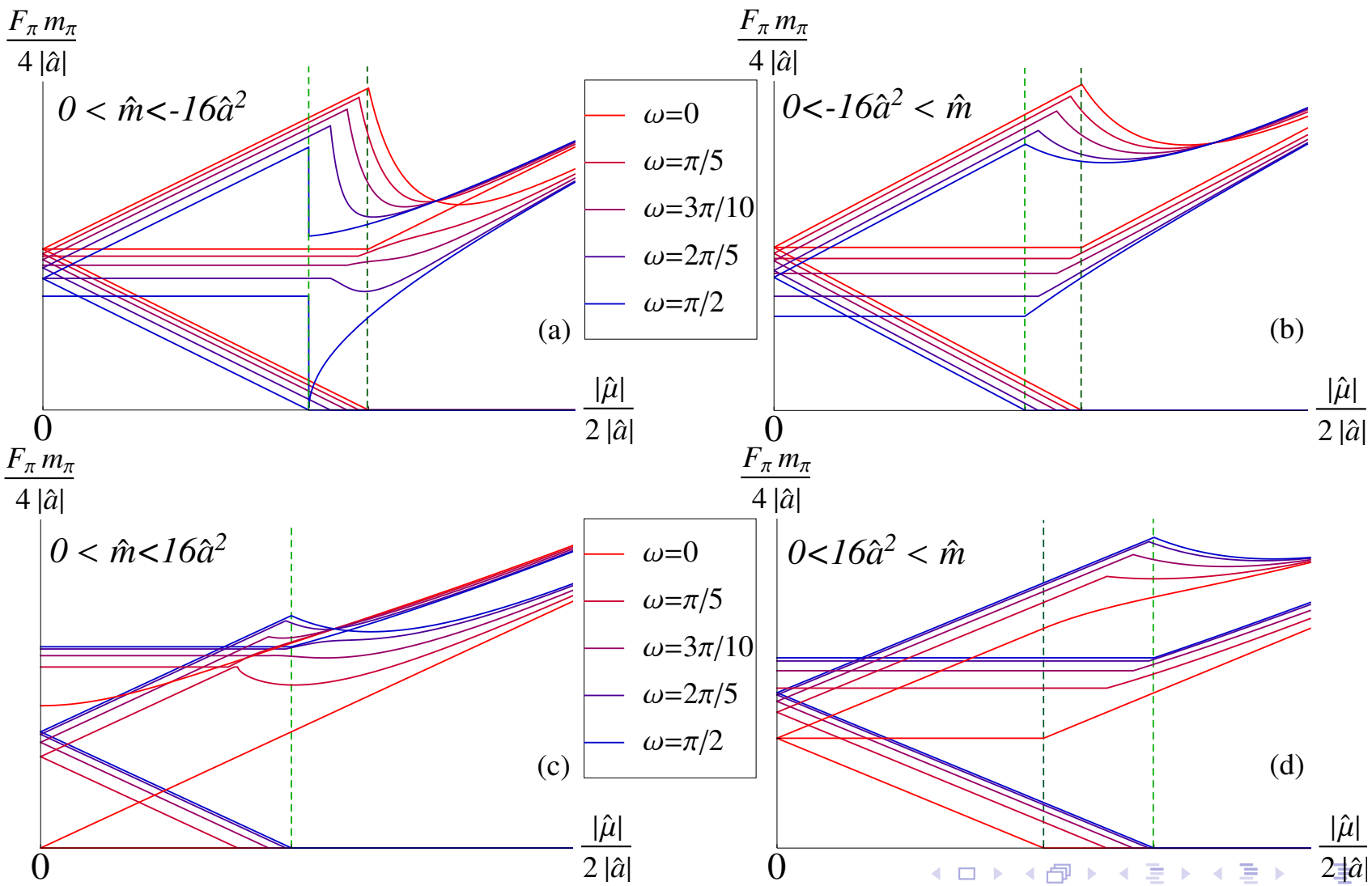
What is also done:

- phase diagram at arbitrary twist



What is also done:

- phase diagram at arbitrary twist
- dependence of the pion masses on m, a, μ, ω



There is still much work to do!

Thank you for your attention!