Signal/noise enhancement strategies for stochastically estimated correlation functions

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Correlation functions in Euclidean spacetime

$$C_{ij}(\tau) = \langle\Omega|\hat{\mathcal{O}}'_i e^{-\hat{H}\tau}\hat{\mathcal{O}}^\dagger_j|\Omega\rangle$$

N`xN matrix

$$= \sum_n Z'_{in} Z^*_{jn} e^{-E_n\tau}$$

sum over states

$$\hat{H}|n\rangle = E_n|n\rangle \qquad Z'_{in} = \langle\Omega|\hat{\mathcal{O}}'_i|n\rangle \qquad Z_{jn} = \langle\Omega|\hat{\mathcal{O}}_j|n\rangle$$

Extraction of energies

N-dimensional source vector

$$m_{eff}(\tau) = -\frac{1}{\Delta\tau}\log\frac{\psi'^{\dagger}C(\tau+\Delta\tau)\psi}{\psi'^{\dagger}C(\tau)\psi}$$

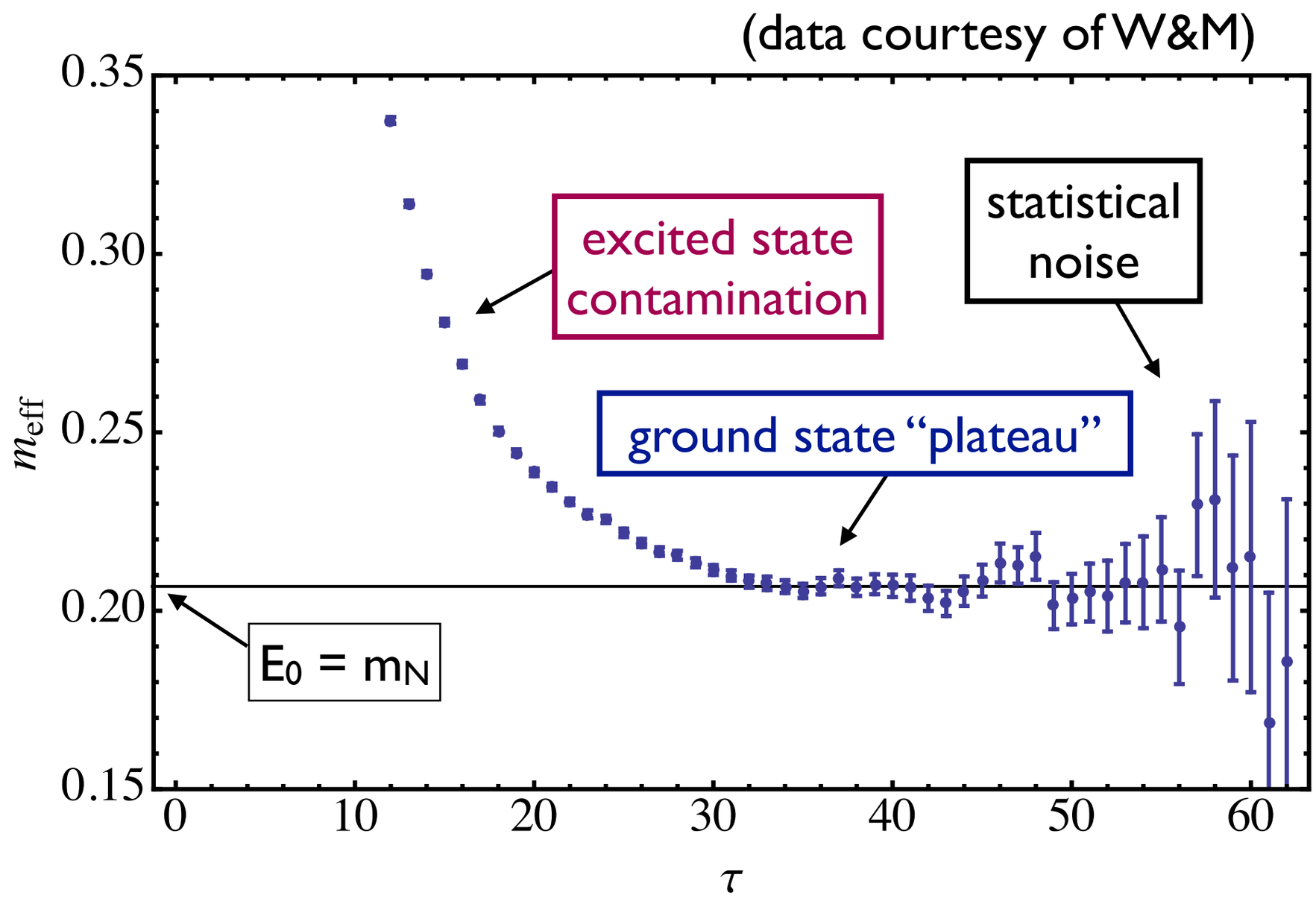
exponential suppression at late times

$$\approx E_0 + \frac{(\psi'^{\dagger}Z_1')(Z_1^{\dagger}\psi)}{(\psi'^{\dagger}Z_0')(Z_0^{\dagger}\psi)}\left[\frac{1-e^{-(E_1-E_0)\Delta\tau}}{\Delta\tau}\right]e^{-(E_1-E_0)\tau}$$

ground state energy

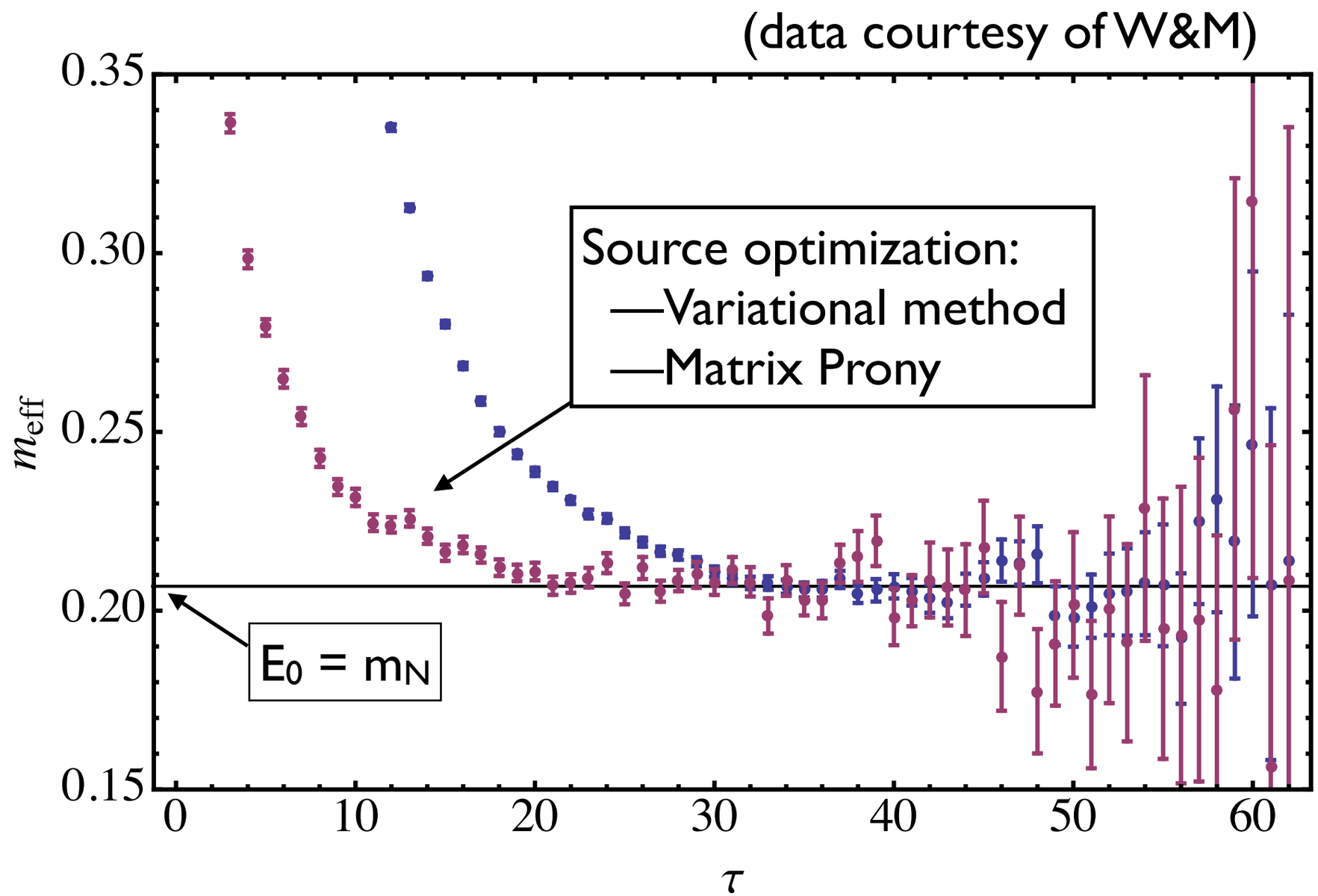
"excited state contamination"

Example: nucleon correlator



"Plateau region" can be short; or worse yet, nonexistent

Example: nucleon correlator

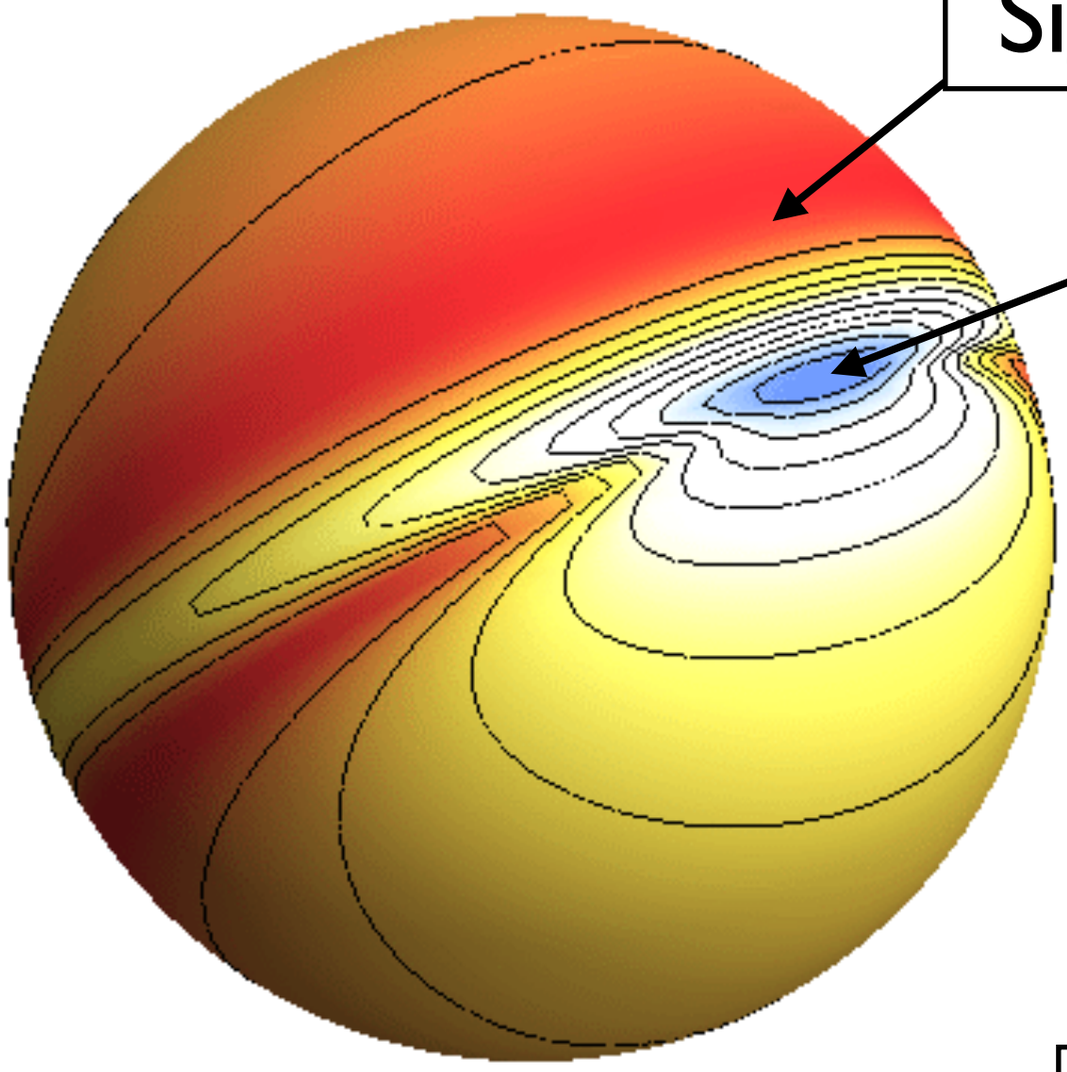


Optimized source yields an earlier plateau for ground state, yet the late time uncertainties seem significantly larger

Source overlap and signal/noise

An investigation of the interplay between excited state contamination and signal/noise

Signal/noise "landscape"



Signal/noise optimized

Source optimized?

$$\frac{S}{N} \sim \theta(\psi', \psi) = \frac{\left|\psi'^{\dagger} C \psi\right|}{\sigma(\psi', \psi)}$$

second moment of correlator distribution

unit length source/sink vectors

Behavior of the variance

The variance of a correlator is itself a correlator

$$\sigma^2(\psi', \psi) = \left(\psi' \otimes \psi'^*\right)^\dagger \Sigma^2 \left(\psi \otimes \psi^*\right)$$

$$\Sigma^2 = \langle \mathcal{C} \otimes \mathcal{C}^* \rangle$$

N`^{2}xN^2 matrix

$$\Sigma^2_{ik;jl}(\tau) = \sum_n \tilde{Z}'_{ik,n} \tilde{Z}^*_{jl,n} e^{-\tilde{E}_n \tau}$$

NxN positive matrix

sum over states with nontrivial valence quantum numbers

Signal/noise at late times

Lepage

At sufficiently late times:

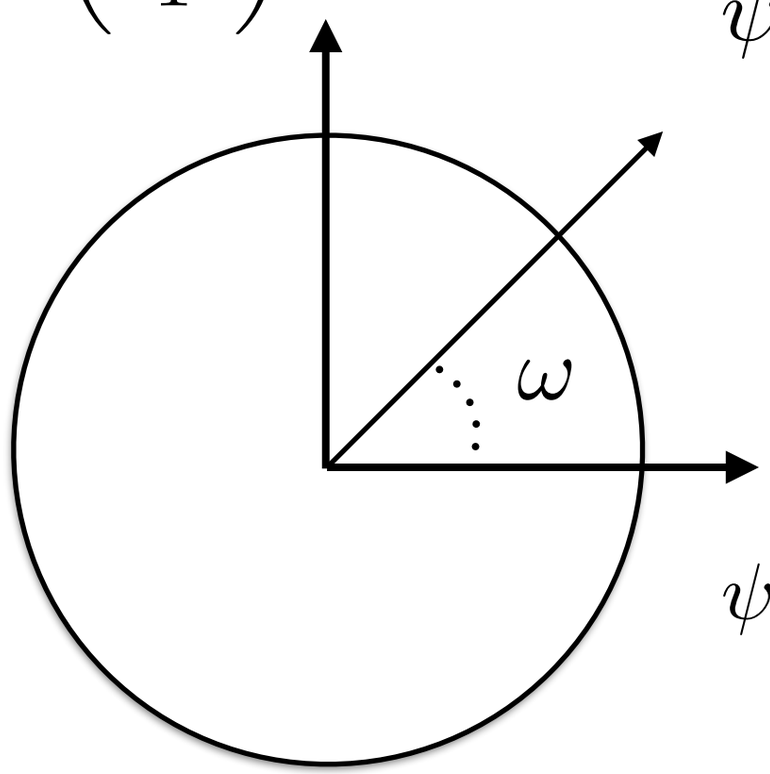
$$\theta(\psi', \psi) \sim \boxed{\frac{|\psi'^{\dagger} Z_0'|}{\sqrt{\psi'^{\dagger} \tilde{Z}_0' \psi'}} \frac{|Z_0^{\dagger} \psi|}{\sqrt{\psi^{\dagger} \tilde{Z}_0 \psi}}} \boxed{e^{-\left(E_0 - \frac{1}{2}\tilde{E}_0\right)\tau}}$$

Exponential degradation is an inherent and unavoidable property of the system…

… but we retain some control over signal/noise via the interplay between ratios of Z-factors

Toy model: a two state system

$$\psi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$\psi'(\omega, \delta) = \begin{pmatrix} \cos\omega \\ \sin\omega\, e^{i\delta} \end{pmatrix}$$

(overall phase is irrelevant)

$$\delta \in [-\pi/2, \pi/2)$$

$$\omega \in [0, \pi)$$

$$\psi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Consider the correlator:

$$\psi'(\omega, \delta)^\dagger C \psi_n \propto e^{-E_n \tau}$$

Pure exponential: NO contamination from the other state!

Toy model: a two state system

Signal/noise, normalized by optimal signal/noise, can be fully parameterized by (the square-root of) a Breit-Wigner-like formula:

$$\hat{\theta}_n(\omega,\delta) = \frac{1}{\sqrt{R_n + (R_n - 1)x_n(\omega)\left[x_n(\omega) - 2\cos(\delta - \delta_n)\right]}}$$

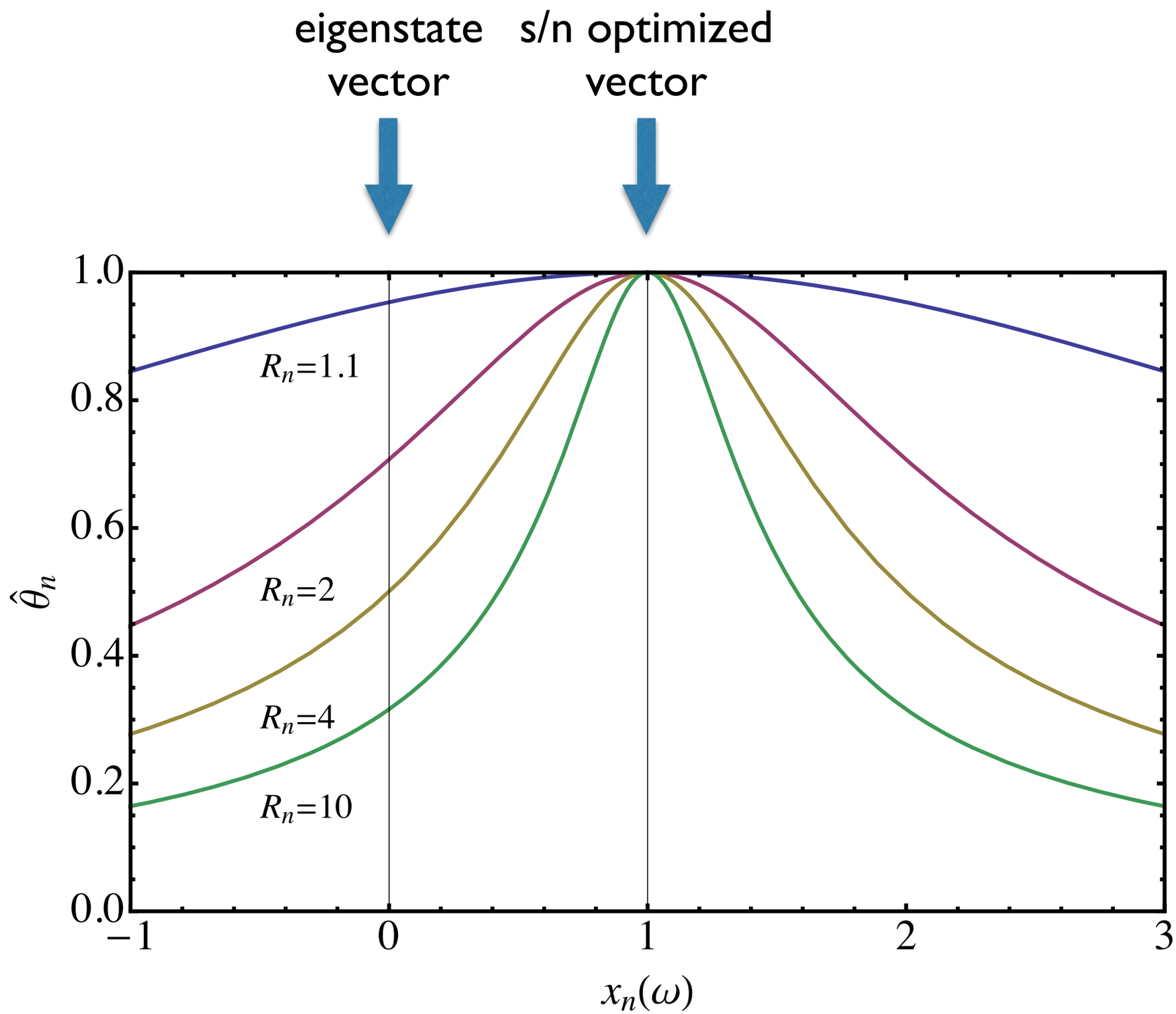
$$x_0(\omega) = \frac{\tan\omega}{\tan\omega_0} \qquad x_1(\omega) = \frac{\cot\omega}{\cot\omega_1}$$

$$\sqrt{R_n} = \text{enhancement factor } (\geq 1)$$

$$\omega_n, \delta_n = \text{optimal mixing angles}$$

System-dependent parameters!

Toy model: a two state system



Signal/noise optimization

Lagrange multiplier

$$\Xi(\psi', \psi) = \log \theta^2(\psi', \psi) + \xi' \left(\psi'^{\dagger} \psi' - 1 \right)$$

unconstrained

fixed

$$\theta(\psi', \psi) = \frac{\left| \psi'^{\dagger} C \psi \right|}{\sigma(\psi', \psi)}$$

Solution:

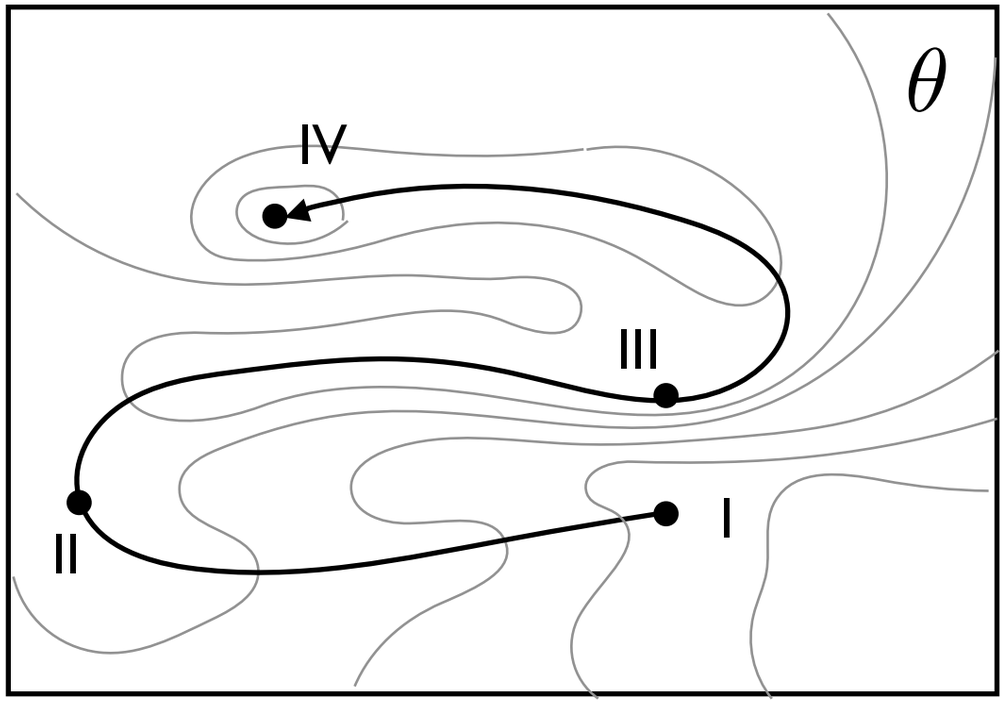
$$\psi'_0 = A'_0(\psi) \sigma_{\psi}^{-2} C \psi \qquad \sigma_{\psi}^2 = \langle \mathcal{C} \psi \psi^{\dagger} \mathcal{C}^{\dagger} \rangle$$

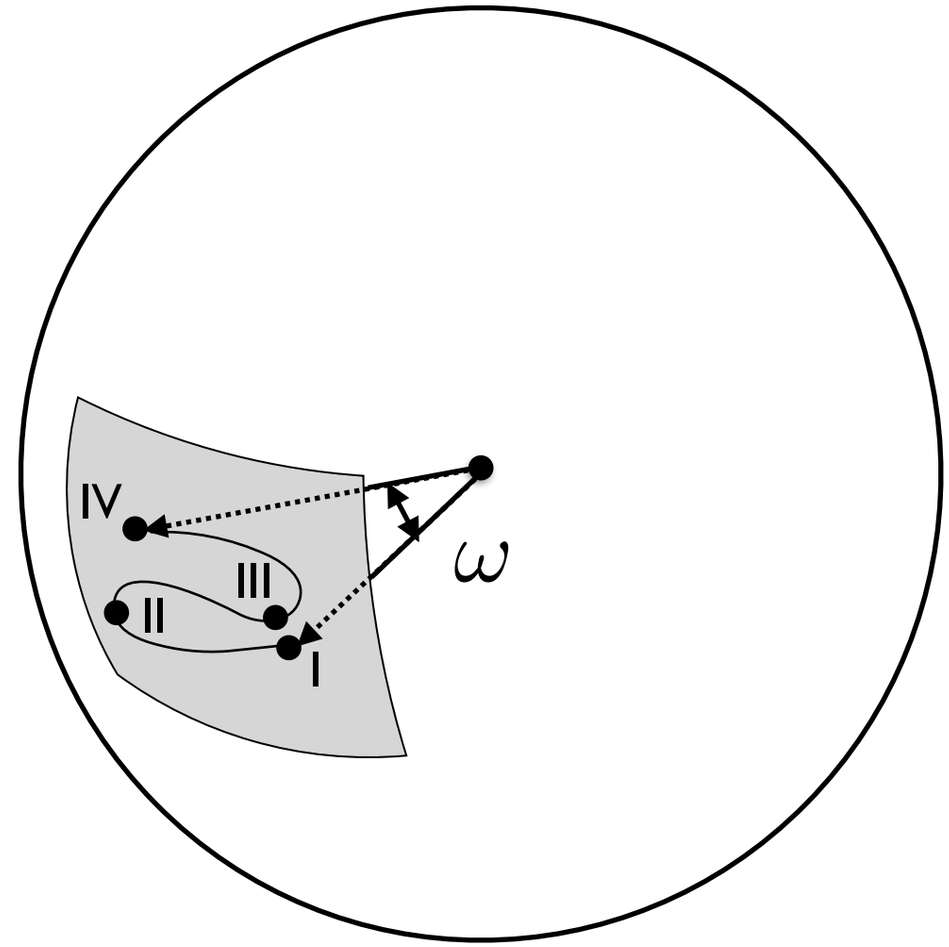
determined by normalization condition

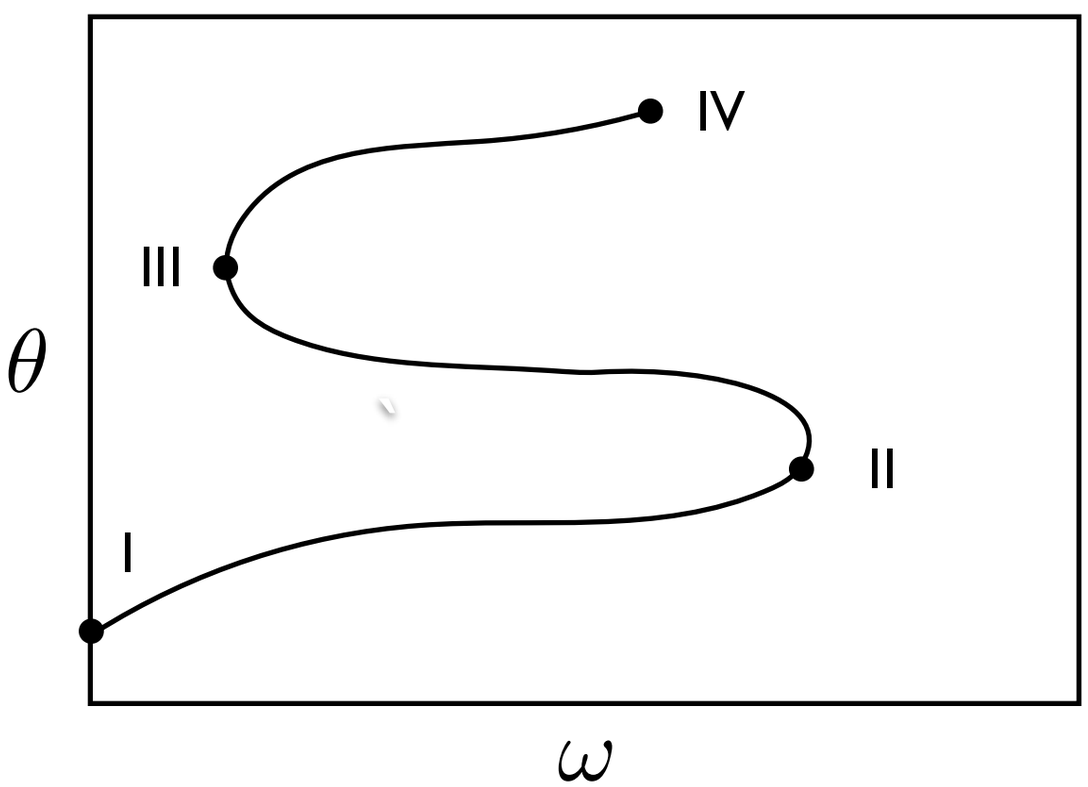
Further extensions

- ✓ S/N optimize both source and sink vectors
- ✓ Impose constraints on the sources and sinks
- ✓ Include correlations between time slices

Steepest ascent

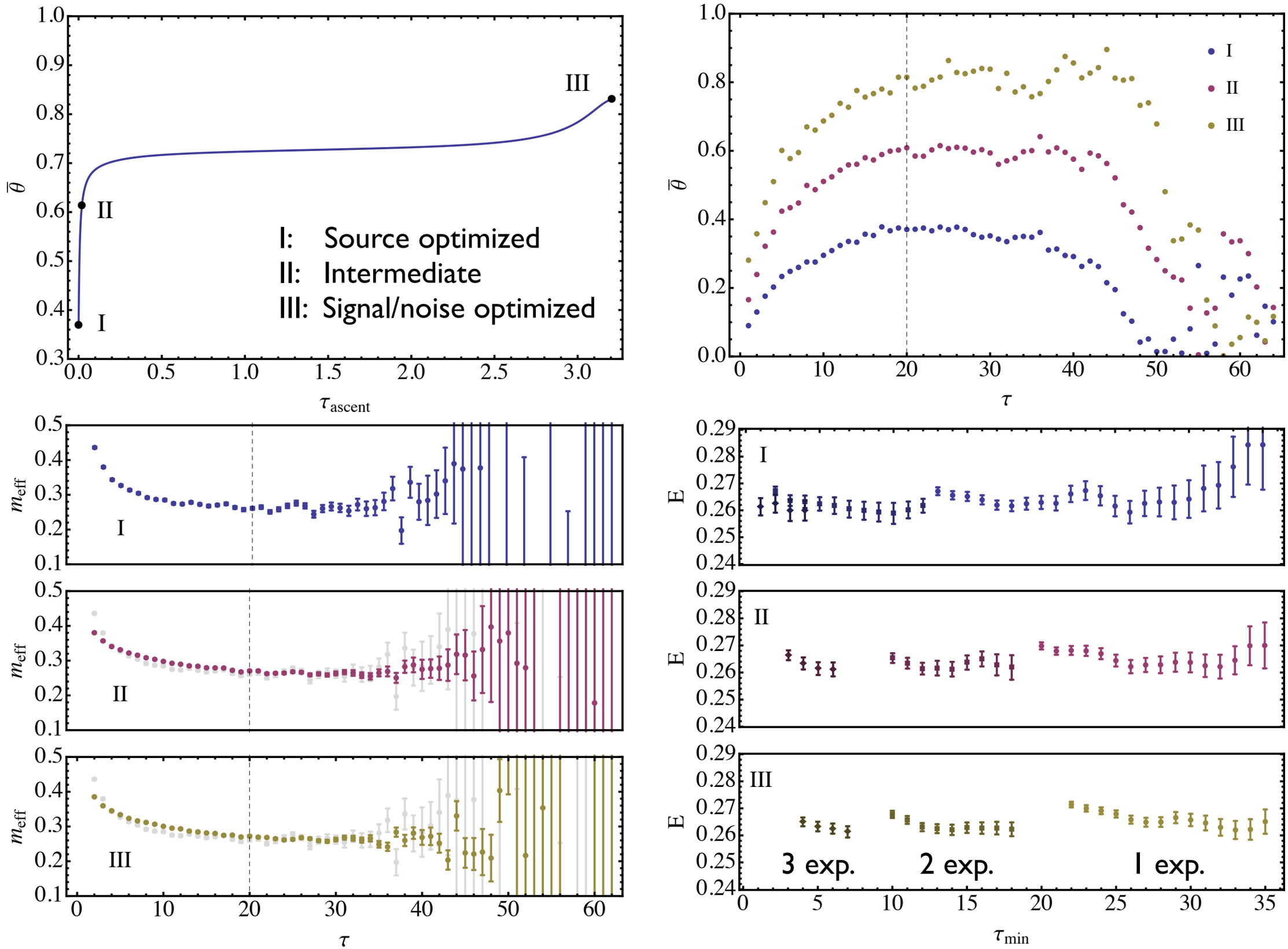






I: Source optimized vector
II, III: Intermediate vectors
IV: Signal/noise optimized vector

Application to QCD: delta



Conclusion and future directions

- Proposed a new avenue for correlator optimization
  - many new ideas (see paper), but it remains a bit unclear whether there exists a context where they might be useful
  - idea is general, applicable to systems beyond QCD
  - applicable to excited states
- Many unexplored direction
  - multi-nucleon systems
  - disconnected diagrams
  - three-point functions