Signal/noise enhancement strategies for stochastically estimated correlation functions

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Correlation functions in Euclidean spacetime

\[ C_{ij}(\tau) = \langle \Omega | \hat{O}_i' e^{-\hat{H} \tau} \hat{O}_j^\dagger | \Omega \rangle \]

\[ = \sum_n Z_{in}' Z_{jn}^* e^{-E_n \tau} \]

\[ \hat{H} | n \rangle = E_n | n \rangle \quad Z_{in}' = \langle \Omega | \hat{O}_i' | n \rangle \quad Z_{jn} = \langle \Omega | \hat{O}_j | n \rangle \]
Extraction of energies

\[
m_{\text{eff}}(\tau) = -\frac{1}{\Delta \tau} \log \frac{\psi^{\dagger} C(\tau + \Delta \tau) \psi}{\psi^{\dagger} C(\tau) \psi}
\]

\[
\approx E_0 + \frac{(\psi^{\dagger} Z_1)(Z_1^{\dagger} \psi)}{(\psi^{\dagger} Z_0)(Z_0^{\dagger} \psi)} \left[ \frac{1 - e^{-(E_1 - E_0)\Delta \tau}}{\Delta \tau} \right] e^{-(E_1 - E_0)\tau}
\]

N-dimensional source vector

exponential suppression at late times

ground state energy

“excited state contamination”
Example: nucleon correlator

- Excited state contamination
- Statistical noise
- Ground state “plateau”

"Plateau region” can be short; or worse yet, nonexistent
Example: nucleon correlator

Optimized source yields an earlier plateau for ground state, yet the late time uncertainties seem significantly larger
Source overlap and signal/noise

An investigation of the interplay between excited state contamination and signal/noise
Signal/noise “landscape”

\[ \frac{S}{N} \sim \theta(\psi', \psi) = \frac{|\psi'^\dagger C\psi|}{\sigma(\psi', \psi)} \]

- Signal/noise optimized
- Source optimized?
- Second moment of correlator distribution
- Unit length source/sink vectors
The variance of a correlator is itself a correlator

\[
\sigma^2(\psi', \psi) = (\psi' \otimes \psi'^*)^\dagger \sum^2 (\psi \otimes \psi^*)
\]

\[
\sum^2 = \langle \mathcal{C} \otimes \mathcal{C}^* \rangle
\]

\[
\Sigma^2_{ik;jl}(\tau) = \sum_n \tilde{Z}'_{ik,n} \tilde{Z}^*_{jl,n} e^{-\tilde{E}_n \tau}
\]

sum over states with nontrivial valence quantum numbers

\(N^2 \times N^2\) matrix

\(N \times N\) positive matrix
Signal/noise at late times

At sufficiently late times:

\[
\theta(\psi', \psi) \sim \frac{|\psi'^\dagger Z_0'|}{\sqrt{\psi'^\dagger \tilde{Z}_0' \psi'}} \frac{|Z_0^\dagger \psi|}{\sqrt{\psi^\dagger \tilde{Z}_0 \psi}} e^{-\left(E_0 - \frac{1}{2} \tilde{E}_0\right) \tau}
\]

Exponential degradation is an inherent and unavoidable property of the system…

… but we retain some control over signal/noise via the interplay between ratios of Z-factors
Toy model: a two state system

\[ \psi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \psi' (\omega, \delta) = \begin{pmatrix} \cos \omega \\ \sin \omega e^{i\delta} \end{pmatrix} \]

(overall phase is irrelevant)

\[ \delta \in [-\pi/2, \pi/2) \]

\[ \omega \in [0, \pi) \]

Consider the correlator:

\[ \psi'(\omega, \delta)\dagger C \psi_n \propto e^{-E_n \tau} \]

Pure exponential: NO contamination from the other state!
Toy model: a two state system

Signal/noise, normalized by optimal signal/noise, can be fully parameterized by (the square-root of) a Breit-Wigner-like formula:

\[
\hat{\theta}_n(\omega, \delta) = \frac{1}{\sqrt{R_n + (R_n - 1)x_n(\omega) [x_n(\omega) - 2 \cos(\delta - \delta_n)]}}
\]

\[
x_0(\omega) = \frac{\tan \omega}{\tan \omega_0} \quad x_1(\omega) = \frac{\cot \omega}{\cot \omega_1}
\]

\[
\sqrt{R_n} = \text{enhancement factor} \ (\geq 1)
\]

\[
\omega_n, \delta_n = \text{optimal mixing angles}
\]

System-dependent parameters!
Toy model: a two state system

eigenstate vector

s/n optimized vector

\[ R_n = 10 \]
\[ R_n = 4 \]
\[ R_n = 2 \]
\[ R_n = 1.1 \]

\[ \hat{\theta}_n \]

\[ x_n(\omega) \]
Signal/noise optimization

\[ \Xi(\psi', \psi) = \log \theta^2(\psi', \psi) + \xi' \left( \psi'^\dagger \psi' - 1 \right) \]

\[ \theta(\psi', \psi) = \frac{\left| \psi'^\dagger C \psi \right|}{\sigma(\psi', \psi)} \]

Solution:

\[ \psi_0' = A_0'(\psi) \sigma_{\psi}^{-2} C \psi \]

\[ \sigma_{\psi}^2 = \langle C \psi \psi'^\dagger C'^\dagger \rangle \]

determined by normalization condition

Lagrange multiplier
Further extensions

- S/N optimize both source and sink vectors
- Impose constraints on the sources and sinks
- Include correlations between time slices
Steepest ascent

I: Source optimized vector
II, III: Intermediate vectors
IV: Signal/noise optimized vector
Application to QCD: delta

I: Source optimized
II: Intermediate
III: Signal/noise optimized

\[ \tau_{\text{ascent}} \]

\[ m_{\text{eff}} \]

\[ \tau \]

\[ \tau_{\text{min}} \]

3 exp. 2 exp. 1 exp.
Conclusion and future directions

• Proposed a new avenue for correlator optimization
  • many new ideas (see paper), but it remains a bit unclear whether there exists a context where they might be useful
  • idea is general, applicable to systems beyond QCD
  • applicable to excited states

• Many unexplored direction
  • multi-nucleon systems
  • disconnected diagrams
  • three-point functions