Small-x physics

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CFNS Summer School on the Physics of the Electron-Ion Collider,
Stony Brook-on-the-web

Outline

- Lecture 1 (Wed): DIS in dipole picture
- Lecture 2 (Wed): Gluon saturation, CGC
- Lecture 3 (Thu): MV model
- Lecture 4 (Thu): BK equation
- Tutorial: (Thu): Discuss a few calculations that come up during lectures.
- Schedule goal: finish lecture material early on Thu to discuss tutorial problems

Exercise questions **as separate sheet in indico**

Ideology

- Not infinitely many slides: intended as lectures, not a talk!
- Some calculations on the way
- Interrupt and ask!

Literature

Very good books, basis of these lectures

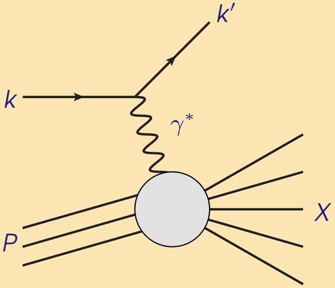
- Y. V. Kovchegov and E. Levin, "Quantum chromodynamics at high energy," Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **33** (2012).
- V. Barone and E. Predazzi, "High-Energy Particle Diffraction," Springer 2002

Reviews:

- E. Iancu and R. Venugopalan, "The Color glass condensate and high-energy scattering in QCD," In *Hwa, R.C. (ed.) et al.: Quark gluon plasma* 249-3363 [hep-ph/0303204]
- F. Gelis, E. Iancu, J. Jalilian-Marian and R. Venugopalan, "The Color Glass Condensate," Ann. Rev. Nucl. Part. Sci. **60** (2010) 463 [arXiv:1002.0333 [hep-ph]]
- H. Weigert, "Evolution at small x(bj): The Color glass condensate," Prog. Part. Nucl. Phys. **55** (2005) 461 [hep-ph/0501087]
- J. Jalilian-Marian and Y. V. Kovchegov, "Saturation physics and deuteron-Gold collisions at RHIC," Prog. Part. Nucl. Phys. **56** (2006) 104 [hep-ph/0505052]
- F. Gelis, T. Lappi and R. Venugopalan, "High energy scattering in Quantum Chromodynamics," Int. J. Mod. Phys. E **16** (2007) 2595 [arXiv:0708.0047 [hep-ph]]

1 DIS in the dipole picture

DIS kinematics

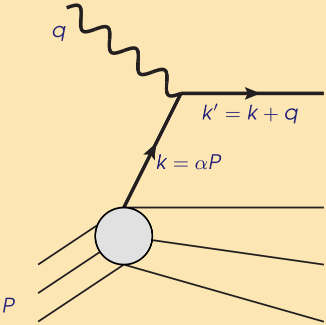


$$
\begin{aligned}
s &= (k+P)^2 \\
q &= k-k' \quad q^2 \equiv -Q^2 \\
W^2 &= (P+q)^2 \\
\nu &= P\cdot q/m_N \\
x &= \frac{Q^2}{2P\cdot q} = \frac{Q^2}{2\nu m_N} = \frac{Q^2}{W^2+Q^2-m_N^2} \\
y &= \frac{2P\cdot q}{2P\cdot k} = \frac{W^2+Q^2-m_N^2}{s m_N^2}
\end{aligned}
$$

Topic of these lectures: high energy limit $x \to 0$

- This is when $W^2 \to \infty$; $\nu \to \infty$; i.e. the virtual photon-target c.m.s. energy is high.
- Now Q^2 is "fixed". (DGLAP limit is x fixed, Q^2 large)
- Useful to think of W as longitudinal momentum, Q as transverse

Infinite momentum frame IMF



Assume photon hits quark carrying fraction α of target momentum. On-shell outgoing quark:

$$0 = (q+\alpha P) = -Q^2 + 2\alpha P\cdot q + \overbrace{\alpha^2 m_N^2}^{\approx 0}$$

$$\Longrightarrow \quad \alpha = \frac{Q^2}{2P\cdot q} = x$$

$x =$ momentum fraction

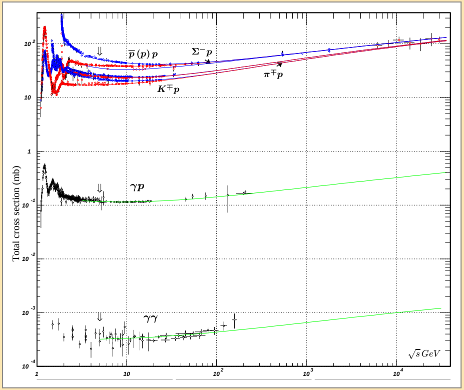
Identify kinematical variable x (measured from electron) with momentum fraction in IMF, @LO
(In rest frame $k = \alpha P$ not good approx)

Cross sections vs. energy

I want to convince you that the γ^* is the theorist's favorite hadron!

γ scattering behaves just like p scattering — apart from extra $\frac{1}{137}$

Same should be true for γ^*

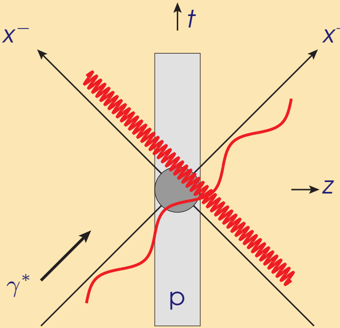


Target Rest Frame TRF kinematics

Light cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(t \pm z)$
(Notes: boldface $\mathbf{x}_\perp$ is 2d transverse, photon right-moving)

$$\text{Target } P^\mu = (\overset{0}{m}, \overset{\perp}{\mathbf{0}}, \overset{z}{0}) \Longrightarrow (\overset{+}{m/\sqrt{2}}, \overset{-}{m/\sqrt{2}}, \overset{\perp}{\mathbf{0}})$$

$$\text{Photon } q^\mu = (\overset{0}{\nu}, \overset{\perp}{\mathbf{0}}, \overset{z}{\sqrt{\nu^2 + Q^2}}) \Longrightarrow (\overset{+}{q^+}, \overset{-}{-Q^2/(2q^+)}, \overset{\perp}{\mathbf{0}})$$



High energy: $q^+ \approx \sqrt{2}\nu$ **big**
Look at γ^* wavefunction $e^{-i(q^+x^- + q^-x^+)}$

- Very accurate resolution in x^-
- No resolution in x^+
  Scattering instantaneous in x^+
  compared to natural timescale of γ^*

In particular γ^* cannot change into a hadronic final state **inside** proton; it has to fluctuate into hadrons before.

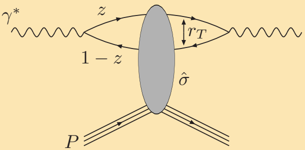
Physical picture in IMF vs TRF

- Standard parton model: γ^* takes "picture" of target partons instantaneously in x^-
  (If you think this way you switch $z \to -z$)
- **Here:** color field of target takes "picture" of partons in γ^* instantaneously in x^+.
- Kinematics: fluctuations in γ^* long-lived in x^+
  $\Longrightarrow$ γ^* fluctuates into partons before collision.
  Partons interact with gluon **shockwave** ($\sim \delta(x^+)$)

Cross section is Lorentz-invariant
— physical picture is very frame-dependent!

Dipole picture of DIS

Simplest hadronic state in the interacting γ^*: quark-antiquark dipole.



$$\sigma_{T,L}^{\gamma^* p} = \int \mathrm{d}^2\mathbf{r}_\perp \, \mathrm{d}z \left|\psi^{\gamma^* \to q\bar{q}}(\mathbf{r}_\perp, z)_{T,L}\right|^2 2\mathrm{Im}\mathcal{A}$$

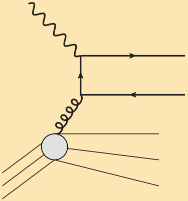
High energy: we assume (lifetime/timescale) factorization between

- $\left|\psi^{\gamma^* \to q\bar{q}}(\mathbf{r}_\perp, z)_{T,L}\right|^2$: probability for photon to fluctuate into $\bar{q}q$
- $2\mathrm{Im}\mathcal{A}$ imaginary part of the forward elastic scattering amplitude, i.e. the total cross section; optical theorem

Remarks: same process in IMF

In usual parton model picture one would draw same diagram as:

- Looks like formally higher order (NLO DIS)
- Does not describe valence quarks



This is however the leading contribution at small-x

- The valence quark distribution is small
- DGLAP sea quarks come from gluons: $xq(x,Q^2) \sim \alpha_s \ln Q^2 xg(x,Q^2)$
  $\Longrightarrow$ power counting depends on $\alpha_s \ln Q^2 \sim 1$ vs $\alpha_s \ln Q^2 \sim \alpha_s$
- Same diagram can be read as both
  - $\gamma^* + g \to q + \bar{q}$: a NLO DIS process $\sim \alpha_s$
  - 1 DGLAP split $g \to q + \bar{q}$, $\sim \alpha_s$ + LO DIS process $\gamma^* + q \to q$
- At NLO split btw quark and gluon distributions scheme dependent
- Matching dipole picture to collinear $xg(x,Q^2)$ and $xq(x,Q^2)$ only done at leading log (where only $xg(x,Q^2)$ matters).

Quantizing the photon: LCPT

- Recall: want to understand the **partonic content of the photon**
- Unlike proton, γ^* is a perturbative object: calculable
- The theoretical tool of choice is Light Cone Perturbation Theory

What is Light Cone Perturbation Theory (LCPT)?

- Heisenberg quantization: time-dependent operators $\hat{A}(t)$, **equal-time** commutation relations $[\hat{A}(t), \hat{B}(t)]$ known.
- Then solve equation of motion $\partial_t \hat{A}(t) = -i[\hat{A}(t), \hat{H}(t)]$
- LCPT: choose the "time" variable to be light-like: x^+

Advantages and disadvantages

+ Only physical degrees of freedom $\Longrightarrow$ partonic interpretation
+ Maximal number of commuting Lorentz generators
+ Longitudinal boosts are easy $\Longrightarrow$ high energy
- 3d rotational invariance is hard
- Connection to lattice, hadron rest frame difficult

Idea of LCPT calculation

- Know free particle Fock states: $|\gamma^*\rangle_0, \quad |q\bar{q}\rangle_0, \quad |q\bar{q}g\rangle_0$ etc.
- **Interacting** states are superpositions of these:

$$|\gamma^*\rangle = (1+\dots)|\gamma^*\rangle_0 + \psi^{\gamma^*\to q\bar{q}} \otimes |q\bar{q}\rangle_0 + \psi^{\gamma^*\to q\bar{q}g} \otimes |q\bar{q}g\rangle_0 + \dots$$

- Calculate in QM perturbation theory, e.g. ground state $|0\rangle$ wavefunction:

$$\psi^{0\to n} = \sum_n \frac{\langle n|\,\hat{V}\,|0\rangle}{E_n - E_0} + \dots$$

  - Here $1/\Delta E$ is $\sim$ the lifetime of the quantum fluctuation from 0 to n
- "Energy" E is conjugate to "time", LC time is $x^+ \implies$ LC energy k^-
- Note: energy not "conserved," only 3-momentum $\vec{k} = (k^+, \mathbf{k}_\perp)$ is

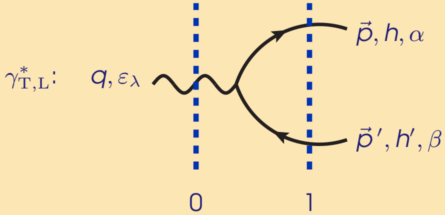
Connection to Feynman perturbation theory

- Matrix elements $\langle n|\,\hat{V}\,|m\rangle$ are vertices in Feynman rules
- LC energy denominators from propagators, integrating over k^- with pole

Let's calculate $\psi^{\gamma^* \to q\bar{q}}$

Steps for calculating light cone wave function:

$$\psi^{\gamma^*_{\mathrm{T/L}} \to q\bar{q}} = \frac{-ee_f \delta_{\alpha\beta}}{q^- - p^- - p'^-} \left[\bar{u}_h(p) \not\epsilon_{\lambda,\mathrm{T/L}}(q) v_{h'}(p') \right]$$



- Color factor $\delta_{\alpha\beta}$ and electric charge ee_f
- Energy denominator $k_0^- - k_1^-$
- Matrix element $\bar{u} \not\epsilon v_{h'}$
- Fourier transform to transverse coordinate space
  (Why coordinate space? Eikonal scattering, will come back to this in next lecture)

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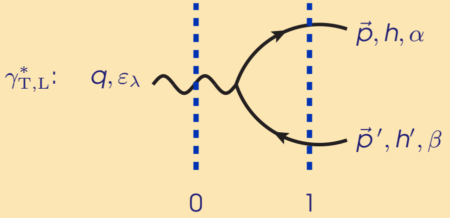

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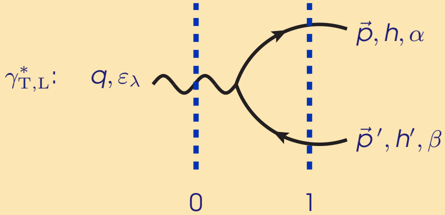

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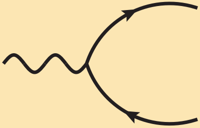
Matrix element

Matrix element
$\bar{u}_h(p)\not{\varepsilon}_{\lambda,\mathrm{T/L}}(q)v_{h'}(p')$ with



$$h, h' = \pm\frac{1}{2}; \quad \lambda = 0 = L, \quad \lambda = \pm 1 = T$$

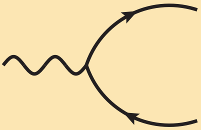
- Transv. polarization vector (LC gauge $\varepsilon^+ = 0$!!)
  $\varepsilon^\mu_{\lambda=\pm}(q) = (0, \frac{\mathbf{q}_\perp \cdot \boldsymbol{\varepsilon}_{\perp\lambda}}{q^+}, \boldsymbol{\varepsilon}_{\perp\lambda}) \overset{\mathbf{q}_\perp = \mathbf{0}}{\longrightarrow} (0, 0, \boldsymbol{\varepsilon}_{\perp\lambda}),$
- Circularly polarized 2d polarization vectors $\boldsymbol{\varepsilon}_{\perp\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mp 1 \\ -i \end{pmatrix}$
- Longit. polarization effectively
  $\varepsilon^\mu_{\lambda=0}(q) = (0, \frac{\sqrt{Q^2 - \mathbf{q}_\perp^2}}{q^+}, \frac{\mathbf{q}_\perp}{\sqrt{Q^2 - \mathbf{q}_\perp^2}}) \overset{\mathbf{q}_\perp = \mathbf{0}}{\longrightarrow} (0, \frac{Q}{q^+}, \mathbf{0}),$
- Using tables for matrix elements

$$\bar{u}\not{\varepsilon}v = \frac{2\delta_{h,-h'}}{\sqrt{z(1-z)}}\left(z\delta_{\lambda,2h} - (1-z)\delta_{\lambda,-2h}\right)\boldsymbol{\varepsilon}_{\perp\lambda}\cdot\mathbf{k}_\perp + \delta_{h,h'}\delta_{\lambda,2s}\frac{\sqrt{2}m}{\sqrt{z(1-z)}}.$$

Remarks:

- Only relative center-of-mass momentum $\mathbf{k}_\perp = \mathbf{p}_\perp - z\mathbf{q}_\perp$
- Quark helicity conserving $\sim \mathbf{k}_\perp$ + helicity-flip $\sim m$

Energy denominator



Energy denominator $(q^- - k^- - k'^-)^{-1}$ (On-shell momenta!)

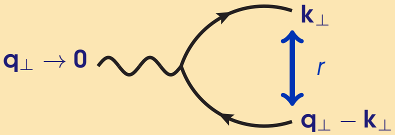
$$q^- - p^- - p'^- = -\left(\overbrace{\frac{Q^2}{2q^+}}^{-q^-} + \overbrace{\frac{\mathbf{p}_\perp{}^2 + m^2}{2zq^+}}^{p^-} + \overbrace{\frac{(\mathbf{p}_\perp - \mathbf{q}_\perp)^2 + m^2}{2(1-z)q^+}}^{p'^-} \right)$$

$$\frac{1}{q^- - p^- - p'^-} = \frac{-2q^+ z(1-z)}{\underbrace{Q^2 z(1-z) + m^2}_{\equiv \bar{Q}^2} + \mathbf{k}_\perp^2}.$$

- Also only relative center-of-mass momentum $\mathbf{k}_\perp = \mathbf{p}_\perp - z\mathbf{q}_\perp$

Fourier transform

Scattering at high energy is **eikonal**: transverse **position** of parton does not change $\Longrightarrow$ Fourier-transform $\mathbf{k}_\perp \to \mathbf{r}_\perp$



$$\text{L:} \quad \int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \frac{1}{\mathbf{k}_\perp^2 + \bar{Q}^2} \sim K_0(r\bar{Q}) \qquad \text{exercise}$$

$$\text{T:} \quad \int d^2\mathbf{k}_\perp e^{i\mathbf{k}_\perp \cdot \mathbf{r}_\perp} \frac{k^i}{\mathbf{k}_\perp^2 + \bar{Q}^2} \sim \frac{r^i}{r} K_1(r\bar{Q})$$

- Recall $\bar{Q}^2 \equiv z(1-z)Q^2 + m^2$.
- Note asymptotics $K_{0,1}(x) \sim e^{-x}$ $\Longrightarrow$ enforces $r \sim 1/Q$.

DIS dipole frame: summary

- Picture DIS as γ^* scattering on target
- At high energy (in TRF) γ^* fluctuates into $q\bar{q}$

$$\sigma^{\gamma^* p}_{T,L} = \int \mathrm{d}^2\mathbf{r}_\perp \, \mathrm{d}z \left|\psi^{\gamma^*\to q\bar{q}}(r,z)_{T,L}\right|^2 2\mathrm{Im}\mathcal{A}$$

$$\left|\psi_T^{\gamma^*\to q\bar{q}}\right|^2 = \frac{\alpha_{\mathrm{e.m.}}}{2\pi^2} N_c e_f \left(\left[z^2+(1-z)^2\right] K_1^2(\varepsilon r) + m_f^2 K_0^2(\varepsilon r)\right)$$

$$\left|\psi_L^{\gamma^*\to q\bar{q}}\right|^2 = \frac{\alpha_{\mathrm{e.m.}}}{2\pi^2} N_c e_f 4Q^2 z^2 (1-z)^2 K_0^2(\varepsilon r)$$

- Typical dipole size: $r \sim 1/Q$
- Used optical theorem: $2\mathrm{Im}\mathcal{A}$ is total cross section
  - can also take $|\mathcal{A}|^2$: elastic scattering (diffractive DIS)
- We are assuming that fixed-size dipoles are the basis that diagonalizes the imaginary part of the T-matrix
  - This makes sense in an eikonal approximation for the scattering
  - In general: high energy/eikonal approximation: particles fly through target at fixed $\mathbf{x}_\perp$; does not imply zero momentum transfer!

2 Gluon saturation and the CGC

CGC basics

- Until now: $\gamma^* \to q\bar{q}$. But what is the target?
- CGC separation of scales:
  - small x: classical field
  - large x: color charge

2 topics for this lecture

1. A high energy particle scatters off a color field: Wilson line
2. What does the color field look like in spacetime?

Next lecture: a concrete calculable model: the McLerran-Venugopalan model

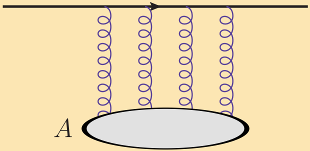
What is the target made of?

- So far we have not specified anything about the degrees of freedom in the target.
- At high energy the target consists dominantly of gluons
  - Experimentally: small x gluon distribution is larger than the quark one.
  - High $\sqrt{s}$: QCD radiation builds up the target by adding gluons to it.

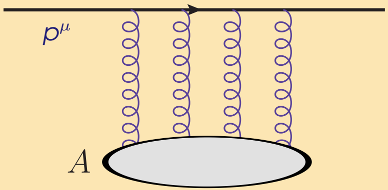
Color Glass Condensate (CGC)

We assume that there are so many gluons in the target, that it can be described by a classical gluon field. This is the heart of the CGC effective theory.

Many gluons = large color field A_μ
Sum all diagrams with n gluon lines
— but we can assume gluons are a classical field



Scattering off CGC target



Quark propagating in classical color field: Dirac equation!

$$(i\partial\!\!\!/ - gA\!\!\!/)\psi(x) = 0$$

(Note: $A\!\!\!/ = A_a^\mu \gamma_\mu t^a$ is $N_c \times N_c$-matrix)

Want to dig out the dominant contribution: **eikonal** approximation

- Gluon is spin 1: it couples to a vector: $\sim p^\mu A_\mu$
- For high energy particle the only vector available is p^μ
- p^μ has one large component: $p^+ \implies p^\mu A_\mu \sim p^+ A^-$

$\implies$ only need A^-

Ansatz for DE: $\psi(x) = V(x) e^{-ip\cdot x} u(p)$, plug in eq. **$N_c \times N_c$-matrix!**

$$\implies \partial_+ V(x^+, x^-, \mathbf{x}_\perp) = -igA^-(x^+, x^-, \mathbf{x}_\perp) V(x^+, x^-, \mathbf{x}_\perp)$$

This is solved by path-ordered exponential

$$V(x^+, x^-, \mathbf{x}_\perp) = \mathbb{P} \exp \left\{ -ig \int^{x^+} dy^+ A^-(y^+, x^-, \mathbf{x}_\perp) \right\}$$

Eikonal propagation

- Now know how high energy quark propagates in a classical field.
- Thus know the scattering S-matrix for many-quark states

E.g. incoming free quark $|q_i(\mathbf{x}_\perp)\rangle$ at $x^+ \to -\infty$ is, at $x^+ \to \infty$

$$|q_i(\mathbf{x}_\perp)\rangle_{\text{in}} = \left[\mathbb{P}\exp\left\{-ig\int_{-\infty}^{\infty} \mathrm{d}y^+ A^-(y^+, x^-, \mathbf{x}_\perp)\right\}\right]_{ji} |q_j(\mathbf{x}_\perp)\rangle_{\text{out}}$$

a linear superposition of color rotated outgoing quarks.

- In scattering problem integrate $x^+ \in [-\infty, \infty]$
- In the high energy limit quark wavefunction oscillates like $e^{ip^+x^-}$ with large p^+ $\Longrightarrow$ x^--dependence negligible compared to this $\Longrightarrow$ approximate $x^- = 0$

Scattering described by 2-d field of SU(N_c)-matrices

$$V(\mathbf{x}_\perp) \equiv \mathbb{P}\exp\left\{-ig\int_{-\infty}^{\infty} \mathrm{d}x^+ A^-(x^+, x^- = 0, \mathbf{x}_\perp)\right\}$$

— These is known as the **Wilson lines**

Dipole amplitude and Wilson lines

Incoming dipole becomes
(color neutral, average over colors! ; $V(\mathbf{y}_\perp)^\dagger_{jk} = V(\mathbf{y}_\perp)^*_{kj}$ for antiquark)

$$|\text{in}\rangle = \frac{\delta_{ii'}}{N_c} |q_i(\mathbf{x}_\perp)\bar{q}_{i'}(\mathbf{y}_\perp)\rangle_{\text{in}} = \frac{\delta_{ii'}}{N_c} V_{ji}(\mathbf{x}_\perp)V^\dagger_{i'j'}(\mathbf{y}_\perp)\, |q(\mathbf{x}_\perp)_j\bar{q}(\mathbf{y}_\perp)_{j'}\rangle_{\text{out}}$$

Total cross section: imaginary part of **forward elastic scattering amplitude**; i.e. count outgoing dipoles in this state

$$S = {}_{\text{out}}\langle q_k(\mathbf{x}_\perp)\bar{q}_k(\mathbf{y}_\perp)|\ \text{in}\rangle = \frac{\delta_{ii'}}{N_c}\delta_{kj}\delta_{kj'} V_{ji}(\mathbf{x}_\perp)V^\dagger_{i'j'}(\mathbf{y}_\perp) = \frac{1}{N_c}\operatorname{Tr} V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)$$

Dipole amplitude in the CGC

Relate DIS amplitude $\mathcal{N}$ to a **microscopical description of the target**:

$$\mathcal{N}_{q\bar{q}} = 1 - \frac{1}{N_c}\operatorname{Tr} V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)$$

Note conventions

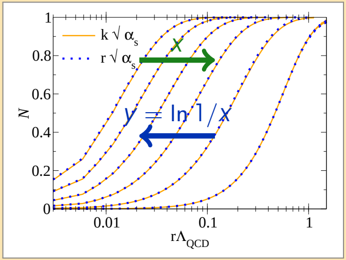
$$S_{fi} = \langle f|\hat{S}|f\rangle = 1 + iT_{fi} \quad \sigma_{\text{tot}} = 2\operatorname{Im}T_{ii} \quad \mathcal{N} \equiv \operatorname{Im}T_{ii} \quad S_{ii} = \delta_{ii} - \mathcal{N} + \text{imag}$$

Saturation of the dipole cross section

Basic features following from

$$\mathcal{N}_{q\bar{q}} = 1 - \frac{1}{N_c} \mathrm{Tr}\, V(\mathbf{x}_\perp) V^\dagger(\mathbf{y}_\perp)$$

- Small dipoles:
  - At $\mathbf{x}_\perp = \mathbf{y}_\perp$: $V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp) = \mathbb{1}$.
  - At $r = 0$ color neutral system, should not scatter by the strong interaction!
  - $\mathcal{N}_{q\bar{q}}(r) \sim r^2 \implies$ perturbative limit
- Large dipoles
  - Fully uncorrelated Wilson lines: $\langle \mathrm{Tr}\, V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)\rangle = 0$
  - Large dipoles $r \gtrsim 1/Q_s$ scatter with $\mathcal{N}_{q\bar{q}} \lesssim 1$



- Know that $\mathcal{N}_{q\bar{q}}$ grows at $x \to 0 \implies$ turnover from $\mathcal{N} \ll 1$ to $\mathcal{N} \sim 1$ happens at smaller $r \sim R_s$ when $\sqrt{s} \to 0$

Nonperturbative weak coupling unitarization = Saturation

Saturation scale $Q_s = 1/R_s$ = inverse distance for turnover

Classical field and equation of motion

- We were describing the high energy nucleus as a classical field: A^- $\Longrightarrow$ Wilson line
- What does this imply for the partonic content of the nucleus?
- The physical picture of "gluons as partons" requires two things
  - Infinite momentum frame: nucleus moving fast
    Change direction: nucleus moves now in $+z$**-direction** with large p^+.
    Means we have large A^+ component.
  - Light cone gauge: have to gauge transform to $A^+ = 0$
- But let us start with the "classical" part.

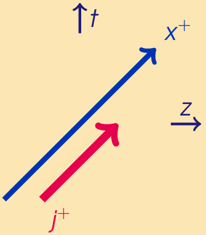
Classical field $\equiv$ from equation of motion

$$[D_\mu, F^{\mu\nu}] = J^\mu$$

What remains is

$$\nabla_\perp^2 A^+ = J^+$$

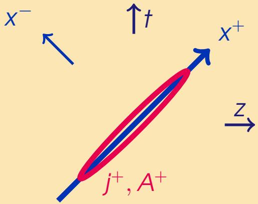
This is nice, the big $+$-field corresponds to a **color** current in the $+$-direction.



Spacetime structure of the field: covariant gauge

I Current lives on the light cone.

1. Naive explanation: Nucleus is Lorentz-contracted to $\Delta z \sim 2R_A m_A/\sqrt{s}$
2. Real explanation: Current represents large x degrees of freedom
   - Current: large p^+, field small p^+
   - Current more localised in x^- than field.



II Current independent of LC time x^+;

glass!

Argument is as above:

1. Time is dilated for the nucleus
2. Any probe will have larger k^- than color current $\Longrightarrow$ probe will oscillate faster in x^+ and see current as static.

Extreme approximation:

$$
\begin{aligned}
j^+(x^-,\mathbf{x}_\perp) &\approx \delta(x^-)\rho(\mathbf{x}_\perp)\\
A^+(x^-,\mathbf{x}_\perp) &\approx \delta(x^-)\frac{1}{\boldsymbol{\nabla}_\perp^2}\rho(\mathbf{x}_\perp)
\end{aligned}
$$

Spacetime picture in LC gauge

Recall: partonic interpretation needs LC gauge — Gauge transform:

$$A^+ \quad \Rightarrow \quad V^\dagger(\mathbf{x}_\perp, x^-) A^+ U(\mathbf{x}_\perp, x^-) - \frac{i}{g} V^\dagger(\mathbf{x}_\perp, x^-) \partial_- V(\mathbf{x}_\perp, x^-) = 0$$

$$A^- \quad \Rightarrow \quad -\frac{i}{g} V^\dagger(\mathbf{x}_\perp, x^-) \partial_+ V(\mathbf{x}_\perp, x^-) = 0, \text{ still}$$

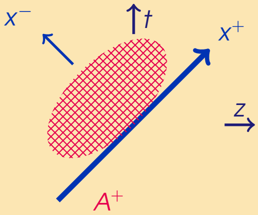
$$A^i \quad \Rightarrow \quad \frac{i}{g} V^\dagger(\mathbf{x}_\perp, x^-) \partial_i V(\mathbf{x}_\perp, x^-)$$

transverse pure gauge

This is solved by familiar Wilson line

$$V(\mathbf{x}_\perp, x^-) = \mathbb{P} \exp \left[-ig \int^{x^-} dy^- A^+ \right]$$

Since $A^+_{\text{cov}} \sim \delta(x^-)$, the integral $A^i_{\text{LC}} \sim \theta(x^-)$ — delocalized in x^-, just like small k^+ physical gluons should be.



Weizsäcker-Williams gluon distribution

In LC quantization (Now of nucleus, not γ^*) number distribution of gluons:

$$\frac{dN}{d^2\mathbf{k}_\perp\, dy} \sim \left\langle A_a^i(\mathbf{k}_\perp) A_a^i(-\mathbf{k}_\perp) \right\rangle$$

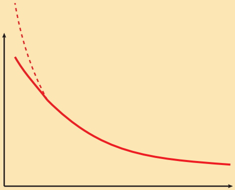
- $A_a^i(\mathbf{k}_\perp)$ is obtained from the Wilson line
- Wilson line is related to DIS dipole cross section, BK equation
- One can express this **Weizsäcker-Williams** gluon distribution as:

$$\frac{dN}{d^2\mathbf{k}_\perp\, dy} = \varphi^{WW}(\mathbf{k}_\perp) = \frac{C_F}{2\pi^3}\frac{1}{\alpha_s}\int d^2\mathbf{b}_\perp \int d^2\mathbf{r}_\perp \frac{e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}}{\mathbf{r}_\perp{}^2}\widetilde{\mathcal{N}}(\mathbf{b}_\perp,\mathbf{r}_\perp)$$

($\widetilde{\mathcal{N}}$ is the adjoint representation Wilson line dipole correlator)

- Gluon saturation in $\varphi^{WW}(\mathbf{k}_\perp)$ at $\mathbf{k}_\perp \lesssim Q_s$
- $\varphi^{WW}(\mathbf{k}_\perp) \sim 1/\alpha_s \implies$ gluon "**condensate**"

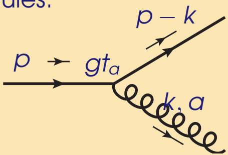
Now we have a **C**olor **G**lass **C**ondensate.



Exercise 2

(Kovchegov & Levin, exercise 5.1)

Let's calculate the gluon field of a high-energy quark as a function of spacetime from Feyman rules.



$$A^a_\mu(k) = -igt^a \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} \bar{u}_\sigma(p-k)\gamma^\nu u_\sigma(p)(2\pi)\delta((p-k)^2)$$

3 McLerran-Venugopalan model

McLerran-Venugopalan model

Basic idea:

- Want explicit model for color charges, easy to calculate things
- Based on simple physical argument: independent, uncorrelated color charges (CLT: Gaussian)
- Single phenomenological parameter: color charge density μ^2

MV model charge density $\rho(\mathbf{x}_\perp)$

- stochastic, Gaussian random variable
- local in x^- and $\mathbf{x}_\perp$ (nucleus still moving in $+$ direction)

$$\left\langle \rho^a(\mathbf{x}_\perp, x^-)\rho^b(\mathbf{y}_\perp, y^-)\right\rangle = g^2\delta^{ab}\mu^2(x^-)\delta(x^- - y^-)\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)$$

(Although $\rho^a(\mathbf{x}_\perp, x^-) \sim \delta(x^-)$, we need to spread the delta function out a bit to make sense of the path ordering.)

(My conventions: μ charge carrier number density, $g\mu$ charge density $\sim$ field A_μ, thus what appears in exponent of Wilson line is $\sim gA_\mu \sim g^2\mu \sim Q_s$. But convention varies.)

Dipole amplitude in MV model

Start with definition of dipole operator

$$S(\mathbf{x}_\perp - \mathbf{y}_\perp) \equiv \frac{1}{N_c} \left\langle \mathrm{Tr}\, V(\mathbf{x}_\perp) V^\dagger(\mathbf{y}_\perp) \right\rangle \qquad V(\mathbf{x}_\perp, x^-) = \mathbb{P} \exp\left[-ig \int_{-\infty}^{x^-} \mathrm{d}y^- A^+ \right]$$

$$\langle A_a^+(\mathbf{x}_\perp, x^-) A_b^+(\mathbf{y}_\perp, y^-) \rangle = g^2 \delta^{ab} \mu^2(x^-) \delta(x^- - y^-) L(\mathbf{x}_\perp - \mathbf{y}_\perp)$$

Remember, $A_a^+(\mathbf{x}_\perp, x^-)$ is the covariant gauge field that satisfies

$$\nabla_\perp^2 A_a^+(\mathbf{x}_\perp, x^-) = \rho(\mathbf{x}_\perp, x^-)$$

We'll calculate $L(\mathbf{x}_\perp - \mathbf{y}_\perp)$ later.
Discretize x^- to make sense of path ordering

$$x^- = n\Delta^- \quad \Longrightarrow \quad \langle A_{ma}^+(\mathbf{x}_\perp) A_{nb}^+(\mathbf{y}_\perp) \rangle = g^2 \delta^{ab} \mu_n^2 \frac{1}{\Delta^-} \delta_{mn} L(\mathbf{x}_\perp - \mathbf{y}_\perp)$$

$$S(r) = \lim_{N\to\infty} \frac{1}{N_c} \mathrm{Tr} \Big\langle e^{-ig\Delta^- A_N^+(\mathbf{x}_\perp)} \ldots e^{-ig\Delta^- A_n^+(\mathbf{x}_\perp)} \ldots e^{-ig\Delta^- A_{-N}^+(\mathbf{x}_\perp)}$$
$$\times\, e^{ig\Delta^- A_{-N}^+(\mathbf{y}_\perp)} \ldots e^{ig\Delta^- A_n^+(\mathbf{u}_\perp)} \ldots e^{ig\Delta^- A_{-N}^+(\mathbf{y}_\perp)} \Big\rangle$$

Note: in V ordering is large x^- to small, in $V^\dagger$ inverse

Infinitesimal Wilson line

Steps in x^- independent and next to each other in product:
$\Longrightarrow$ expectation value for each separately:

$$\left\langle e^{-ig\Delta^- A_n^+(\mathbf{x}_\perp)} e^{ig\Delta^- A_n^+(\mathbf{y}_\perp)} \right\rangle \approx \Big\langle 1 - ig\Delta^- A_n^+(\mathbf{x}_\perp) + ig\Delta^- A_n^+(\mathbf{y}_\perp)$$

$$+g^2(\Delta^-)^2 A_n^+(\mathbf{x}_\perp) A_n^+(\mathbf{y}_\perp) - \frac{1}{2} g^2 (\Delta^-)^2 \left(\left(A_n^+(\mathbf{x}_\perp)\right)^2 + \left(A_n^+(\mathbf{y}_\perp)\right)^2 \right) \Big\rangle + \mathcal{O}(\Delta^-)^{3/2})$$

$\langle A^+ \rangle = 0$, and thus

$$\left\langle e^{-ig\Delta^- A_n^+(\mathbf{x}_\perp)} e^{ig\Delta^- A_n^+(\mathbf{y}_\perp)} \right\rangle$$

$$\approx 1 + g^4 (\Delta^-)^2 \frac{\mu_n^2}{\Delta^-} \left[L(\mathbf{x}_\perp - \mathbf{y}_\perp) - \frac{1}{2} L(\mathbf{x}_\perp - \mathbf{x}_\perp) - \frac{1}{2} L(\mathbf{y}_\perp - \mathbf{y}_\perp) \right] \overbrace{t^a t^a}^{= C_F \mathbb{I}_{N_c \times N_c}}$$

$$\approx e^{g^4 \Delta^- C_F \mu_n^2 \left[L(\mathbf{x}_\perp - \mathbf{y}_\perp) - \frac{1}{2} L(\mathbf{x}_\perp - \mathbf{x}_\perp) - \frac{1}{2} L(\mathbf{y}_\perp - \mathbf{y}_\perp) \right]}$$

(Remember Δ^- infinitesimal)

Put pieces together: dipole operator

We have taken the traces and the expectation values $\Longrightarrow$ all the infinitesimal values are just commuting numbers.

$$S(r) = \lim_{N\to\infty} \overbrace{\frac{1}{N_c}\mathrm{Tr}}^{=1} \prod_{n=-N}^{N} e^{g^4 \Delta^- C_F \mu_n^2 [L(\mathbf{x}_\perp - \mathbf{y}_\perp) - L(\mathbf{0})]}$$

$$= \exp\left\{ g^4 C_F \int_{-\infty}^{\infty} \mathrm{d}x^- \mu^2(x^-) \overbrace{[L(\mathbf{x}_\perp - \mathbf{y}_\perp) - L(\mathbf{0})]}^{\equiv -\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp)} \right\}$$

Note, the MV model is defined by two things:

1. The $\rho\rho$ correlators are Gaussian
2. The $\rho\rho$ correlators are delta functions in tranverse coordinate

Until now we have only used property 1.
Just choosing some desired functional form for $\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp)$ is the "Nonlinear Gaussian" approximation.

Functional form in MV model

Now what are $L(\mathbf{x}_\perp - \mathbf{y}_\perp)$ and $\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) = L(\mathbf{0}) - L(\mathbf{x}_\perp - \mathbf{y}_\perp)$?

$$A^+(k_T) = \frac{1}{\mathbf{k}_\perp^2}\rho(\mathbf{k}_\perp)$$

$$\left\langle \rho^a(\mathbf{k}_\perp)\rho^b(\mathbf{p}_\perp)\right\rangle = g^2\delta^{ab}\mu^2 \int \mathrm{d}^2\mathbf{x}_\perp\, \mathrm{d}^2\mathbf{y}_\perp\, e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp + i\mathbf{p}_\perp\cdot\mathbf{y}_\perp}\delta^{(2)}(\mathbf{x}_\perp - \mathbf{y}_\perp)$$
$$= g^2\delta^{ab}\mu^2(2\pi)^2\delta^{(2)}(\mathbf{k}_\perp + \mathbf{p}_\perp)$$

$$\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) = \int \frac{\mathrm{d}^2\mathbf{k}_\perp}{(2\pi)^2}\frac{\mathrm{d}^2\mathbf{p}_\perp}{(2\pi)^2}(2\pi)^2\frac{\delta^{(2)}(\mathbf{k}_\perp + \mathbf{p}_\perp)}{{p_T}^2{k_T}^2}\left[1 - e^{-i\mathbf{k}_\perp\cdot\mathbf{x}_\perp - i\mathbf{p}_\perp\cdot\mathbf{y}_\perp}\right]$$
$$= \frac{1}{2\pi}\int \mathrm{d}k\,\frac{1 - J_0(kr)}{k^3} \approx \frac{1}{8\pi}r^2 \ln \overbrace{\frac{1}{r\Lambda}}^{\gg 1}$$

Dipole operator in the MV model

Final result

$$S(r) \equiv \frac{1}{N_c}\left\langle \mathrm{Tr}\, V(\mathbf{x}_\perp)V^\dagger(\mathbf{y}_\perp)\right\rangle = \exp\left\{-\frac{g^4 C_F}{8\pi}\left[\int_{-\infty}^{\infty} \mathrm{d}x^- \mu^2(x^-)\right] r^2 \ln\frac{1}{r\Lambda}\right\}$$

From which we identify (up to log) the saturation scale as

$$Q_s^2 \sim \frac{g^4 C_F}{4\pi}\left[\int_{-\infty}^{\infty} \mathrm{d}x^- \mu^2(x^-)\right]$$

- Note: we spread out delta function $\delta(x^-)$ within the calculation, but in the end only the integral over x^- matters.
- We can also Fourier-transform this to get the "dipole gluon distribution"

$$\varphi_{\mathrm{dip.}}(k_T) \sim {k_T}^2 \int \mathrm{d}^2\mathbf{r}_\perp e^{i\mathbf{r}_\perp\cdot\mathbf{k}_\perp} S(r)$$

- Thanks to the logarithm of r in the MV model expression, it behaves like $1/{k_T}^2$ at large k_T.

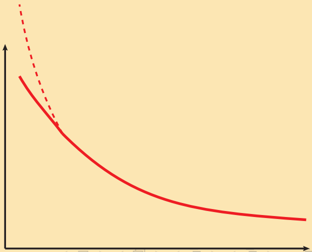
Weizsäcker-Williams distribution

By a similar, but more complicated, calculation one can also calculate the WW-distribution

$$\frac{dN}{d^2\mathbf{k}_\perp\, dy} = \varphi^{WW}(\mathbf{k}_\perp) \sim \left\langle A^i_a(\mathbf{k}_\perp) A^i_a(-\mathbf{k}_\perp) \right\rangle \qquad A^i_a(\mathbf{x}_\perp) = \frac{i}{g} V^\dagger(\mathbf{x}_\perp) \partial_i V(\mathbf{x}_\perp)$$

$$\left\langle A^i_a(\mathbf{x}_\perp) A^i_b(\mathbf{y}_\perp) \right\rangle = \delta^{ab} \frac{2C_F}{g^2(N_c{}^2 - 1)} \frac{\boldsymbol{\nabla}^2_\perp \Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp)}{\Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp)} \times \left(\exp\left\{ -g^4 N_c \int_{-\infty}^{\infty} dx^- \mu^2(x^-) \Gamma(\mathbf{x}_\perp - \mathbf{y}_\perp) \right\} - 1 \right)$$

$$\varphi_{WW}(k_T) \sim \left(\begin{array}{ll} \frac{1}{\alpha_s} \ln k_T, & k_T \ll Q_s^2 \\ \frac{1}{\alpha_s} \frac{Q_s^2}{k_T{}^2}, & k_T \gg Q_s^2 . \end{array} \right.$$



Some comments: two gluon distributions

- Factor in exponent of the WW distribution is an adjoint representation Casimir C_A, compared to $C_F = (N_c{}^2 - 1)/(2N_c)$ in the fundamental representation dipole. Expected for gluons.
- Both distributions $\sim 1/k_T{}^2$ at large $k_T \gg Q_s \implies$

$$xG(x, Q^2) = \int^{Q^2} d^2\mathbf{k}_\perp \varphi(k_T) \sim \ln Q^2$$

The expected behavior for a gluon distribution. Result of starting from independent point charges

- Dipole and WW distributions look quite different at at small k_T. Both exhibit saturation: the behavior changes at $k_T \lesssim Q_s$; the dipole distribution changes to $\sim k_T{}^2$ and the WW distribution to $\sim \ln k_T$.
- Both distributions have different operator definitions and appear in different processes. In traditional pQCD factoriation they would need to be measured separately. The MV model (or rather: the nonlinear Gaussian assumption) allows to calculate one from the other.

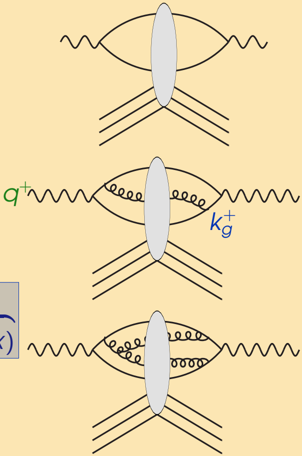
4 Balitsky-Kovchegov equation

Power counting

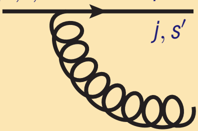
- Leading order (LO)
- Add one **soft** gluon $\Longrightarrow$ Leading Log (LL), resum by BK evolution
- Add one gluon, but **not** necessarily soft: Next-to-Leading Order (NLO)
  (need to subtract the soft gluon!)
- Add two gluons, one of them soft: NLL $\Longrightarrow$ resum by NLO BK equation

$$\sigma \sim \overbrace{\mathcal{O}(1)}^{\text{LO}} + \overbrace{\mathcal{O}(\alpha_s \ln 1/x)}^{\text{LL}} + \overbrace{\mathcal{O}(\alpha_s)}^{\text{NLO}} + \overbrace{\mathcal{O}(\alpha_s^2 \ln 1/x)}^{\text{NLL}}$$

- Current research: NLO & NLL
- Previous lectures: LO
- This lecture: LL



What happens if one radiates a gluon?



Light cone wavefunction

$$\psi^{q\to qg}(z, \mathbf{k}_\perp) = \frac{1}{\frac{\mathbf{p}_\perp{}^2}{2p^+} - \frac{\mathbf{k}_\perp^2}{2k^+} - \frac{\mathbf{p}'_\perp{}^2}{2p'^+}} \bar{u}_{s'}(p')(-g)t^a_{ji}\not{\varepsilon}^*(k)u_s(p)$$

Full matrix element (take quark mass $m = 0$)

$$\bar{u}_{s'}(p')(-g)t^a_{ji}\not{\varepsilon}^*(k)u_s(p) = \frac{-2gt^a_{ji}}{z\sqrt{1-z}}(\delta_{\lambda,2s}+(1-z)\delta_{\lambda,-2s})\mathbf{q}_\perp\cdot\varepsilon_{\perp}{}^*_\lambda, \quad \mathbf{q}_\perp = \mathbf{k}_\perp - z\mathbf{p}_\perp$$

Focus on the **soft** (or slow) gluon limit $z \to 0$:

$$\psi^{q\to qg}(k^+, \mathbf{k}_\perp) \approx \frac{-2zp^+}{\mathbf{k}_\perp^2}\frac{-2gt^a_{ji}\delta_{ss'}}{z}\mathbf{k}_\perp\cdot\varepsilon_{\perp}{}^*_\lambda = \frac{4gt^a_{ji}p^+}{\mathbf{k}_\perp^2}\mathbf{k}_\perp\cdot\varepsilon_{\perp}{}^*_\lambda$$

IR divergences in gluon emission

Squaring the wave function we get a "probability for gluon emission"

$$dP_{q\to qg} = \left|\psi^{q\to qg}(k^+, \mathbf{k}_\perp)\right|^2 \frac{dk^+\, d^2\mathbf{k}_\perp}{2k^+(2\pi)^3} \sim \frac{dz}{z}\,\frac{d^2\mathbf{k}_\perp}{\mathbf{k}_\perp^2} \qquad \left(\sum_{\lambda=\pm 1} \varepsilon_i \varepsilon_j^* = \delta_{ij}\right)$$

This has 2 types of divergences:

collinear $\int_0 \frac{d^2\mathbf{k}_\perp}{\mathbf{k}_\perp^2}$ This will cancel when we consider emission from color neutral dipole.

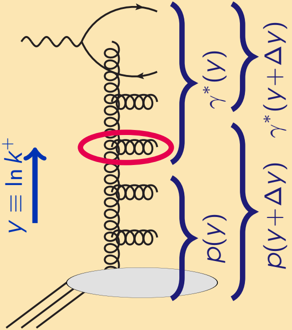
soft $\int_{\sim 0} \frac{dz}{z}$ Does not cancel, but needs to be resummed

The soft divergence is not really a divergence.

- The limit $z \to 0$: large $q\bar{q}g$ invariant mass: $M_{q\bar{q}g} \to \infty$
- We are working in the "$s = \infty$" eikonal approximation
- Physically, one must however have $M^2_{q\bar{q}g} < s \implies z \gtrsim [\perp]/s$
  (where "$\perp$" is some relevant transverse momentum scale.)
- Thus "divergence" is a sign of a large log $\sim \alpha_s \ln s \sim \alpha_s \ln 1/x$

We will resum this large logarithm using the Balitsky-Kovchegov renormalization group equation

Soft gluons and large logs, idea of RGE



- Emitted gluons have z between 1 and x: each gluon contributes $\sim \alpha_s \ln 1/x$
- For x small $\alpha_s \ln 1/x \sim 1$ $\Longrightarrow$ all n gluon emissions contribute same $\Longrightarrow$ resum
- Done by Renormalization Group Eq

Is the gluon at y a part of γ^* or of p?

You have to decide!
Physical cross section is the same.

$$\sigma^{\gamma^* p} = \overbrace{\left|\psi^{\gamma^*\to q\bar q}\right|^2_y \otimes 2\mathrm{Im}\mathcal{A}_y^{q\bar q p} + \left|\psi^{\gamma^*\to q\bar q g}\right|^2_y \otimes 2\mathrm{Im}\mathcal{A}_y^{q\bar q g p}}^{\text{gluons up to } y \text{ are part of proton}} + \dots$$

$$= \underbrace{\left|\psi^{\gamma^*\to q\bar q}\right|^2_{y+\Delta y} \otimes 2\mathrm{Im}\mathcal{A}_{y+\Delta y}^{q\bar q p} + \left|\psi^{\gamma^*\to q\bar q g}\right|^2_{y+\Delta y} \otimes 2\mathrm{Im}\mathcal{A}_{y+\Delta y}^{q\bar q g p} + \dots}_{\text{gluons up to } y+\Delta y \text{ are part of proton}}$$

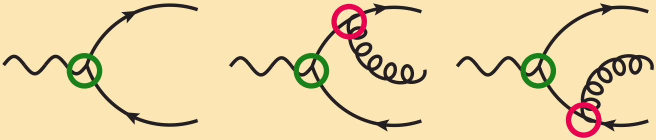
Can calculate $\left|\psi^{\gamma^*\to q\bar q}\right|^2_y$'s $\Longrightarrow$ get differential equation for unknown $\mathcal{A}$

Quick derivation of the BK equation

Let's put this idea into practice. We will

- Calculate $\psi^{\gamma^* \to q\bar{q}g}(z)$
- Take soft gluon limit $z \to 0$
- Reabsorb the gluon to become a part of the target
- Get evolution equation for $q\bar{q}$ cross section

We need:



In the soft gluon limit $z \to 0$ this calculation simpfies, because

$$\psi^{\gamma^* \to q\bar{q}g} \approx \psi^{\gamma^* \to q\bar{q}} \left(\psi^{q \to qg} + \psi^{\bar{q} \to \bar{q}g} \right)$$

This is true only for limit $z \to 0$ where $k^-_{q\bar{q}g} - k^-_{\gamma^*} \approx k^-_g \sim 1/z$

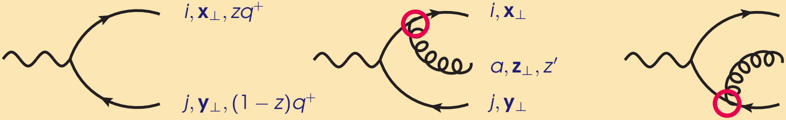
(In the full kinematics the gluon emission knows about the γ^*, not just the emitting parent $q/\bar{q}$ $\Longrightarrow$ this makes full NLO cross section computation much complex)

Gluon emission from coordinate space dipole

First step: Fourier-transform the gluon emission wavefunction to coordinate space

$$\psi^{q\to qg}(k^+,\mathbf{r}_\perp) = \int \frac{d^2\mathbf{k}_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp}\psi^{q\to qg}(k^+,\mathbf{k}_\perp) = -i2p^+\frac{2gt^a_{ji}}{2\pi}\frac{\varepsilon_\perp\cdot\mathbf{r}_\perp}{\mathbf{r}_\perp{}^2}\delta_{s,s'}$$

Second step: sum emission from quark and antiquark (note relative sign!)



$$\mathbf{r}_\perp = \mathbf{x}_\perp - \mathbf{y}_\perp \quad \mathbf{r}'_\perp = \mathbf{x}_\perp - \mathbf{z}_\perp \quad \mathbf{z}_\perp - \mathbf{y}_\perp = \mathbf{r}_\perp - \mathbf{r}'_\perp$$

$$|\gamma^*\rangle_{\text{int}} = |\gamma^*\rangle + \int_{z,\mathbf{r}_\perp} C(\mathbf{r}_\perp)\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r}_\perp)\,|q_i(\mathbf{x}_\perp,z)\bar{q}_i(\mathbf{y}_\perp,1-z)\rangle$$

$$+\int_{z,\mathbf{r}_\perp,\mathbf{r}'_\perp}\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r}_\perp)\int\frac{dz'}{4\pi z'}\frac{-i2g}{2\pi}t^a_{ji}\left[\frac{(\mathbf{x}_\perp-\mathbf{z}_\perp)\cdot\varepsilon_\perp}{(\mathbf{x}_\perp-\mathbf{z}_\perp)^2}-\frac{(\mathbf{y}_\perp-\mathbf{z}_\perp)\cdot\varepsilon_\perp}{(\mathbf{y}_\perp-\mathbf{z}_\perp)^2}\right]$$

$$\times\left|q_i(\mathbf{x}_\perp,z)\bar{q}_j(\mathbf{y}_\perp,1-z)g_a(\mathbf{z}_\perp,z')\right\rangle,$$

Virtual term

Here: take a "unitarity" shortcut instead of calculating loop diagram

$$|\gamma^*\rangle_{\mathrm{int}} = |\gamma^*\rangle + \int_{z,\mathbf{r}_\perp} C(\mathbf{r}_\perp)\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r}_\perp)\,|q_i(\mathbf{x}_\perp,z)\bar{q}_i(\mathbf{y}_\perp,1-z)\rangle$$

$$+\int_{z,\mathbf{r}_\perp,\mathbf{r}'_\perp}\psi^{\gamma^*\to q\bar{q}}(z,\mathbf{r}_\perp)\int\frac{\mathrm{d}z'}{4\pi z'}\frac{-i2g}{2\pi}t^a_{ji}\left[\frac{(\mathbf{x}_\perp-\mathbf{z}_\perp)\cdot\boldsymbol{\varepsilon}_\perp}{(\mathbf{x}_\perp-\mathbf{z}_\perp)^2}-\frac{(\mathbf{y}_\perp-\mathbf{z}_\perp)\cdot\boldsymbol{\varepsilon}_\perp}{(\mathbf{y}_\perp-\mathbf{z}_\perp)^2}\right]$$
$$\times\left|q_i(\mathbf{x}_\perp,z)\bar{q}_j(\mathbf{y}_\perp,1-z)g_a(\mathbf{z}_\perp,z')\right\rangle,$$

The gluon emission forces us to correct the normalization of the original dipole by a factor $C(\mathbf{r}_\perp)=1+\mathcal{O}(\alpha_s)$

$$N_c\,|C(\mathbf{r}_\perp)|^2 =$$

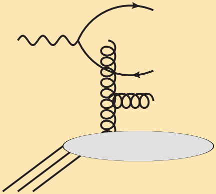
$$N_c-\frac{(2g)^2}{(2\pi)^2}\frac{1}{4\pi}t^a_{ij}t^a_{ji}\int\frac{\mathrm{d}z'}{z'}\int\mathrm{d}^2\mathbf{r}'_\perp\sum_{\lambda=\pm1}\left|\frac{(\mathbf{x}_\perp-\mathbf{z}_\perp)\cdot\boldsymbol{\varepsilon}_{\perp\lambda}}{(\mathbf{x}_\perp-\mathbf{z}_\perp)^2}-\frac{(\mathbf{y}_\perp-\mathbf{z}_\perp)\cdot\boldsymbol{\varepsilon}_{\perp\lambda}}{(\mathbf{y}_\perp-\mathbf{z}_\perp)^2}\right|^2$$

$$=N_c-\frac{\alpha_s}{\pi^2}\frac{{N_c}^2-1}{2}\Delta y\int\mathrm{d}^2\mathbf{r}'_\perp\frac{{\mathbf{r}_\perp}^2}{{\mathbf{r}'_\perp}^2(\mathbf{r}_\perp-\mathbf{r}'_\perp)^2}\qquad\sum_{\lambda=\pm1}\varepsilon_i^{(\lambda)}\varepsilon_j^{(\lambda)*}=\delta_{ij}$$

Crucial step: move the gluon to the target

Scattering amplitude is $\mathrm{Im}\mathcal{A}(\mathbf{r}_\perp) = \int d^2\mathbf{b}_\perp \mathcal{N}(\mathbf{b}_\perp, \mathbf{r}_\perp)$.
Absorb corrections from gluon to a redefinition of the $q\bar{q}$ amplitude

$$\mathcal{N}_{q\bar{q}}^{y+\Delta y} = \mathcal{N}_{q\bar{q}}^{y} + \frac{\alpha_s}{\pi^2}\frac{N_c^2-1}{2N_c}\int_y^{y+\Delta y} d\ln 1/z' \int d^2\mathbf{r}'_\perp \frac{\mathbf{r}_\perp^2}{\mathbf{r}'^2_\perp(\mathbf{r}_\perp - \mathbf{r}'_\perp)^2}\left[\mathcal{N}_{q\bar{q}g}^{\ln 1/z'} - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}\right]$$



Dipole scattering on new target $\mathcal{N}_{q\bar{q}}^{y+\Delta y}$ is

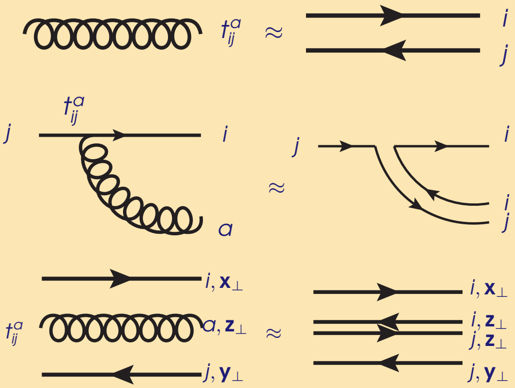
- Dipole scattering off original target $\mathcal{N}_{q\bar{q}}^{y}$
- Dipole emits a gluon into rapidity interval $[y, y+\Delta y]$, which scatters off target
- Normalization of original dipole is corrected
  (There are now less dipoles in γ^*)

Almost there

Want an equation for $\mathcal{N}_{q\bar{q}}$: but enocuntered new quantity $\mathcal{N}_{q\bar{q}g}$.
Needs to be related to $\mathcal{N}_{q\bar{q}}$ $\Longrightarrow$ Use the large N_c approximation

Gluon at large N_c

- At large N_c
  $\Longrightarrow$ gluon = $q\bar{q}$ pair
  (not dipole!)
- $N_c{}^2 - 1$ gluon colors $\approx N_c{}^2$ quark-antiquark pair colors.
- Had $|q(\mathbf{x}_\perp)\bar{q}(\mathbf{y}_\perp)g(\mathbf{z}_\perp)\rangle$
- Approximate by $|q(\mathbf{x}_\perp)\bar{q}(\mathbf{z}_\perp)q(\mathbf{z}_\perp)\bar{q}(\mathbf{y}_\perp)\rangle$



Now, instead of $\mathcal{N}_{q\bar{q}g}$, we need $\mathcal{N}_{q\bar{q}q\bar{q}}$:
amplitude for simultaneous scattering of two dipoles.

(Note: the gluon is not becoming a new dipole, but the common end of two new dipoles.)

Two dipole scattering amplitude

- $\mathcal{N}$ is really **scattering probability**;
- $S = 1 - \mathcal{N}$ is probability **not to scatter**

For two dipoles:

- No scattering: neither dipole scatters $\Longrightarrow S_{q\bar{q}q\bar{q}} = S_{q\bar{q}} S_{q\bar{q}}$
- Scattering probability $\mathcal{N}_{q\bar{q}q\bar{q}} = 1 - S_{q\bar{q}q\bar{q}} = 1 - (1 - \mathcal{N}_{q\bar{q}})(1 - \mathcal{N}_{q\bar{q}})$

Thus we end up with the approximation:

$$\mathcal{N}_{q(\mathbf{x}_\perp)\bar{q}(\mathbf{y}_\perp)g(\mathbf{z}_\perp)} \approx \mathcal{N}_{q(\mathbf{x}_\perp)\bar{q}(\mathbf{z}_\perp)} + \mathcal{N}_{q(\mathbf{z}_\perp)\bar{q}(\mathbf{y}_\perp)} - \mathcal{N}_{q(\mathbf{x}_\perp)\bar{q}(\mathbf{z}_\perp)}\mathcal{N}_{q(\mathbf{z}_\perp)\bar{q}(\mathbf{y}_\perp)}$$

and our equation is

$$\mathcal{N}_{q\bar{q}}^{y+\Delta y} = \mathcal{N}_{q\bar{q}}^{y} + \frac{\alpha_s}{\pi^2}\frac{N_c{}^2 - 1}{2N_c}\int_y^{y+\Delta y} \mathrm{d}\ln 1/z' \int \mathrm{d}^2\mathbf{z}_\perp \frac{(\mathbf{x}_\perp - \mathbf{y}_\perp)^2}{(\mathbf{x}_\perp - \mathbf{z}_\perp)^2(\mathbf{z}_\perp - \mathbf{y}_\perp)^2}$$
$$\times \Big[\mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x}_\perp, \mathbf{z}_\perp) + \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{z}_\perp, \mathbf{y}_\perp) - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x}_\perp, \mathbf{z}_\perp)\mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{z}_\perp, \mathbf{y}_\perp) - \mathcal{N}_{q\bar{q}}^{\ln 1/z'}(\mathbf{x}_\perp, \mathbf{y}_\perp)\Big]$$

Which is easy to write differentially in y

Summary

Balitsky-Kovchegov equation (∼1995)

$$\partial_y \mathcal{N}(\mathbf{r}_\perp) = \frac{\alpha_s N_c}{2\pi^2} \int d^2\mathbf{r}'_\perp \frac{{\mathbf{r}_\perp}^2}{{\mathbf{r}'_\perp}^2(\mathbf{r}'_\perp - \mathbf{r}_\perp)^2} \times \left[\mathcal{N}(\mathbf{r}'_\perp) + \mathcal{N}(\mathbf{r}_\perp - \mathbf{r}'_\perp) - \mathcal{N}(\mathbf{r}'_\perp)\mathcal{N}(\mathbf{r}_\perp - \mathbf{r}'_\perp) - \mathcal{N}(\mathbf{r}_\perp)\right]$$

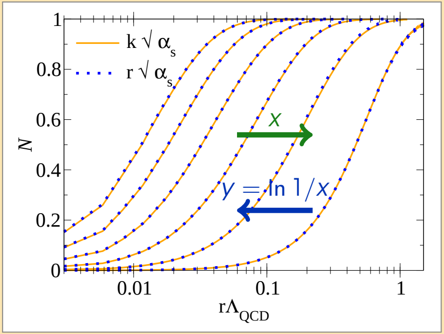
This is the basic tool of modern small-x physics.

- Given initial condition $\mathcal{N}(\mathbf{r}_\perp)$ at $y = y_0$ the equation predicts the scattering amplitude at larger y = smaller x = higher $\sqrt{s}$.
- Drop nonlinear term: BFKL equation
- Divergences at $\mathbf{r}'_\perp \to 0$ and $\mathbf{r}'_\perp \to \mathbf{r}_\perp$ regulated because $\mathcal{N}(0) = 0$ due to color neutrality.
- Enforces black disk limit $\mathcal{N} < 1$
- For practical work coupling α_s should depend on distance: some combination of $\mathbf{r}_\perp, \mathbf{r}'_\perp, \mathbf{r}_\perp - \mathbf{r}'_\perp$

What the solution of BK looks like

BK can be solved numerically

- Small dipoles $r \lesssim 1/Q_s$ scatter very little
- Large dipoles $r \gtrsim 1/Q_s$ scatter with probability almost one, but not more
- **Saturation** scale in between grows with $\ln 1/x$.



(Actually cheating, this plot is a solution of JIMWLK, which generalizes BK)

Remember, for F_2, F_L convolute this with the $\psi^{\gamma^* \to q\bar{q}}$

$$\sigma_{T,L}^{\gamma^* p} = \int d^2\mathbf{b}_\perp \, d^2\mathbf{r}_\perp \, dz \left|\psi^{\gamma^* \to q\bar{q}}(r,z)_{T,L}\right|^2 2\mathcal{N}(\mathbf{r}_\perp, \mathbf{b}_\perp, x)$$

Fits HERA data ($x < 0.01$ & Q^2 moderate) extremely well

Exercise

(Kovchegov & Levin, 4.5 b) Solve the BK equation in zero transverse dimensions:

$$\partial_y N = \alpha_s N - \alpha_s N^2, \quad N(y=0) = N_0 \ll 1$$